Dana Scott's proof of Brouwer's continuity principle

We analyze the topos $Sh(\mathbb{N}^{\mathbb{N}})$, the category of sheaves over the Baire-space $\mathbb{N}^{\mathbb{N}}$: the product of countably many copies of the countably infinite discrete space.

$\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the space of irrational numbers, with the subspace topology induced from the standard space of the reals.

We will find out that Brouwer's continuity theorem holds in it: the statement that every function from the reals to the reals is continuous is true internally in this topos.

Some generalities first. Given any topological space T , let $H = O(T)$ = the cHa of opens of T ; $\mathbf{E} = Sh(T) = Sh(H)$. Then the real number object $\mathcal{R}$ in $\mathbf{E}$ can be identified as the sheaf of continuous functions $U \longrightarrow \mathbb{R}$ from opens U of T to $\mathbb{R}$, the space of the standard reals. In other words, $\mathcal{R}$ as an H-set is as follows:

$$|\mathcal{R}| = \{s: U \longrightarrow \mathbb{R} : U \in H\}$$

and, with writing $|s| = dom(s)$,

$$\| s = t \| = \| s =_{\mathcal{R}} t \| = int\{x \in |s| \cap |t| : s(x) = t(x)\}$$

($int(S)$ is the interior (largest open subset) of the set $S \subset T$).
We have $Es = |s|$.

The relation $<$ on $\mathcal{R}$ is given by

$$\| s < t \| = \{x \in |s| \cap |t| : s(x) < t(x)\}$$

(since s , t are continuous, the last set is open), and the operations $+$, $\cdot$, $-$, $|\ \ |$ (absolute value) (and, indeed, all the usual continuous operations such as exp , cos , sin , etc.) are defined pointwise:

$$\| u = s + t \| = \{x \in |s| \cap |t| \cap |u| : u(x) = s(x) + t(x)\} ,$$

etc. The closed interval $[s,t]$ is defined by the condition

$$u \in [s,t] \iff (\neg(u < s) \wedge (\neg(t < u)) ,$$

and thus

$$\|u \in [s,t]\| = \mathrm{Int}\{x \in T : u(x) \in [s(x),t(x)]\} .$$

The rational numbers are identified, under the above identification of the reals, with the constant functions with values the rationals. The object of rational numbers, a subobject of $\mathbb{R}$, is denoted by $\mathbb{Q}$.

(As a reminder, let me mention that the above are not definitions; in the topos E , the concept of (Dedekind) reals, with all the usual relations and operations on it, has a fixed meaning, derived from the axiomatic (intuitionistic) concept. Thus, the above are facts to be verified.)

In the case of the Baire-space, we can make a simplification: we may restrict attention to functions s with the full domain T . Consider the sub-H-set X of $\mathbb{R}$ consisting of those s for which $|s| = T$; we put $\| s =_X t \| = \| s =_{\mathbb{R}} t \|$. We have the inclusion $i : X \longrightarrow \mathbb{R}$, a morphism of H-sets, represented (strongly represented in the sense of the notes on H-sets, p. 4.5) by the ordinary inclusion $|X| \longrightarrow |\mathbb{R}|$. We claim that i is an isomorphism in H-**Set** . It is clearly a monomorphism; it remains to show that it is an epi, that is

$$\| \forall s \in \mathbb{R}\ \exists u \in X\ s = i(u) \| = 1 .$$

The truth-value in question is

$$\bigwedge_{s \in |\mathcal{R}|} (|s| \longrightarrow \bigvee_{u \in |X|} \|s = u\|) \;;$$

thus, we have to show

$$|s| \leq \bigvee_{u \in |X|} \|s = u\| \qquad (1)$$

for any $s \in |\mathcal{R}|$. But, the Baire-space is totally disconnected, i.e., every open set is a union of clopen (closed and open) sets: in fact, a basis of the topology is given by the sets of the form

$$U = \{x \in \mathbb{N}^{\mathbb{N}} : x(i_1) = n_1 \;, \; \ldots \;, \; x(i_k) = n_k\}$$

with k, i_j and $n_j \in \mathbb{N}$, which are also closed (since U is the union of all $\{x \in \mathbb{N}^{\mathbb{N}} : x(i_1) = m_1 \;, \; \ldots \;, \; x(i_k) = m_k\}$, with $m_j \neq n_j$ for at least one j). Let C be any clopen subset of $|s|$. Define $u : T \longrightarrow \mathbb{R}$ by putting $u(x) = s(x)$ for $x \in C$, and $u(x) = 0$ for all $x \in T - C$; u is continuous since C is clopen. Clearly, $\|s = u\| = C$. Since the union of all clopen $C \subset |s|$ is $|s|$, (1) follows.

Since we have the isomorphism $X \cong \mathcal{R}$, we can take $\mathcal{R}$ to be X; the arithmetical operations remain to be defined in the pointwise manner. Note that (the new) $\mathcal{R}$ is *full*: $Es = 1$ for all $s \in |\mathcal{R}|$.

In [Scott I], the reals are *defined* directly as continuous functions $T \longrightarrow \mathbb{R}$; it is then shown that this new notion of "real number" satisfies the axioms of intuitionistic analysis.

Next, we identify the exponential $\mathcal{R}^{\mathcal{R}}$ in a convenient way. First of all, according to page 4.36 of [Notes], we have for $\mathcal{R}^{\mathcal{R}}$ the H-set F with $|F| = \mathrm{Pