Notes for Mathematics 189-705B

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The goal of the first part of the course is an exposition of André Joyal's theorem that says (roughly) that every function from the reals to the reals in the free topos is continuous. This theorem of Joyal's will serve as a focus for a rather large body of general material with obvious importance in categorical logic. Later we will build on this general material to go in other specific directions.

Remark / June 1980: the notes as they exist here contain only some general material; they do not contain Joyal's thm. This and other things will be written up later.

§1. The interpretation of formulas in categories

1.1 Higher order logic and toposes.

In Lecture 5 [Lambek, '79 Fall], the language L_1 of higher order logic was introduced. We will adopt and use L_1 extensively. It will be important for us to be able to use "arbitrary types, functions and relations", and accordingly, we adopt a minor amendment to the formulation of L_1. Remember that L_1 was used to construct the free topos freely generated by the empty graph. Our modification amounts essentially to what is necessary to give a similar description of the topos freely generated by an arbitrary graph.

The 'arbitrary graph' is replaced by what we now call a frame of reference, (f. of r.), something slightly more general than an arbitrary graph. A f. of r. consists of:

(i) a (n arbitrary) set of type parameters. In order to be able to describe the other ingredients of a f. of r., namely relation and operation symbols, we define, the set of types just as they were defined loc. cit except that all type parameters are also types. Thus, $1, \Omega$, N are types (but not type parameters); e.g. $P(A \times \Omega)$ is a type where A is a type parameter, etc. (we write $P(\ )$ for the power-set operation, written P( ) loc. cit.). We might want, at some stage, to extend slightly the ways of type-formation. One obvious such extension would be to allow the exponential A^B for any types A and B.

Now, in the frame of ref., we also have

(ii) a stock of relation symbols

and

(iii) a stock of <u>operation symbols</u>;

with the following further specifications: Every relation symbol R has a number say n, its <u>arity</u>, associated with it (n may be 0). Furthermore, we have an n-tuple of types $A_1,\ldots,A_n$ also associated once for all with R. Intuitively, we want that R be a sub$\left\{\begin{matrix}\text{set}\\ \\ \text{object}\end{matrix}\right.$ of $A_1\times\ldots\times A_n$; we in fact write: $R \text{ " } \subset \text{ " } A_1\times\ldots\times A_n$ (1)
to mean that R has been thus specified. It is important here that the A_i can be any types, not just type parameters. E.g., we might want to consider a relation-symbol $R \text{ " } \subset \text{ " } A\times N\times\mathcal{P}(N)$, i.e., an unspecified ternary relation whose 'first place' is of type A (a type parameter), second place is to be filled by a natural number, and third place by a set of nat. no's. — We have a similar specification for operation symbols. The corresponding notation is $f : A_1\times\ldots\times A_n \text{ " } \longrightarrow \text{ " } A$. (2)

The language $\mathcal{L}$, based on a fixed but arbitrary frame of ref., is now defined as before except that we allow (i) $R(a_1,\ldots,a_n)$ as a formula (= term of type Ω !) if a_i is a term of type $A_i (i = 1,\ldots,n)$, for R as in (1), and (ii) $f(a_1,\ldots,a_n)$ as a term of type A if a_i is a term of type A_i $(i = 1,\ldots,n)$, for f as in (2). The axioms and rules concerning entailment ($\vdash_X$) are given exactly as before; what we have defined is intuitionistic higher order logic with respect to an arbitrary frame of reference.

Next we turn to interpreting $\mathcal{L}_1$ in a topos; we follow Lecture 6 loc. cit. If we denote the f of r. adopted by $\mathcal{F}$, then we have to start with an <u>interpretation</u>

$$M : \mathcal{F} \longrightarrow \mathcal{E}$$

of $\mathcal{F}$ in $\mathcal{E}$, a topos. M associates an object $M(A)$ of $\mathcal{E}$ with each type parameter A in $\mathcal{F}$; before we explain what else M should be specified to do, note that now $M(A)$ is defined for any arbitrary type A (not just a parameter) in a natural way: e.g., $M(N)$ = the natural number object in $\mathcal{E}$, $M(\mathcal{P}(A)) = \mathcal{P}(M(A))$ where $\mathcal{P}(\ )$ on the right-hand side denotes the power-set operation in $\mathcal{E}$. Now, M must specify a subobject

$$M(R) \hookrightarrow M(A_1)\times\ldots\times M(A_n)$$

or equivalently, an arrow

$$M(R) : M(A_1)\times\ldots\times M(A_n) \longrightarrow \Omega$$

(some abuse of notation to identify the two; we'll be more careful only if it is necessary), for any $R \in \mathcal{F}$ as in (1), and an arrow

$$M(f) : M(A_1)\times\ldots\times M(A_n) \longrightarrow M(A)$$

for f as in (2). This finishes the definition of an interpretation $M : \mathcal{F} \longrightarrow \mathcal{E}$.

The interpretation of arbitrary terms (including formulas) in $\mathcal{E}$ by M is now defined as before; we alter the notation somewhat, however. Recall that a term t of type A with exactly the distinct free variables $x_1,\ldots,x_m$ was interpreted as an arrow $\tilde{t} : 1 \longrightarrow M(A)$ ($M(A)$ was written simply A before) in the predogma $\mathcal{E}[x_1,\ldots,x_m]$, where x_i is an indeterminate of type $M(A_i)$ (!) . By functional completeness, $\tilde{t}$ gives rise to a unique arrow denoted $M(t): M(A_1)\times\ldots\times M(A_n) \longrightarrow M(A)$ in $\mathcal{E}$ (so that $\tilde{t}$ equals the composite

$$1 \xrightarrow{<x_1,\ldots,x_m>} M(A_1)\times\ldots\times M(A_m) \xrightarrow[M(t)]{} M(A)) \ .$$

Furthermore, in case of a formula ϕ , of course $M(\phi)$ will classify a subobject $M(\phi) \hookrightarrow M(A_1)\times\ldots\times M(A_m)$, (also denoted by $M(\phi)$, unless more precise notation is forced upon us). Now, as we said, we can take over the definition of $\tilde{t}$ with the straightforward amplification required by the 'new' terms, *or* , we can also give a direct definition of $M(t)$. What we would like to have, in final analysis, is a slight variant: given a term t and a list of district variables $\vec{x} = \langle x_1,\ldots,x_m \rangle$ of respective types $B_1,\ldots,B_m$, including, *but not necessarily identical to*, the set of free variables in t , we want to define

$$M_{\vec{x}}(t) : \underbrace{M(B_1)\times\ldots\times M(B_m)}_{M[\vec{x}]} \longrightarrow M(A)$$

(here t is of type A) . Here is one clause of the definition. Given a term $t \cdot = \cdot f(a_1,\ldots,a_n)$, with f an operation symbol as in (2), and $\vec{x}$ as above, we have (by induction hypothesis) the arrows

$$M_{\vec{x}}(a_i) \;:\; M[\vec{x}] \longrightarrow M(A_i) .$$

$M_{\vec{x}}(t)$ is now defined as the composite

$$M[\vec{x}] \xrightarrow{< M_{\vec{x}}(a_1),\ldots,M_{\vec{x}}(a_n)} \prod_{i=1}^{n}(M(A_i)) \xrightarrow[M(f)]{} M(A) .$$

The case of a formula $t \cdot = \cdot \phi \cdot = \cdot R(a_1,\ldots,a_n)$ (with R as in (1) is no different since this is the special case when $A = \Omega$. We will not give the other clauses of the definition. They can be inferred from Lecture 6, loc. cit. without difficulty. Also, the interpretation of the 'new' terms could be recast in the form of arrows in predogma-extensions; then the old clauses can be taken over verbatim. For an exposition of interpreting terms t as $M_{\vec{x}}(t)$ as we did (*except* for arbitrary relation and operation symbol, and for the subscript $\vec{x}$), see also Section 5.4, pp.152 – ... in [P.T. Johnstone, Topos Theory]. The convenience of having arbitrary relation and operation symbols will soon be apparent; since examples will abound later, we won't give any now.

Given a *sentence* σ (a formula without free variables), we say that σ *is valid in the interpretation* M (or: in $\mathcal{E}$ *by* M), and write $M \models \sigma$, or $M \models_{\mathcal{E}} \sigma$, if $M(\sigma) : (= M_{\emptyset}(\sigma)!) : 1 \to \Omega$ equals the arrow 'true' : $1 \to \Omega$ in $\mathcal{E}$, *or*, in the subobject interpretation (note that then $M(\sigma)$ is meant to be a subobject of 1), if $M(\sigma)$ is the 'total' (maximal) subobject 1 of 1. This is in agreement with loc. cit.; specifically, it coincides with the entailment 'empty' $\vdash_{\text{'empty'}} \sigma$ being valid as defined there. More generally, given formulas $\phi_1, .., \phi_n, \psi$ and variables $\vec{x}$ containing all free variables of the formulas, we could define the validity of the entailment

$$\phi_1, \ldots, \phi_n \vdash_{\vec{x}} \psi$$

as in loc. cit., or equivalently, by

$$M_{\vec{x}}(\phi_1) \wedge \ldots \wedge M_{\vec{x}}(\phi_n) \quad \leq \quad M_{\vec{x}}(\psi)$$

intersection of subobjects of $M[\vec{x}]$

order of subojects of $M[\vec{x}]$.

or equivalently again, by

$$M \models_{\mathcal{E}} \forall \vec{x} \; ((\phi_1 \wedge \ldots \wedge \phi_n) \to \psi)$$

with reference to validity of *sentences.* (*Exercise*).

The fundamental fact is that all logical and non-logical axioms (defined to be specific entailments in Lecture 5) will be always valid in any interpretation in any topos, as well as all (structural) rules will <u>preserve</u> validity; of course, this was proved in Lecture 5 and our changes and additions are inessential. — this fact is expressed by saying that <u>intuitionistic logic is valid in any topos</u>.

The most important choice of the frame is when it is 'identical' to the topos $\mathcal{E}$ we interpret it in. With any topos $\mathcal{E}$, the canonical frame of ref. associated with $\mathcal{E}$ has all the objects of $\mathcal{E}$ as types; in addition, it only has unary operation symbols $f : A$ "$\longrightarrow$" B, one for each morphism $f : A \longrightarrow B$; the <u>identical</u> <u>interpretation</u> of this frame in $\mathcal{E}$ does what its name suggests. With this M, we write $[\vec{x} : \phi]$ or $[x_1 \in A_1, \ldots : \phi])$ for $M_{\vec{x}}(\phi)$; or even, $[\phi]$ if the free variables meant are precisely the ones in ϕ. E.g., for $f : A \to B$ in $\mathcal{E}$, the image of f, $\mathrm{im}\, f$, a subobject of B, is the same as $[y \in B : \exists_{x \in A} (f(x) = y)]$ (<u>exercise</u>).

We write

$$\overset{\mathcal{E}}{\models} \sigma$$

for $M_{can} \overset{\mathcal{E}}{\models} \sigma$, with M_{can} the identical interpretation of the canonical frame in $\mathcal{E}$; also, we write

$$T \overset{\mathcal{E}}{\models} \sigma$$

to mean: if $\overset{\mathcal{E}}{\models} \bigwedge T$, then $\overset{\mathcal{E}}{\models} \sigma$.

Warning: $T \overset{\mathcal{E}}{\models} \sigma$ is not the same as

$$\overset{\mathcal{E}}{\models} (\bigwedge T \rightarrow \sigma)$$

the second is stronger!

The canonical frame is slightly but usefully extended in the following way. Given objects $A_1,\dots,A_n$ in $\mathcal{E}$, and a subobject $R \hookrightarrow A_1\times\dots\times A_n$, we introduce the *relation-symbol* $R \text{ "}\hookrightarrow\text{" } A_1\times\dots\times A_n$ into the language; of course its canonical intrepretation being the subobject we started with. Similarly, an operation-symbol $f : A_1\times\dots\times A_n \text{ "}\longrightarrow\text{" } A$ comes from an arrow $f : A_1\times\dots\times A_n \rightarrow A_n \rightarrow A$. ——— It looks like we have not done anything, but consider the following *example*. We define a subobject $R \hookrightarrow A\times A$ of $A\times A$ to be an *equivalence relation* (Grothendieck) on A if

$$\overset{\mathcal{E}}{\models} \text{'}R \text{ is reflexive, symmetric and transitive'}$$

i.e., $\overset{\mathcal{E}}{\models} Rxx$

$$\overset{\mathcal{E}}{\models} Rxy \rightarrow Ryx$$

$$\models (Rxy \wedge Ryz) \rightarrow Rxz$$

(we have omitted $\forall_{x\in A}\ \forall_{y\in A}$ in front as usual). Without the 'extended' canonical language, this could be said only in a considerably more complicated manner.

Another situation in which we have an interpretation $M : \mathcal{F} \rightarrow \mathcal{E}$ arises as follows. Imagine that we have a functor $F : \mathcal{E}' \rightarrow \mathcal{E}$; assume at least that F is left exact. Then F can be regarded an interpretation of the canonical frame $\mathcal{F}_{\mathcal{E}'}$ associated with $\mathcal{E}'$ in $\mathcal{E}$; this is quite obvious if we consider $\mathcal{F}_{\mathcal{E}'}$ only in the restricted sense, namely, with types and unary operations only. But even in the extended sense of $\mathcal{F}_{\mathcal{E}'}$ this is OK. If e.g. R "$\hookrightarrow$" $A_1\times\ldots\times A_n$ is a relation symbol based on a subobject $R \hookrightarrow A_1\times\ldots\times A_n)$ in $\mathcal{E}'$, then, since F is left exact, $F(A_1\times\ldots\times A_n) = FA_1\times\ldots\times FA_n$, and $F(R \hookrightarrow A_1\times\ldots\times A_n)$ is a subobject $FR \hookrightarrow FA_1\times\ldots\times FA_n$ (since a left exact F takes a monomorphism into a monomorphism). Thus, F *interprets* the *symbol* R "$\hookrightarrow$" $A_1\times\ldots\times A_n$ as a subobject $FR \hookrightarrow F(A_1)\times\ldots\times F(A_n)$ as it should.

Returning to the example above, let $R \hookrightarrow A \times A$ be an equivalence relation *in $\mathcal{E}'$* . F(R) being an equivalence relation on F(A) (in $\mathcal{E}$) is equivalent to saying that the three axioms above are valid *in $\mathcal{E}$* with R replaced by F(R) and A by F(A) . This is equivalent to saying that in the *interpretation* $F : \mathcal{E}' \longrightarrow \mathcal{E}$ those three sentences, *now as they are, without R and A being replaced* by anything, are valid (!) . It can be shown (see MR as well as [Fourman-Scott]) that the following is true. Whenever σ is a Horn sentence in the language of $\mathcal{E}'$ (σ of the form $\forall\vec{x}((\pi_1 \ldots \pi_n) \rightarrow \pi)$, with π_i , π first order atomic), then a left exact $F : \mathcal{E}' \rightarrow \mathcal{E}$ preserves the validity of σ :

$$\overset{\mathcal{E}'}{\models} \sigma \Rightarrow F \underset{\mathcal{E}}{\models} \sigma .$$

$\uparrow$ validity in $\mathcal{E}$ by F

$\uparrow$

validity in the
identical interpretation.

So, we conclude that a left exact functor <u>preserves equivalence relations</u>.

1.2. Extensions and restrictions

An important extension of L_1 is the consideration of infinite conjunctions and disjunctions. Syntactically, we adopt $\bigwedge\Sigma$ and $\bigvee\Sigma$ as formulas whenever Σ is a set of formulas satisfying the restriction that there should be altogether finitely many free variables in formulas in Σ. The interpretation of the new class of formulas will be defined by the following additional clauses: if $M : F \to E$ is as before, free variables of E are in $\vec{x}$, then

$$M_{\vec{x}}(\bigwedge\Sigma) = \underset{\uparrow}{\bigwedge}\{M_{\vec{x}}(\sigma) : \sigma \in \Sigma\}$$

inf (intersection)(g.ℓ.b.) of a set of subobjects of $M[\vec{x}]$ in E

$$M_{\vec{x}}(\bigvee\Sigma) = \underset{\uparrow}{\bigvee}\{M_{\vec{x}}(\sigma) : \sigma \in \Sigma\}$$

sup (union)(ℓ.u.b.) of a set of subobjects of $M[\vec{x}]$ in E .

These inf's and sup's don't necessarily exist in an (elementary) topos E ; but they do exist e.g., in Grothendieck topoi (see below). If they do, the corresponding formulas become <u>interpretable</u> in E ; more precisely, for a formula to become interpretable, all the infinite conjunctions and disjunctions in the formula have to be interpretable.

One also considers parts of the language we have introduced so far. *Infinitary first order logic* is the following 'fragment' of the full language. A f. of r. $\mathcal{F}$ now can have relation and operation symbols with type specifications that can be type parameters only. Terms are built up from operation symbols and variables only (no $\{ \mid \dots \}$, etc.) ; *atomic* formulas are those of the form $R(a_1,\dots,a_n)$, with R " $\subset$ " $A_1 \times \dots \times A_n$, a_i term of type A_i , and $a_1 = a_2$ with terms of identical types. Arbitrary formulas are built-up from atomic ones by finitary and infinitary $(\bigvee, \bigwedge)$ propositional operations, as well as $\forall_{x \in A}$, $\exists_{x \in A}$, with A a type (or 'sort') in the frame of ref. adopted. *Coherent* (*geometric*) *logic* has formulas built from atomic ones using only $\underset{\substack{\uparrow \\ \text{finitary conj.}}}{\wedge}$, $T(= \text{true})$, $\underset{\substack{\uparrow \\ \text{inf. disj.}}}{\bigvee}$, $\exists$. *Finitary first order logic* has no infinitary operations; *finitary coherent logic* is meant accordingly.

The reference [Makkai & Reyes: First order Categorical logic] (MR) has various results concerning the interpretation of first order logic in categories. We mention only two. A *coherent axiom* is a sentence of the form $\forall \vec{x} (\phi \rightarrow \psi)$, with ϕ and ψ coherent formulas; it is *countable* if only countable disjunctions are used in ϕ and ψ ; it is *finitary* if only finite disjunctions are used. A (*finitary*) (*countable*) *coherent theory* is a set (in the case of 'countable', a countable set) of (finitary)(countable) axioms. We have the ordinary (SET $-$) interpretation of first order (or even, higher order) logic, which is actually the one we explained above for the special case $\mathcal{E}$ = SET = the category of all sets and functions. For a theory T , and an axiom σ , we write

$T \models_{\mathcal{E}} \sigma$ if for all $M : \mathcal{F} \to \mathcal{E}$, interpretation of $\mathcal{F}$ in $\mathcal{E}$, $M \models_{\mathcal{E}} T$ ($M \models_{\mathcal{E}} \tau$ for $\forall \tau \in T$) implies that $M \models_{\mathcal{E}} \sigma$. Then 'ordinary' (model-theoretical) logical consequence is the same as $T \models_{\mathrm{SET}} \sigma$.

Theorem 1.1. (i) For finitary coherent T and σ, $T \models_{\mathrm{SET}} \sigma$ implies $T \models_{\mathcal{E}} \sigma$ for any topos $\mathcal{E}$.

(ii) For countable coherent T and σ, the same is true for toposes $\mathcal{E}$ in which all infinitary disjunctions in T and σ are interpretable.

The proof of this theorem is given by combining Theorem 3.5.4 (ii) (p. 129 in MR) and Corollary 5.2.3 (p. 162 in MR), together with the easy observation that the coherent fragment of $L_{\infty\omega}$ is "stable with respect to $M : \underset{\uparrow}{\mathcal{F}} \to \mathcal{E}$"

(called L in MR, p.129).

Instead of explaining this, we refer to MR where some minimal conditions for 'adequate' interpretations of formulas in categories (not necessarily toposes) are explained, chief among them being "stability under pullback". Actually, we will occasionally refer to interpretation of formulas when we don't know if the category is a topos; then the definitions in MR must be used.

Another result is

Theorem 1.2 Suppose that $\mathcal{E}$ is a finitely complete category (has finite left limits). Let $R \hookrightarrow A \times B$ be given. Then R is the graph of some arrow $f : A \longrightarrow B$ (i.e. $\overset{\mathcal{E}}{\models} \forall_{x\in A} \forall_{y\in B}(Rxy \leftrightarrow fx = y))$ if and only if $\overset{\mathcal{E}}{\models}$ 'R is functional', i.e.

$$\overset{\mathcal{E}}{\models} \forall_{x\in A} \; \exists_{y\in B} \quad Rxy$$

$$\overset{\mathcal{E}}{\models} \forall_{x\in A\ ,\ y_1 y_2\in B} \; ((Rxy_1 \wedge Rxy_2) \to y_1 = y_2) \ .$$

For this theorem, see Thm. 2.4.4 (p.89) in MR. For a topos $\mathcal{E}$, the same theorem was proved in [J. Lambek, Fall '79] .

1.3 Remarks concerning the informal uses of logic.

The rather obvious significance of Theorem 1.1 can be spelled out as follows. Suppose we have made an assumption concerning a few objects and morphisms in a topos $\mathcal{E}$ in the form of (a typically finite set say T of) certain sentences σ_i in the canonical frame associated with $\mathcal{E}$) being valid in $\mathcal{E}$. We'll see many examples of this situation. Imagine that we have an 'internal topological space' or a 'group object' in $\mathcal{E}$, etc.; all these notions will be introduced formally later, in the form just mentioned. Now we want to deduce a new property of the same objects and morphisms, formulated in the form of the sentence say σ being valid in $\mathcal{E}$. Note that what we want is: $T \overset{\mathcal{E}}{\models} \sigma$. Typically, we would have ample knowledge of the situation if $\mathcal{E}$ were the category of sets. If, e.g., the original assumption set T concerns one or more 'group objects', and σ is a further statement about these groups, then ordinary group theory would tell us that "$T \models \sigma$ is true in the ordinary sense", which is equivalent to saying that $T \models_{\mathrm{SET}} \sigma$ holds.

Now, if it so happens that both T and σ are both finitary coherent, or both countable coherent, then Theorem 1.1 allows us to conclude, without further work, that indeed $T \models^{\mathcal{E}} \sigma$ is true, for the original (arbitrary) topos $\mathcal{E}$ in which T and σ were formulated (in the 'countable' case we would need the appropriate additional assumption). [Notice that $T \models_{\mathcal{E}} \sigma$ implies $T \models^{\mathcal{E}} \sigma$!] . A similar procedure can be followed in the general case of T and σ being from the full language $\mathcal{L}_1$, even in the infinitary sense, although in this case a clear-cut statement is less easy to make. The fact of intuitionistic logic being valid in $\mathcal{E}$ could be used as follows. We would examine our informal proof of $T \models_{\mathrm{SET}} \sigma$ and could find that it could be formalized into a proof '$T \vdash \sigma$' in the formal system mentioned above (see Lecture 5, [J. Lambek, Fall'79]) . Then, of course, we could conclude that $T \models_{\mathcal{E}} \sigma$. But, typically, it would be a more-than-tedious task to show the formal probability of $T \vdash \sigma$.

There is, however, an informal sense of intuitionistic validity. For an excellent exposition of informal intuitionism, I refer to [A. Heyting, "Intuitionism, An Introduction", North Holland, 3rd. revised ed. 1972] . In intuitionistic mathematics, existence is taken seriously; something to exist is meant that it is possible to exhibit that something. This strong notion of existence is carried over into the interpretation of disjunction: a disjunction $\bigvee_{i \in I} \phi_i$ being true means that there exists an $i \in I$ such that ϕ_i

is true, i.e., that we can exhibit an $i \in I$ such that ϕ_i is true. So, the truth of the disjunction $\phi \vee \psi$ means that I can actually pinpoint one of the two formulas ϕ and ψ and assert that it is true. Therefore, the disjunction

Fermat's last theorem $\vee \neg$ Fermat's last theorem

has not been shown to be true, since I cannot pinpoint either of the two statements being ture. [Obsolete! (2008)]

Let us give a few more examples. Consider the following sentences in infinitary propositional logic:

$$(\bigvee_{i\in I} \phi_i) \wedge \psi \leftrightarrow \bigvee_{i\in I}(\phi_i \wedge \psi)$$

$$(\bigwedge_{i\in I} \phi_i) \vee \psi \leftrightarrow \bigwedge_{i\in I}(\phi_i \vee \psi)$$

You can convince yourself that, in ordinary (classical) logic, both are valid [technically, they are SET-valid]. Now, the first one is intuitionistically valid, but the second one is not. If, e.g., $(\bigvee_{i\in I} \phi_i) \wedge \psi$ is true, then we can find $i \in I$ such that ϕ_i is true; also, we know that ψ is true; hence $\phi_i \wedge \psi$ is true; so, we have found $i \in I$ such that $\phi_i \wedge \psi$ is true, as required. — The validity of the second sentence fails as follows: Suppose $\bigwedge_{i\in I}(\psi_i \vee \psi)$ is valid. Then for each i, we can actually find that one of ϕ_i and ψ is true. But now we are stuck; can we actually claim that either for all $i \in I$ (an infinite set, say) we have found ϕ_i true, or that at least once, we have found ψ true? To say so would require examining possibly infinitely many indices i that we could never complete. — The following example is instructive. Let, for $i \in \mathbb{N}$, α_i be the statement: "in the decimal expansion of π, starting with the i^{th} decimal place, there are at least 10 consecutive 7's", and let ϕ_i be $\neg \alpha_i$. Let ψ be $\bigvee_{i\in\mathbb{N}} \alpha_i$, i.e., "there is at least one i with the property ...". Now, nobody knows whether $\neg\psi$ or ψ is the case. This means that nobody knows if $\underbrace{\bigwedge_{i\in\mathbb{N}} \phi_i}_{\text{this being equivalent to } \neg\psi} \vee \psi$ is intuitionistically valid. But we do know $\bigwedge_{i\in I} (\phi_i \vee \psi)$: given any $i \in \mathbb{N}$, we can decide if α_i is true. If not, ϕ_i is true. If yes, then $\bigvee_{i\in\mathbb{N}} \alpha_i$ is true; that's it. — You can find many examples in [Heyting, loc. cit.] of this nature that suggest that certain classical truths <u>should</u> <u>not</u> be true intuitionistically. Naturally, to claim that the above is actually not valid intuitionistically requires a precise definition of intuitionistic truth. Once we have such, as we do for finitary logic as formulated in Lecture 5, or for infinitary logic as formulated in Chapter 5, MR, then we can actually prove that the above is not valid (see also later).

It is quite safe to say that informal intuitionistic validity implies formal intuitionistic validity. Although one cannot make this quite as precise as the corresponding statement in classical logic (where it is formulated as Godel's completeness theorem), it can be accepted similarly to Church's thesis, actually in the context of the full, even infinitary, higher order logic L_1. Thus, returning to our starting situation, if we can establish by informal but constructively/intuitionistically valid arguments that the assumptions T imply σ, we can more-or-less safely conclude that $T \overset{E}{\models} \sigma$.

Glossary of notation

$$\boxed{M \underset{\mathcal{E}}{\models} \sigma}$$

makes sense for an interpretation $M : \mathcal{F} \longrightarrow \mathcal{E}$ and it means that the sentence σ is (made) $\left\{\begin{matrix}\text{valid}\\ \text{true}\end{matrix}\right\}$ in $\mathcal{E}$ by M .

For a set T of sentences,

$$\boxed{M \underset{\mathcal{E}}{\models} T} \iff \forall \sigma \in T \text{ , } M \underset{\mathcal{E}}{\models} \sigma \text{ .}$$

$$\boxed{T \underset{\mathcal{E}}{\models} \sigma}$$

makes sense for any language in which T & σ are formulated, and any topos (category) $\mathcal{E}$; and it means: for all M , $M \underset{\mathcal{E}}{\models} T$ implies $M \underset{\mathcal{E}}{\models} \sigma$. Warning: (even for finite T)

$$\underset{\mathcal{E}}{\models} \bigwedge T \to \sigma \quad \text{might be stronger}$$

than $T \underset{\mathcal{E}}{\models} \sigma$; the former is equivalent to saying that $\underbrace{M(\bigwedge T)}_{\uparrow} \le \underbrace{M(\sigma)}_{\uparrow}$ for all M

a sub-
object of $\mathcal{E}$

(although for $\mathcal{E} = SET$, they are equivalent (why?)).

$$\boxed{\overset{\mathcal{E}}{\models} \sigma}$$

makes sense only if σ is in the canonical frame associated with $\mathcal{E}$, and then it means that σ is true in the identical interpretation.

$$\boxed{T \overset{\mathcal{E}}{\models} \sigma}$$

means: $\overset{E}{\models} \bigwedge T$ implies $\overset{E}{\models} \sigma$

(and again, it might be weaker than

$\overset{E}{\models} \bigwedge T \rightarrow \sigma$).

§ 2. Grothendieck toposes.

Historically, Grothendieck toposes came before elementary ones. Without any regard to history, here we give a brief summary of basic definitions and facts.

Unless otherwise stated, every category is assumed to be finitely complete. Also, every category is locally small, i.e. each hom-set is a set rather than a class.

Definition 2.1. (i) A geometric morphism

$$\begin{tikzcd}\mathcal{E}_1 \arrow[r, "u"] & \mathcal{E}_2\end{tikzcd}$$

between categories $\mathcal{E}_1$ and $\mathcal{E}_2$ is a pair

$$\begin{tikzcd}\mathcal{E}_1 \arrow[r, shift right, "u_*"'] & \mathcal{E}_2 \arrow[l, shift right, "u^*"']\end{tikzcd}$$

of adjoint functors: $\underset{\substack{\uparrow \\ \text{left-adjoint}}}{u^*} \dashv u_*$, with the additional property of u^* being left exact (preserving finite left limits).

(ii) A (geometric) inclusion is a geometric morphism u such that u_* is full and faithful.

Remark. Limits, and "everything else", are not 'distinguished' usually; they are defined only up to isomorphism. Preservation of limits is meant accordingly.

Definition 2.2. (i) Let $\mathcal{S}$ be a topos. An $\mathcal{S}$-topos is a geometric morphism $\mathcal{E} \to \mathcal{S}$, with $\mathcal{E}$ a topos. We also say: $\mathcal{E}$ is a topos over $\mathcal{S}$, with structure map $\mathcal{E} \to \mathcal{S}$.

(ii) The geometric morphism $\mathcal{E} \to \mathcal{S}$ is bounded if there is an object $G \in |\mathcal{E}|$ such that for any $E \in |\mathcal{E}|$ there are: $S \in |\mathcal{S}|$, a subobject $\Sigma \hookrightarrow u^{*}(S) \times G$ and an epi $\Sigma \twoheadrightarrow E$.

(iii) A Grothendieck topos is a topos over SET with a bounded structure map.

Remarks. For any topos $\mathcal{E}$, there is at most one geometric morphism $\mathcal{E} \to \mathrm{SET}$ up to a unique isomorphism; there is one precisely when copowers of any object in $\mathcal{E}$, indexed by any set, exist in $\mathcal{E}$. (see Prop. 4.41, p. 119 in TT). So, a topos $\mathcal{E}$ is a Grothendieck topos iff $\mathcal{E}$ has arbitrary copowers, and the (essentially unique) geometric morphism $\mathcal{E} \to \mathrm{SET}$ is bounded; the latter is (now) equivalent to the existence of a set $\mathcal{G}$ of objects (a set of generators) in $\mathcal{E}$ such that for any $E \in |\mathcal{E}|$, the family of all morphisms $G \to E$, with $G \in \mathcal{G}$, is an epimorphic family (the corresponding morphism $\coprod G_i \to E$ is an epimorphism).

The classical definition of Grothendieck toposes is given by the notions of presheaf, Grothendieck topology and sheaf.

Let $\mathcal{C}$ be a small category, now not necessarily finitely complete. The category of presheaves over $\mathcal{C}$ is the category of all functors $\mathcal{C}^{op} \to \mathrm{SET}$, denoted $(\mathcal{C}^{op}, \mathrm{SET})$, or even $\hat{\mathcal{C}}$.

Theorem 2.3. $\hat{\mathcal{C}}$ is a Grothendieck topos.

The (easy) proof is essentially contained in the proof of 1.12 (pp. 24-25) in TT. It is important to know how the (elementary) topos structure is computed in $\hat{\mathcal{C}}$; this is described in loc. cit.

Let $\mathcal{C}$ be as above. A Grothendieck topology on $\mathcal{C}$ is defined by specifying, for each object $A \in |\mathcal{C}|$, a set $\mathrm{Cov}(A)$ of families $\{A_i \to A: i \in I\}$ of morphisms, the specification satisfying the closure conditions (i) - (iv) specified below. An element of $\mathrm{Cov}(A)$ is a covering of A.

(i) $\{A \xrightarrow{\mathrm{Id}_A} A\} \in \mathrm{Cov}(A)$.

(ii) (pulling back coverings) if $\{A_i \to A: i \in I\} \in \mathrm{Cov}\ A$, $B \to A$ is a morphism, then the family of all morphisms $B' \to B$ such that for some $i \in I$ and $B' \to A_i$ we have that

$$\begin{tikzcd}
A_i \arrow[r] & A \\
B' \arrow[u] \arrow[r] & B \arrow[u]
\end{tikzcd}$$

is commutative ($B' \to B$ factors through at least one member of the given covering) is a covering of B.

Remark. If $\mathcal{C}$ has pullbacks, then the following simpler version of (ii) can be used: for $\{A_i \to A: i \in I\} \in \mathrm{Cov}\ A$, and any $B \to A$, the family $\{A_i \times_A B \to B: i \in I\}$ is a covering of B.

(iii) (composing coverings) if $\{A_i \to A: i \in I\} \in \mathrm{Cov}(A)$, and $\{A_{ij} \to A_i: j \in J_i\} \in \mathrm{Cov}(A_i)$ for all $i \in I$, then $\{A_{ij} \to A_i \to A: j \in J_i, i \in I\} \in \mathrm{Cov}(A)$.

(iv) (monotonicity). Suppose $\{A_i \to A: i \in I\} \in \mathrm{Cov}(A)$, and $\{A'_j \to A: j \in J\}$ is such that for all $i \in I$ there is $j \in J$ such that $A_i \to A$ factors through $A'_j \to A$. Then $\{A'_j \to A: j \in J\} \in \mathrm{Cov}(A)$.

A site is a category with a specified Grothendieck topology on it.

Remark. See SGA4. MR contains an almost self-contained introductory chapter (Ch. I) to Grothendieck toposes; there, however, only sites with finitely complete underlying categories are considered.

Let us fix a site $\mathcal{C}$. Let $\{A_i \xrightarrow{\alpha_i} A: i \in I\} \in \mathrm{Cov}(A)$ and let F be a presheaf $F: \mathcal{C}^{\mathrm{op}} \to \mathrm{SET}$. Let $\langle \zeta_i : i \in I \rangle$ be a family of elements (sections) $\zeta_i \in F(A_i)$ $(i \in I)$. We say this is a compatible family of sections if the following condition holds: whenever $i, j \in I$ and $B \to A_j$ are morphisms such that

$$\begin{tikzcd}
 & A_i \arrow[dr, "\alpha_i"] & \\
B \arrow[ur, "f"] \arrow[dr, "g"'] & & A \\
 & A_j \arrow[ur, "\alpha_j"'] &
\end{tikzcd}$$

commutes, we have that $F(f)(\xi_i) = F(g)(\xi_j)$ (both are elements of $F(B)$). F is a sheaf if for every covering $\{A_i \xrightarrow{\alpha_i} A: i \in I\}$ of any $A \in |\mathcal{C}|$, and any compatible family of sections $\xi_i \in F(A_i)$, there is a unique section $\xi \in F(A)$ such that $F(\alpha_i)(\xi) = \xi_i$ for all $i \in I$. The category of sheaves $\mathrm{Sh}(\mathcal{C})$, or $\widetilde{\mathcal{C}}$, is the full subcategory of $\hat{C}$ whose objects are the sheaves.

Theorem 2.4. The inclusion functor $\widetilde{C} \xrightarrow{i} \hat{C}$ is a geometric inclusion, i.e. it has a left exact left adjoint, called the associated sheaf functor.

For the proof, see MR, Chapter I, Section 2; or, in a more general context, Section 3.3, pp. 84 -, in TT; or again, in SGA4, Vol. I, Exposé II, pp. 228-.

Theorem 2.5. $\widetilde{C}$ is a (Grothendieck) topos.

For the easy but important proof, see 1.12 Proposition in TT, pp. 24-25.

There is another set of "exactness conditions" that can replace those defining elementary toposes in the definition of Grothendieck topos. A geometric category is one that

(i) has finite left limits

(ii) has stable sups (of arbitrary families of subobjects of a given object)

(iii) has stable images;

these are called ∞-logical in MR (see Definition 3.4.4 there), and these are the ones in which the infinitary coherent language can be 'adequately' interpreted (see 3.5.4(ii) in MR). A geometric functor (called an ∞-logical functor in MR, 3.4.5) is one that preserves the structure listed under (i) - (iii) above. An equivalence relation $R \hookrightarrow A \times A$ in $\mathcal{R}$ was defined (as an example) in § 1. A quotient of $R \hookrightarrow A \times A$ is a morphism $A \xrightarrow{p} B$ (with B possibly written as A/R) such that

$$\models^{R} \forall_{y\in B} \exists_{x\in A} (px = y). \quad \text{(p is an (extremal regular) epi)}$$

$$\models^{R} \forall_{x,x'\in A} (Rxx' \leftrightarrow p(x) = p(x')).$$

A family of morphisms $\alpha_i : A_i \to B$ $(i \in I)$ (with B possibly written as $\coprod_{i\in I} A_i$) is a <u>disjoint</u> <u>sum</u> if

$$\models^{R} \alpha_i(x) = \alpha_i(x') \to x = x' \ (i \in I) (\alpha_i \text{ is a mono})$$

$$\models^{R} \forall_{x\in A_i, y\in A_j} (\alpha_i(x) = \alpha_j(y) \to \bot)$$

(the subobjects $A_i \hookrightarrow B$, $A_j \hookrightarrow B$ are disjoint)

$$\models^{R} \forall_{y\in B} \bigvee_{i\in I} \exists_{x\in A_i} (\alpha_i(x) = y)$$

(the family of the α_i's is jointly surjective).

A ∞-<u>pretopos</u> is a geometric category satisfying the following two additional axioms:

(iv) every equivalence relation has a quotient;

(v) every set of objects has a disjoint sum.

Finally, a Giraud topos is an ∞-pretopos with a set of generators (c.f. the Remark after the definition of Grothendieck topos).

The above definition is a variant of the 'classical' Giraud definition, see 1.4.3 in MR. It is designed to emphasize the connection with infinitary coherent logic. Proposition 3.4.8 in MR is the statement of the equivalence of the two variants.

Theorem 2.6. (Giraud's theorem) The following are equivalent for any category $\mathcal{E}$.

(i) $\mathcal{E}$ is a Grothendieck topos

(ii) $\mathcal{E}$ is a Giraud topos

(iii) There is a small category $\mathcal{C}$ and a geometric inclusion $\mathcal{E} \xrightarrow{i} \hat{\mathcal{C}}$.

(iv) $\mathcal{E}$ is equivalent to the category of sheaves over a small site.

The implication (iv) ⇒ (iii) is Thm 2.4 above. The equivalence of (ii) ⇒ (iv) is proved in SGA4, and in MR, Theorem 1.4.5. The equivalence of conditions (i) and (iii) is proved, in a more general form, in TT, 4.46 Theorem (p. 123).

Theorem 2.7. A functor $F: \mathcal{E} \to \mathcal{E}'$ between Grothendieck toposes is u^* for some geometric morphism $u: \mathcal{E}' \to \mathcal{E}$ if and only if F is geometric.

For the proof, see 3.4.10 in MR (p.124).

Definition 2.8. Let $\mathcal{C}, \mathcal{D}$ be sites with finitely complete underlying categories. A functor $F: \mathcal{C} \to \mathcal{D}$ is continuous if it is left exact and transforms any covering into a covering.

Remark. This terminology is at variance with that in SGA4, and TT; we call 'continuous' a functor that they would call 'continuous and left exact'.

The canonical topology on any category can be defined (see MR and SGA4); on a Grothendieck topos $\mathcal{E}$ it is the same as the one defined by

$$\{E_i \xrightarrow{f_i} E\colon i \in I\} \in \mathrm{Cov}(E) \Leftrightarrow \bigvee_{i\in I} I_m(f_i) = I_E$$

$$\Leftrightarrow \models^{\mathcal{E}} \forall_{x\in E} \bigvee_{i\in I} \exists_{y\in E_i} (f_i(y) = x).$$

Any Grothendieck topos is considered a site with its canonical topology; a functor between Gr. toposes is continuous iff it is geometric (3.4.10 in MR).

Theorem 2.9. Let $\mathcal{C}$ be a finitely complete site, let $\varepsilon\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ be the composite $\mathcal{C} \xrightarrow{y} \hat{\mathcal{C}} \xrightarrow{a} \widetilde{\mathcal{C}}$ of the Yoneda embedding followed by the associated sheaf functor. Then ε has the following universal property: for any continuous $M\colon \mathcal{C} \to \mathcal{E}$ into a Grothendieck topos $\mathcal{E}$, there is a geometric functor $\widetilde{M}\colon \widetilde{\mathcal{C}} \to \mathcal{E}$, unique up to a unique isomorphism, such that the following commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\varepsilon"] \arrow[dr, "M"'] & \widetilde{\mathcal{C}} \arrow[d, "\widetilde{M}"] \\
 & \mathcal{E}
\end{tikzcd}$$

More strongly, the functor

$$\mathrm{Con}(\widetilde{\mathcal{C}},\mathcal{E}) \to \mathrm{Con}(\mathcal{C},\mathcal{E})$$

defined by composition by ε (from continuous $\widetilde{\mathcal{C}} \to \mathcal{E}$ to continuous $\mathcal{C} \to \mathcal{E}$) is an equivalence of categories.

For the <u>proof</u>, see 1.3.15, although the part after 'more strongly' is not stated there. See also Section 4.9, Exposé IV, SGA4, p. 354.

§ 3. Complete Heyting Algebras.

We follow here FS closely.

3.1. The category of complete Heyting algebras.

Definition 3.1.1. (i) A complete Heyting algebra, cHa, or local lattice, is a poset (partially ordered set) $(H, \leq)$ with a maximal element, denoted 1, such that any two elements $x,y \in H$ have a greatest lower bound $x \wedge y$, and any set Σ of elements of H has a least upper bound denoted $\bigvee\Sigma$, and such that the following identity holds

$$(\bigvee\Sigma) \wedge x = \bigvee\{y \wedge x\colon y \in \Sigma\}.$$

(ii) A morphism of cHa's, $h\colon H \to K$, is a map between the underlying sets of the cHa's H, K that preserves $\leq, 1, \wedge, \bigvee$: $h(x \wedge y) = h(x) \wedge h(y)$, $h\bigvee\Sigma = \bigvee\{h(x)\colon x \in \Sigma\}$ (preservation of $\leq, 1$ being a consequence).

(iii) The category cHa of cHa's is the category whose objects are the cHa's, and morphisms are as specified, with composition being the composition of maps between two underlying sets.

Remark. We will use the same latter, say H, for the cHa and its underlying set. For emphasis, we might write $|H|$ for the underlying set. Also, we shall write something like $\bigvee^{(H)}$ for sup in H, etc.; we won't do this unless forced to.

Main Example. Let E be a topological space, and let $H = \mathcal{O}(E)$ be the collection of all open sets of E. (A space E is understood to be given by $\mathcal{O}(E)$ such that $\mathcal{O}(E)$ is closed under finite intersection and arbitrary union). Then, with set-theoretic containment as the ordering $\leq$, H is a cHa: $\wedge$ is ordinary set-theoretic intersection, $\bigvee$ is ordinary set theoretic union. - It is far from being true that every cHa is isomorphic to one of the form $\mathcal{O}(E)$.

The importance of this example comes from the following further connection. Given any continuous function $f: E \to E'$ between top. spaces, one has the inverse-image map: $f^{-1}: \mathcal{O}(E') \to \mathcal{O}(E)(f^{-1}(V) = \{x \in E: f(x) \in V\}$ for any $V \in \mathcal{O}(E'))$; it is immediate that f^{-1} is a cHa-morphism. For a significant class of spaces, indeed, for most spaces of interest, there is a converse to this statement: every cHa-morphism between the cHa's of opens gives rise to a unique continuous map (in the opposite direction) between the spaces themselves. We will explain this in detail a bit later.

A very special case of open set algebras is a 'power-set algebra'. For any set X, the power-set $\mathbb{P}(X)$ of X is a cHa with containment as the order; this is the special case of the above $\mathcal{O}(E)$ with E the discrete space on X (every subset of X is open). Classically, $\mathbb{P}(X)$ is in fact a complete Boolean algebra (see below). We say 'classically' because we have in mind a context, the context of a given ambiant topos, when this will not be true any more, but when we still will want to be able to talk about 'internal' cHa's etc. Although we don't want to make any formal statement about facts being proved intuitionistically, we want to point out distinctions informally that will help us later to formally internalize things when we come to that.

As a matter of fact, intuitionistically $\mathbb{P}(X)$ is not a Boolean algebra, and indeed, much worse is the case.

Let's consider the power set $\mathbb{P}(\mathbb{1})$ of the one-element set $\mathbb{1} = \{*\}$. In ordinary (classical) mathematics, $\mathbb{1}$ has two subsets, namely $\emptyset$ and $\mathbb{1}$ itself, hence $\mathbb{P}(\mathbb{1})$ is a 2-element set. Intuitionistically, this is not a valid statement any more. One can e.g. define the subset X of $\mathbb{1}$ by the following specifications:

$* \in X \Leftrightarrow$ Format's last theorem is true. This is a perfectly good definition of the subset $X \subset \mathbb{1}$, but we don't *know* if X is empty or $X = \mathbb{1}$. [Recall, formally, that $P(\mathbb{1})$ in a topos is the algebra of truth values]. In other words, in intuitionism, it is perfectly legitimate to talk about $\mathbb{P}(\mathbb{1})$, *but* $\mathbb{P}(\mathbb{1})$ is not the same as a two-element set!

Now, let us return to the familiar context of classical mathematics. Then we see that $\mathbb{P}(\mathbb{1})$, as a cHa, is an initial object of *cHa*: given any cHa, H, there is a unique cHa-morphism

$$\mathbb{P}(\mathbb{1}) \to H ,$$

namely the one that assigns 1_H to $\mathbb{1}(=\mathbb{1}_{\mathbb{P}(\mathbb{1})})$ and $0_H(=\bigvee^{(H)}\emptyset)$ to $\emptyset = (0_{\mathbb{P}(\mathbb{1})})$. Actually, $\mathbb{P}(\mathbb{1})$ is an initial object of *cHa* on purely intuitionistic grounds as well. We will return to this point when it will be necessary; officially, we stay in classical mathematics for a while.

cHa's are defined in an 'algebraic' manner, with operations satisfying identities ($\leq$ is redundant since $x \leq y \Leftrightarrow x \wedge y = x$), although one of the operations, $\bigvee$, is an 'infinitary' one (it operates on *sets* of elements). As a consequence, projective limits of arbitrary diagrams of cHa's is computed 'as usual'; technically, we say that the underlying set-functor *cHa* → SET *creates* (projective) limits (see CWM). E.g., the product $\prod_{i\in I} H_i$ of a family $\{H_i : i \in I\}$ of cHa's has the cartesian product $\prod_{i\in I}|H_i|$ as its underlying set, and operations are defined element-wise: e.g., $(\bigvee_{j\in J}\sigma_j)(i) = \bigvee_{j\in J}^{(H_i)}\sigma_j(i)$ for any family $\{\sigma_j : j \in J\} \subset \prod_{i\in I}|H_i|$ (*exercise*).

Considerations of colimits, and the related question of free algebras, will be important for us. For this, and other purposes as well, we will now discuss a method of presenting cHa's. This method will be closely related to sites; in fact, we now define a particular kind of site. A p-site, by definition, is a poset $(P,\leq)$ with 1 and $\wedge$, together with a Grothendieck topology on $(P,\leq)$, the latter considered a category in the following (familiar) way: the objects are the elements of P, $|\hom(p,p')| \leq 1$, and $|\hom(p,p')| = 1$ iff $p \leq p'$; it is clear that this uniquely defines a category. In particular, what we have is the relation $\mathrm{Cov} \subset \mathcal{P}(P) \times P$ of covering: $(\Sigma,p) \in \mathrm{Cov} \Leftrightarrow \Sigma \in \mathrm{Cov}(p)$ with our earlier notation.

The definition of Grothendieck topology restated yields now:

(i) $\{p\} \in \mathrm{Cov}(p)$;

(ii) $\Sigma \in \mathrm{Cov}(p) \Rightarrow \Sigma(\wedge)q \in \mathrm{Cov}(q)$ (here $\Sigma(\wedge)q = \{x \wedge q: x \in \Sigma\}$);

(iii) if $\{p_i: i \in I\} \in \mathrm{Cov}(p)$, and $\Sigma_i \in \mathrm{Cov}(p_i)$, for all $i \in I$, then $\bigcup_{i\in I} \Sigma_i \in \mathrm{Cov}(p)$.

(iv) if $\Sigma \in \mathrm{Cov}(p)$, Φ consists of elements $\leq p$, and Φ dominates Σ in the sense that for all $x \in \Sigma$ there is $y \in \Phi$ with $x \leq y$, then $\Phi \in \mathrm{Cov}(p)$.

For later reference, we note the following as a consequence:

(v) Suppose Φ consists of elements $\leq p$, $\Sigma \in \mathrm{Cov}(p)$, and for every $q \in \Sigma$, we have $\Phi(\wedge)q \in \mathrm{Cov}(q)$. Then $\Phi \in \mathrm{Cov}(p)$.

Proof: by (iii), the set $\Phi(\wedge)\Sigma \underset{df}{=} \{x \wedge q: x \in \Phi, q \in \Sigma\} = \bigcup_{q\in\Sigma} (\Phi(\wedge)q) \in \mathrm{Cov}(p)$. On the other hand, clearly Φ dominates $\Phi(\wedge)\Sigma$, so by (iv) $\Phi \in \mathrm{Cov}(p)$.

□

A *morphism* of p-sites is defined to be the same as a continuous functor between sites: in other words, $f\colon P \to Q$, for p-sites P and Q, is a function between the underlying sets that preserve $\leq$, 1 and $\wedge$, as well as satisfies:

$$\Sigma \in \mathrm{Cov}^{(P)}(p) \Rightarrow \underbrace{f[\Sigma]} \in \mathrm{Cov}^{(Q)}(f(p)).$$

$$= \{f(p)\colon p \in \Sigma\}.$$

We have thus defined the category of p-sites, **p-sites**.

Notice that every cHa is a p-site, in a natural way: $\Sigma \in \mathrm{Cov}(x) \Leftrightarrow \bigvee\Sigma = x$. Also, between cHa's, a p-site-morphism is the same as a cHa-morphism. In other words, **cHa** is a full subcategory of **p-sites**.

Now, every p-site can be regarded as a *presentation* of a uniquely determined cHa. We explain this in the properly categorical spirit.

Theorem 3.1.2. The inclusion $i\colon \mathbf{cHa} \hookrightarrow \mathbf{p\text{-}sites}$ has a left adjoint:

$$\begin{tikzcd} \mathbf{cHa} \arrow[r, hook, shift right, "i"'] & \text{p-sites.} \arrow[l, shift right, "L"'] \end{tikzcd}$$

$$L \dashv i$$

Equivalently(!), for every p-site P there is (a necessarily unique, up to iso.) cHa $L(P)$ together with a morphism of p-sites $\varepsilon\colon P \to L(P)$ with the following universal property: for any cHa H and any p-site morphism $f\colon P \to H$ there is a unique cHa-morphism $\tilde{f}\colon L(P) \to H$ such that

$$\begin{tikzcd} P \arrow[r, "\varepsilon"] \arrow[dr, "f"'] & L(P) \arrow[d, "\tilde{f}"] \\ & H \end{tikzcd}$$

commutes. $L(P)$ is the cHa defined by P (as a 'presentation'). We can make sure that $L(iP) = P$ if P is a cHa; in other words, $L \circ i = \mathrm{Id}_{cHa}$.

Proof. Let P be a p-site. Define the relation $\le$ on $\mathcal{P}(P)$ as follows:

$$\Sigma \le \Phi \iff \forall_{x \in P}(\Sigma(\wedge)x \in \mathrm{Cov}^{(P)}(x) \to \Phi(\wedge)x \in \mathrm{Cov}^{(P)}(x)).$$

Also, define $\sim$ by

$$\Sigma \sim \Phi \iff \Sigma \le \Phi \ \& \ \Phi \le \Sigma.$$

Clearly, $\le$ is a quasi-order, $\sim$ an equivalence, and $\le$ induces a partial ordering also denoted $\le$ on $L \underset{df}{=} \mathcal{P}(P)/\sim$ = the set of equivalence classes of $\sim$. For the equivalence class of Σ we write: $\bigvee^{(f)}\Sigma$ ('formal sup of Σ').

We claim that $(L,\le)$ is a cHa. Clearly, $\bigvee^{(f)}(\mathcal{P}(P))$ is the maximal element of L. We claim that

$$\bigvee^{(f)}\Sigma \wedge \bigvee^{(f)}\Phi = \bigvee^{(f)}(\Sigma(\wedge)\Phi) .$$

It is clear that Σ dominates $\Sigma(\wedge)\Phi$, hence $\Sigma(\wedge)p$ dominates $(\Sigma(\wedge)\Phi)(\wedge)p$, for any p. Hence by condition (iv) defining p-sites, it follows that $\Sigma(\wedge)\Phi \le \Sigma$, i.e. $\bigvee^{(f)}(\Sigma(\wedge)\Phi) \le \bigvee^{(f)}\Sigma$; similarly, $\bigvee^{(f)}(\Sigma(\wedge)\Phi) \le \bigvee^{(f)}\Phi$. Now, assume $\Psi \le \Sigma$ and $\Psi \le \Phi$; to show that $\Psi \le \Sigma(\wedge)\Phi$, assume that $\Psi(\wedge)p \in \mathrm{Cov}(p)$.

We have that $\Sigma(\wedge)p$, $\Phi(\wedge)p \in \mathrm{Cov}(p)$. For any $x \wedge p \in \Sigma(\wedge)p$, we have that $\Phi(\wedge)(x \wedge p) \in \mathrm{Cov}(x \wedge p)$ (see (ii)); hence, applying (iii), we get that $\bigcup_{x\in\Sigma} \Phi(\wedge)(x \wedge p) = \Sigma(\wedge)\Phi(\wedge)p$ is a covering of p. It follows that $\Psi \leq \Sigma(\wedge)\Phi$ as desired. This shows (1).

We claim that

$$\bigvee_{i\in I} (\bigvee^{(f)} \Sigma_i) = \bigvee^{(f)}(\bigcup_{i\in I} \Sigma_i) . \tag{2}$$

It is clear that $\Sigma_i \leq \bigcup_{i\in I} \Sigma_i$ for all $i \in I$. Let Ψ be such that $\Sigma_i \leq \Psi$ for all $i \in I$, and assume that $(\bigcup_{i\in I} \Sigma_i)(\wedge)p \in \mathrm{Cov}(p)$. Let $x \in \bigcup_{i\in I} \Sigma_i$; then $x \in \Sigma_i$ for some $i \in I$, and trivially $\Sigma_i(\wedge)x \in \mathrm{Cov}(x)$ (by (i) and (iv)). By $\Sigma_i \leq \Psi$, it follows that $\Psi(\wedge)x \in \mathrm{Cov}(x)$, hence $(\Psi(\wedge)p)(\wedge)(x \wedge p) = \psi(\wedge)(x \wedge p) \in \mathrm{Cov}(x \wedge p)$. Since $x \wedge p$ is an arbitrary element of the covering family $(\bigcup_{i\in I} \Sigma_i)(\wedge)p$ of p, it follows by (v) that $\Psi(\wedge)p \in \mathrm{Cov}(p)$. We have shown that $(\bigcup_{i\in I} \Sigma_i)(\wedge)p \in \mathrm{Cov}\ p$ implies $\Psi(\wedge)p \in \mathrm{Cov}(p)$, hence $\bigcup_{i\in I} \Sigma_i \leq \Psi$, as required to show (2).

After these computations of $\wedge$ and $\bigvee$, the 'local identity' $(\bigvee\sigma_i) \wedge \eta = \bigvee(\sigma_i \wedge \eta)$ for $L(P)$ is a direct consequence of the trivial equality

$$(\bigcup_{i\in I} \Sigma_i)(\wedge)\Phi = \bigcup_{i\in I} (\Sigma_i(\wedge)\Phi) .$$

This completes showing that $L(P)$ is a cHa.
Also, defining $\varepsilon(p) = \bigvee^{(f)}\{p\}$, we immediately conclude that ε is continuous (morphism of p-sites). For $\Sigma \in \mathcal{P}(P)$, the element $\bigvee^{(f)}\Sigma \in L(P)$ can now be written:

$$\bigvee^{(f)}\Sigma = \bigvee\{\varepsilon(p): p\in\Sigma\} = \bigvee(\varepsilon[\Sigma]) .$$

(why?).

Finally, let $P \xrightarrow{f} H$ be an arbitrary p-site morphism into a cHa H. In order to have $\tilde{f}: L(P) \to H$ a cHa morphism, and also $f = \tilde{f}o\varepsilon$, we must define

$$\tilde{f}(\bigvee^{(f)}\Sigma) = \bigvee^{(H)}(f[\Sigma]). \tag{3}$$

Denoting the right-hand side of (3) temporarily by $\bar{\Sigma}$, we claim that $\Sigma \le \Phi$ implies $\bar{\Sigma} \le \bar{\Phi}$. From $\Sigma \le \Phi$, it follows that for every $x \in \Sigma$, $\Phi(\wedge)x \in Cov(x)$ (why?); hence, since f is continuous, $f(x) = \bigvee^{(H)}\{f(y\wedge x): y\in\Phi\} = \bigvee^{(H)}\{f(y) \wedge f(x): y\in\Phi\} = (\bigvee^{(H)}\{f(y): y\in\Phi) \wedge f(x)$; in other words, $f(x) \le \bigvee^{(H)}f[\Phi]$. Since here $x \in \Sigma$ was arbitrary, it follows that $\bigvee^{(H)}f[\Sigma] \le \bigvee^{(H)}f[\Phi]$, or $\bar{\Sigma} \le \bar{\Phi}$, as claimed. Now, in particular, it follows that (3) is a correct definition of the function $\tilde{f}$, and that it preserves $\le$. Using the identifications (1) and (2) of $\wedge$ and $\bigvee$ in $L(P)$, it is clear that $\tilde{f}$ is, indeed, a cHa-morphism.

This completes the proof of 3.1.2, except for the last sentence in the statement; but it is easy to modify L to make sure that $L(P) = P$ for P a cHa. □

We can use 3.1.2 to construct colimits in cHa. Namely, first we compute the colimit in p-sites; applying L: p-sites → cHa, we'll obtain the colimit in cHa, since L, being a left adjoint, preserves colimits. On the other hand, colimits exists in p-sites, and they are computed as follows. First of all, notice that any system $\mathrm{Cov}_o \subset \mathcal{P}(P) \times P$ of 'coverings' or a poset P with 1 and ∧ gives rise to a well-determined Grothendieck topology, namely the one generated by Cov_o, which is the smallest Grothendieck topology $\mathrm{Cov} \subset \mathcal{P}(P) \times P$ containing Cov_o: $\mathrm{Cov}_o \subset \mathrm{Cov}$. [Note that the intersection of arbitrary Grothendieck topologies on P is again a Grothendieck topology on P.]

The category of posets with 1 and ∧ , being a category of algebras satisfying a certain (finite) set of identities (for ∧ and 1), does have colimits; see e.g. [MacLane, CWM]. This is now the way to compute colimits in p-sites. Given a diagram of p-sites, one first computes the colimit of the underlying posets-with-1-and-∧; then, using the canonical injections from the underlying posets of the p-sites to this limit, one imposes all the coverings from the given p-sites on the limit, and finally one takes the Grothendieck topology generated by the set of all these coverings. Note that this is the smallest topology that makes all canonical injections continuous.

Instead of giving a precise general statement, we deal with the example of coproducts that will be important for us later.

Let L and H be posets of 1 and ∧. Their coproduct, $L \otimes H$, is as follows. Its elements are "formal intersections" $x \otimes y$ ($x \in L$, $y \in H$); the operation ∧ on $L \otimes H$ is:

$$(x \otimes y) \wedge (x' \otimes y') = (x \wedge x') \otimes (y \wedge y').$$

The canonical injection $L \xrightarrow{j} L \otimes H$ is: $x \longmapsto x \otimes 1_H$, and similarly, $H \xrightarrow{j'} L \otimes H$ is: $y \longmapsto 1_L \otimes y$. It is left as an exercise to check that this works. Next, 'transport' all coverings on L and H to $L \otimes H$ by defining

$\mathrm{Cov}_o \subset \mathcal{P}(L\otimes H) \times (L\otimes H)$ to be the set $\{<j[\Sigma], j(\bigvee\Sigma)> : \Sigma \in \mathcal{P}(L)\}$ $\cup \{<j'[\Phi], j'(\bigvee\Phi)> : \Phi \in \mathcal{P}(H)\}$. Note that Cov_o is the smallest 'pretopology' that makes j and j' continuous. Let $L\otimes H$ also denote the site whose Grothendieck topology is generated by Cov_o, and finally, the coproduct in cHa of L and H is: $L(L\otimes H)$, (denoted simply $L\otimes H$); the canonical injections are the composites $L \xrightarrow{j} L\otimes H \xrightarrow{\varepsilon_{L\otimes H}} L(L\otimes H)$, and $H \xrightarrow{j'} L\otimes H \xrightarrow{\varepsilon_{L\otimes H}} L(L\otimes H)$. It is quite easy to check that the coproduct so defined indeed has the requisite universal property.

As we said before, an arbitrary specification of covering families determines a site. More precisely, let P be a poset with 1 and $\wedge$ ("finitely complete"), and for each $x \in P$, let $\mathrm{Cov}_o(x)$ be an arbitrary collection of sets $\{x_i : i\in I\}$ such that $x_i \le x$; $\{x_i : i\in I\} \in \mathrm{Cov}_o(x)$ is read to mean: $\{x_i : i\in I\}$ is a 'prescribed' or 'distinguished' covering. Then there is a Grothendieck topology $<\mathrm{Cov}(x) : x\in P>$ generated by the prescribed families: the reason is that for any family $<T_j : j\in J>$ of Grothendieck topologies on P, T_j given by $<\mathrm{Cov}_j(x) : x\in P>$, the specification

$$\mathrm{Cov}(x) = \bigcap_{j\in J} \mathrm{Cov}_j(x)$$

determines a Grothendieck topology, the 'intersection' of the given topologies T (easy exercise); therefore, the intersection of all topologies $T = <\mathrm{Cov}_T(x) : x\in P>$ s.t. $\mathrm{Cov}_o(x) \subset \mathrm{Cov}_T(x)$ $\forall x\in P$ is the smallest topology $<\mathrm{Cov}(x) : x\in P>$ with $\mathrm{Cov}_o(x) \subset \mathrm{Cov}(x)$ $(\forall x\in P)$; this is the one that we call the topology generated by $<\mathrm{Cov}_o(x) : x\in P>$. If we are forced to call it a name, we call a P as above together with a specification $<\mathrm{Cov}_o(x) : x\in P>$ as above a pre-p-site. Every pre-p-site determines/generates a p-site with the same underlying poset as described. Now, we have the following fundamental fact:

Proposition 3.1.3. Let P be a pre-p-site with prescribed coverings $\mathrm{Cov}_o(x)\,(x \in P)$; let us denote the p-site determined by the pre-p-site by P too. Let Q be another p-site. Then for a 1,∧-preserving function

$$f: P \to Q$$

to be a p-site morphism it suffices to have that f carries prescribed coverings into Q-coverings.

Proof: In other words, what we are saying is that if the left exact f carries prescribed coverings in P into Q-coverings, then f carries any covering of the site P into a Q-covering. We prove this as follows: we consider

$$\mathrm{Cov}^*(x) \underset{\mathrm{df}}{=} \text{the set of those families } \{x_i : i \in I\} \text{ such that } x_i \leq x \ (\forall i \in I) \text{ and such that } \{f(x_i) : i \in I\} \text{ is a Q-covering;}$$

and we prove that $T^* = \langle \mathrm{Cov}^*(x) : x \in P \rangle$ is a Grothendieck topology on P. Once we have done so, then the assertion clearly follows. - The verification that T^* is a Grothendieck topology on P is straightforward (exercise). □

3.2 Spaces, duality, locales.

We now return to our 'main example'. For any topological space E, we have defined $\mathcal{O}(E)$, the cHa of open sets of E. We actually have a functor

$$\mathcal{O}(\cdot): \mathrm{Top} \to \mathrm{cHa}^{\mathrm{op}}$$

from the category of topological spaces to the opposite of cHa, with the definition $\mathcal{O}(f) = f^{-1}$, for $f: E \to E'$ continuous (see above).

Theorem 3.2.1. The functor $\mathcal{O}(\cdot)$ has a right adjoint.

Proof. [Consult CWM, IV.1 for terminology.] Let us call a cHa-map $p: H \to \mathbf{P}(\mathbf{1})$ an (abstract) point of H. Equivalently, such a p can be described by the set $F = \{U \in H: \ p(U) = \mathbf{1}\} \subset H$; F has the following properties: (i) it is a filter, i.e. $1 \in F$, F is closed under $\wedge$, and $x \in F$ and $x \leq y$ imply $y \in F$; (ii) whenever $\bigvee_{i \in I} x_i \in F$, then for some $i \in I$, $x_i \in F_i$. Such a set $F \subset H$ can be called a (completely) prime filter on H. Conversely, every prime filter gives rise to a unique cHa-map $h: H \to \mathbf{P}(\mathbf{1})$. The set $\mathrm{Pt}(H)$ of all points of H has the following natural topology: for any $U \in H$, let $[U] = \{p: \ p(U) = \mathbf{1}\}$; let $\mathcal{O}(\mathrm{Pt}(H)) = \{[U]: \ U \in H\}$ (check that this is topology!). The function $\mathrm{Pt}(-)$

$$H \longmapsto \mathrm{Pt}(H) \in |\mathbf{Top}|$$

can actually be considered a functor $(\mathbf{cHa})^{\mathrm{op}} \to \mathbf{Top}$: with any $h: L \to H$, we associate the function

$$\begin{tikzcd} \mathrm{Pt}(h): & \mathrm{Pt}(H) \arrow[r] & \mathrm{Pt}(L) \\ & p \arrow[r, mapsto] & p \circ h. \end{tikzcd}$$

We claim that $\mathrm{Pt}(-)$ is a right adjoint to $\mathcal{O}(-)$. In fact, for any space E, define $\eta_E: \ E \to \mathrm{Pt}(\mathcal{O}(E))$ as follows: for any $x \in E$, $\eta_E(x)$ is $p: \ \mathcal{O}(E) \to \mathbf{P}(\mathbf{1})$ such that for $U \in \mathcal{O}(E)$, $p(U)$ is the subset of φ of $\mathbf{1}$ for which $* \in \varphi \Longleftrightarrow x \in U$. ("Classically" this is the same as to say that $\varphi = \mathbf{1}$ if $x \in U$, and $\varphi = \emptyset$ if $x \notin U$.) It is easy to verify that $p \in \mathrm{Pt}(\mathcal{O}(E))$ and that η_E is a continuous map. Also, for $H \in |\mathbf{cHa}|$, define $\varepsilon_H\colon\ H \to \mathcal{O}(\mathrm{Pt}(H))$ by $\varepsilon_H(U) = [U]$ (see above); again, it is clear that ε_H is a cHa-map. Moreover, it is immediately checked that $\eta_{(\ )}$ and $\varepsilon_{(\ )}$ are natural transformations:

$$\eta_{(\ )}\colon \mathrm{Id}_{\underset{\sim}{\mathrm{Top}}} \xrightarrow{\cdot} \mathrm{Pt}\circ\mathcal{O},\ \varepsilon_{(\ )}\colon \mathcal{O}\circ\mathrm{Pt} \xrightarrow{\cdot} \mathrm{Id}_{(\underset{\sim}{\mathrm{cHa}})^{\mathrm{op}}}\ (!)\ .$$

We claim that $\eta_{(\ )}$ and $\varepsilon_{(\ )}$ are the unit and the counit, respectively, of an adjunction $\mathcal{O}(-) \dashv \mathrm{Pt}(-)$. This means (for η) that whenever $f\colon E \to \mathrm{Pt}(H)$ is a continuous map, then there is a unique cHa-map $g\colon\ H \to \mathcal{O}(E)$ (!) such that the diagram

$$\begin{tikzcd}
E \arrow[r, "\eta_E"] \arrow[dr, "f"'] & \mathrm{Pt}(\mathcal{O}(E)) \arrow[d, "\mathrm{Pt}(g)"] \\
 & \mathrm{Pt}(H)
\end{tikzcd}$$

commutes. Assume g is such, and let us adopt the 'prime filter' definition of points. Then we have

$$\begin{tikzcd}
x\in E \arrow[r, mapsto, "\eta_E"'] \arrow[dr, "f"'] & \{U\in\mathcal{O}(E):\ x\in U\} \arrow[d, mapsto, "\mathrm{Pt}(g)"] \\
 & f(x) = \{h\in H:\ x\in g(h)\}
\end{tikzcd}$$

in other words, for $h\in H$ and $x\in E$, we have

$$x \in g(h) \Leftrightarrow h \in f(x)\ .$$

This immediately tells us that g is unique, and we have to define $g(h) = \{x \in E\colon\ h \in f(x)\}$ $(h \in H)$. Since for $h \in H$, $[h] = \{p \in Pt(H)\colon h \in p\}$ is open in $Pt(H)$, and f is continuous, we have that $g(h) = f^{-1}([h])$ is open in E, hence g is indeed a map $g\colon H \to \mathcal{O}(E)$. Since both f^{-1} and $h \mapsto [h]$ are cHa-maps, so is their composite g. - We have verified that $\eta_{(\ )}$ is indeed the unit of an adjunction in which $\mathcal{O}(-)$ is the left adjoint of $Pt(-)$. - It is left an exercise to verify that $\varepsilon_{(\ )}$ is the counit of the same adjunction. □

In general (using the notation in CWM, IV.1.), if $\langle F,G,\varphi\rangle\colon X \rightharpoonup A$ is an adjunction, and X' is the full subcategory of X consisting of those objects x for which $\eta_x\colon x \to GFx$ is an isomorphism, and $A' \subset A$ is defined dually, with ε replacing η, then for every $x \in |X'|$, we have $Fx \in |A'|$: since we have that

$$Fx \xrightarrow{F\eta_x} FGFx \xrightarrow{\varepsilon_{Fx}} Fx$$

is the identity (see (8), p.80, loc.cit.), and $F\eta_x$ is an isomorphism (since η_x is), it follows that ε_{Fx} is an isomorphism. Of course, the dual statement also holds. Therefore, $X' \xhookrightarrow{\text{incl.}} X \xrightarrow{F} A$ factors through $A' \xhookrightarrow{\text{incl.}} A$; call the resulting functor $X' \longrightarrow A'$ by F'. Similarly, we have $G'\colon A' \longrightarrow X'$. Now, the very definition of X' says that the composite $G'F'$ is isomorphic to $Id_{X'}$, and the isomorphism $Id_{X'} \overset{\sim}{\to} G'F'$ is $\eta_{(\ )}$ restricted to X'; similarly, $F'G'$ is isomorphic to $Id_{A'}$. In other words, F' and G' are equivalences.

Definition 3.2.2. A space E is called sober is $\eta_E\colon E \to Pt(\mathcal{O}(E))$ is a homeomorphism. The cHa H is said to have enough points if $\varepsilon_H\colon H \to \mathcal{O}(Pt(H))$ is an isomorphism.

Corollary 3.2.3. The category of sober spaces (full subcategory of **Top** whose of objects are the sober spaces) is equivalent to the *opposite of* the category of cHa's with enough points, with equivalence functors $\mathcal{O}(-)$ and $\mathrm{Pt}(-)$ properly restricted. In particular, the category of sober spaces is fully and faithfully embedded into $(\mathbf{cHa})^{\mathrm{op}}$ by the functor $\mathcal{O}(-)$. □

Proposition 3.2.4. Any Haussdorff space is sober.

Proof. Instructive *exercise*; it is *not* true intuitionistically (see FS). □

With 3.2.3 in mind , cHa's can be considered 'generalized spaces'; the continuous maps then should be the same as cHa-maps except that they should be considered to point in the opposite direction. The opposite of the category **cHa**, $(\mathbf{cHa})^{\mathrm{op}}$, is called the category of *locales*. In other words, a locale is the same as a cHa, except that a morphism $f\colon H \to L$ between locales is the same as a morphism $L \to H$ between cHa's. In this context, if E is a locale, the notation $\mathcal{O}(E)$ is used for E itself, if we want to consider it an object of **cHa**. Similarly, we write $f^{-1}\colon \mathcal{O}(E') \to \mathcal{O}(E) \in \mathbf{cHa}$ for $f\colon E \to E' \in \mathbf{Loc}$.

Some further important remarks concerning these notions; we use the notation introduced in the proof of 3.2.1.

Proposition 3.2.5. (i) For any cHa L, $\mathrm{Pt}(L)$ is a sober space.

(ii) For any top.space E, $\mathcal{O}(E)$ is a cHa with enough points.

Remark: This means that the arrows

$$\mathrm{Pt}(L) \xrightarrow[\eta_{\mathrm{Pt}(L)}]{} \mathrm{Pt}(\mathcal{O}(\mathrm{Pt}(E))) \tag{1}$$

$$\mathcal{O}(E) \xrightarrow[\varepsilon_{\mathcal{O}(E)}]{} \mathcal{O}(\mathrm{Pt}(\mathcal{O}(E))) \tag{2}$$

are always isomorphisms. This reminds one of the identities $K^{***} = K^{*}$, $H^{***} = H^{*}$ of Galois correspondence; however it should be noted that the arrows corresponding to (1) and (2) for a general adjunction $X \underset{G}{\overset{F}{\rightleftarrows}} A$ $(F \dashv G)$ are not necessarily isomorphisms (look at $\mathrm{SET} \underset{\text{"underlying set"}}{\overset{\text{Free}}{\rightleftarrows}} \underbrace{\mathrm{Ab}}_{\text{abelian gps}}$).

Proof of 3.2.5. It is a general fact (quoted above) on adjunctions that the composite

$$\begin{tikzcd} \mathrm{Pt}(L) \arrow[r, "\eta_{\mathrm{Pt}(L)}"'] & \mathrm{Pt}(\mathcal{O}(\mathrm{Pt}(L))) \arrow[r, "\mathrm{Pt}(\varepsilon_L)"'] & \mathrm{Pt}(L) \end{tikzcd}$$

is the identity. Therefore, to show (i), it suffices to show that the composite

$$\begin{tikzcd} \mathrm{Pt}(\mathcal{O}(\mathrm{pt}(L))) \arrow[r, "\mathrm{Pt}(\varepsilon_L)"'] & \mathrm{Pt}(L) \arrow[r, "\eta_{\mathrm{Pt}(L)}"'] & \mathrm{Pt}(\mathcal{O}(\mathrm{pt}(L))) \end{tikzcd} \tag{3}$$

is the identity as well. - We will use the 'prime filter' definition of points; for $U \in L$ the open set $\subset \mathrm{Pt}(L)$: $\varepsilon_L(U) = [U]$ was defined so that

$$F \in [U] \Leftrightarrow U \in F \tag{4}$$

for $F \in \mathrm{Pt}(L)$. - Let $F' \in \mathrm{Pt}(\mathcal{O}(\mathrm{pt}(L)))$; let's follow the course of F' along the composite (3):

$$\begin{tikzcd} F' \arrow[r, mapsto, "\mathrm{Pt}(\varepsilon_L)"'] & F \arrow[r, mapsto, "\eta_{\mathrm{Pt}(L)}"'] & F'' \end{tikzcd}$$

where

$$U \in F \Leftrightarrow [U] \in F' \tag{5}$$

$$\begin{aligned}\underbrace{[U]}_{\substack{\text{arb. element of}\\ \mathcal{O}(\mathrm{pt}(L));\quad U\in L}} \in F'' \Leftrightarrow F \in [U] \quad &\underset{\text{by (4)}}{\Leftrightarrow} \quad U \in F \\ &\underset{\text{by (5)}}{\Leftrightarrow} \quad [U] \in F'\end{aligned}$$

$\therefore$ $F'' = F'$, as claimed.

□ for part (i)

Ad(ii): In general, to say that a cHa H has enough points is equivalent to saying that for $U,V \in H$

$$\forall_{x\in Pt(H)} (U\in x \Leftrightarrow V\in x) \quad \text{implies} \quad U = V \tag{6}$$

(check). Let $H = \mathcal{O}(E)$. For $x\in E$, we have the point $\tilde{x} \in Pt(H)$, $\tilde{x} = \eta_E(x)$, for which

$$U \in \tilde{x} \Leftrightarrow x \in U$$

(we have restated the definition of $\eta_E(x)$ in the 'prime filter' style). - If in (6) the assumption holds, then for every $x\in E$, $U\in\tilde{x} \Leftrightarrow V\in\tilde{x}$, hence $x\in U \Leftrightarrow x\in V$; by the axiom of extensionality, $U = V$ as required.

□ for (ii)

□

Sober spaces are important because of the last statement in 3.2.3: for sober E,E', the continuous maps

$$E \xrightarrow{f} E'$$

are in 1-1 correspondence with the cHa-morphisms

$$\mathcal{O}(E) \xleftarrow{h} \mathcal{O}(E')$$

($h = f^{-1} = \mathcal{O}(f)$) [actually, it turns out, for this it suffices for E' to be sober]; the definition of E' being sober is this same statement with E the one-element space (check). - We have also learned that once we have presented a space E in the form $Pt(H)$ for a cHa H, then we know it is sober. - Notice that all we have said in this subsection (except 3.2.4) is intuitionistically valid.

Finally, we want to point out that discrete spaces are sober (even intuitionistically). Let X be any set; the discrete space on X, X_{disc}, is the space whose underlying set is X and whose open sets are all the subsets of X: $\mathcal{O}(X_{disc}) = \mathbb{P}(X)$; we will (sometimes) write $\mathbb{P}(X)$ for the cHa (with ordinary intersection and union) of all subsets of X.

Let's first show that $\mathbb{P}(\mathbb{1}) = \mathbb{P}(\mathbb{1})$ is initial in cHa; this will imply that $\mathbb{1}_{disc}$ is sober (why?). Let $U \in \mathbf{P}(\mathbf{1})$, i.e. $U \subset \mathbf{1} = \{*\}$. Let $L \in |cHa|$. Consider the subset $\{1_L|U\} \underset{df}{=} \{x \in L\colon x = 1_L \text{ and } * \in U\}$ of L; i.e., $x \in \{1|U\} \underset{\substack{\uparrow \\ \text{for all } x\in L}}{\Longleftrightarrow} x = 1_L$ and $* \in U$.

For $L = L_0 = \mathbb{P}(\mathbb{1})$ in particular, we have

$$\{1_{L_0}|U\} \subset L_0 \quad .$$

We claim: $U = \bigvee\{1_{L_0}|U\}$. Indeed, this just means

$$x \geq U \underset{\substack{\uparrow \\ \text{for all } x\in\mathbf{P}(\mathbf{1})}}{\Longleftrightarrow} \forall_{y\in\mathbb{P}(\mathbb{1})} (y \in \{1|U\} \to x \geq y) \quad .$$

But the left hand side is equivalent to:

$$[* \in U \Rightarrow * \in x]$$

and the right hand-side is equivalent to:

$$[(y = 1 \text{ and } * \in U) \Rightarrow x \geq y]$$

$$\text{i.e. } [* \in U \Rightarrow x = 1];$$

hence our claim is clear.

Since $U = \bigvee\{1_{L_0} | U\}$, any $\underbrace{\wedge\bigvee}_{\text{cHa}}$-map $\mathbb{P}(\mathbb{1}) \to L$ has to take U into

$$\bigvee^{(L)}\{1_L | U\};$$

which shows that there can be at most one $\wedge\bigvee$-map $\mathbb{P}(\mathbb{1}) \to L$. On the other hand, we can check that

$$U \longmapsto \bigvee^{(L)}\{1_L | U\}$$

is indeed a cHa-map (exercise); this shows that $\mathbb{P}(\mathbb{1})$ is initial.

Now, let X_{disc} be an arbitrary discrete space, and let u be a prime filter on $\mathcal{O}(X_{\text{disc}}) = \mathbb{P}(X)$; we have to show that there is a unique $x \in X$ such that for any $A \subset X$,

$$A \in u \Leftrightarrow x \in A$$

(this is precisely to say that X_{disc} is sober; check). Since X (the max. subset of X) $\in u$, and

$$\bigcup_{x \in X} \{x\} = X$$

it follows that there is $x \in X$ such that $\{x\} \in u$.

It follows that

$$A \in u \Leftarrow x \in A$$

(why?). - Consider the one-element subset $\{x\}$ of X, and restrict u to it: define $u' \subset \mathbb{P}\{x\}$ by: for $U \in \mathbb{P}\{x\}$

$$U \in u' \Longleftrightarrow U \in u .$$

It is immediate to check that u' is a prime filter on $\mathbb{P}\{x\}$; the condition $1_{\mathbb{P}\{x\}} \in u'$ is precisely the fact $\{x\} \in u$. - But we know that there is a unique prime filter on $\mathbb{P}\{x\} \simeq \mathbb{P}(\mathbb{1})$ (why?), namely the one u_0 for which

$$U \in u_0 \Longleftrightarrow x \in U;$$

hence $U \in u \Longleftrightarrow x \in U \Longleftrightarrow \{x\} \subset U$. Let $A \subset X$ and $A \in u$; then $U = A \cap \{x\} \in u$ (why?); hence $\{x\} \subset A \cap \{x\}$, i.e. $x \in A$; we obtained the other direction

$$A \in u \Rightarrow x \in A;$$

this shows that x is as required. - The uniqueness of x is left to the reader to show. $\square$

3.3. Infinitary propositional logic in a cHa

Infinitary propositional logic is a sublanguage of the full (infinitary) L_1. Its formulas are built up from propositional atoms (0-ary relation symbols) by using $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, infinitary $\bigwedge$ and $\bigvee$. A 'formal' entailment has the form $\varphi \vdash \psi$ with formulas φ and ψ (now, no variables have to be mentioned; we can take the conjunction of the formulas on the left to get the single formula φ). Remember that the specification of the logic L_1 meant that we have given when an entailment was deducible from a set of entailments;

deducibility was defined in terms of axioms and rules of inference. In MR, the symbol '$\vdash$' is replaced by '$\Rightarrow$' and deducibility itself is denoted by '$\vdash$'. Deducibility for intuitionistic infinitary propositional logic is described as part of deducibility for infinitary predicate logic in MR, Chapter V.

not described for intuitionistic version!

Of course, we also have <u>informal</u> intuitionistic infinitary propositional logic, as explained in §1. If one wants to establish a deducibility

$$\varphi_1 \Rightarrow \psi_1,\ \varphi_2 \Rightarrow \psi_2, \ldots \vdash \varphi \Rightarrow \psi$$

informally, one assumes that: "if φ_1, then ψ_1", "if φ_2, then ψ_2", ..., and, using intuitionistically valid argument, one deduces that "if φ, then ψ".

Here is how we interpret propositional logic in a cHa L. We define the operations

$$\wedge,\ \vee,\ \rightharpoonup,\ -,\ \leftrightharpoons, \bigwedge, \bigvee$$

corresponding, respectively, to the connectives

$$\wedge,\ \vee,\ \rightarrow,\ \neg\ ,\ \leftrightarrow, \bigwedge, \bigvee$$

as follows: $x \wedge y$ - has been defined

$x \vee y = \bigvee\{x,y\}$

$x \rightharpoonup y = \bigvee\{z \in L\colon\ z \wedge x \leq y\}$

$-x = x \rightharpoonup \perp$ ($\perp$ = smallest element of $L, = \bigvee\emptyset$)

$x \leftrightharpoons y = (x \rightharpoonup y) \wedge (y \rightharpoonup x)$

$\Sigma \subset |L|\colon \bigwedge\Sigma = \bigvee\{z \in L\colon\ z \leq x$ for all $x \in \Sigma\}$

$\bigvee\Sigma$ - has been defined.

Now, if φ is a formula, and one has an assignment I of an element of L to each propositional atom, then one has a value $|\varphi|_I$ of φ in L, upon interpreting each connective by the corresponding operation in L. We say that an entailment $\varphi \Rightarrow \psi$ is valid in I if $|\varphi|_I \leq |\psi|_I$; we say that

$$\varphi_1 \Rightarrow \psi_1, \varphi_2 \Rightarrow \psi_2, \ldots \vdash \varphi \Rightarrow \psi \qquad (1)$$

holds in L if for all I, whenever I is an interpretation in L, each $\varphi_1 \Rightarrow \psi_1$ is valid in I, then $\varphi \Rightarrow \psi$ is valid in I.

Theorem 3.3.1. (Soundness): Any intuitionistic deducibility holds in any cHa. In other words, if (1) is intuitionistically deducible, then it holds in any cHa.

The proof is a direct verification, by seeing that each axiom is valid in any I (see above!), and that any rule of inference preserves validity in any given I.

Theorem 3.3.1 is mainly a moral support; what is really useful is the 'consequence' that if we establish a deducibility informally (but intuitionistically) then we can 'conclude' that it holds in any cHa.

As usual, the most common situation for an interpretation is that the frame of reference is the cHa L itself, and I is the identical interpretation.

Consider the example:

$$(\bigvee_{i\in I} \varphi_i) \wedge (\bigvee_{j\in J} \psi_j) \vdash \bigvee_{\langle i,j\rangle\in I\times J} (\varphi_i \wedge \psi_j) . \qquad (2)$$

This is intuitionistically valid. If we can point to an $i\in I$ and have that φ_i is true and to a $j\in J$ and have ψ_j true, then we have pointed to an $\langle i,j\rangle \in I\times J$ such that $\varphi_i \wedge \psi_j$ is true. - Therefore, 'it follows' that

$$(\bigvee_{i\in I}\varphi_i) \wedge (\bigvee_{j\in J}\psi_j) \leq \bigvee_{<i,j>\in I\times J}(\varphi_i \wedge \psi_j)$$

holds in any cHa L and any elements $\varphi_i, \psi_j \in L$. In fact, we would have this argument complete if we knew that (2) holds formally. But, actually, it is easier to invent a direct proof:

$$(\bigvee_i \varphi_i) \wedge (\bigvee_j \psi_j) = \bigvee_i (\varphi_i \wedge (\bigvee_j \psi_j)) =$$

$$= \bigvee_i ((\bigvee_j \psi_j) \wedge \varphi_i) = \bigvee_i (\bigvee_j (\psi_j \wedge \varphi_i)) = \bigvee_{<i,j>\in I\times J}(\varphi_i \wedge \psi_j) .$$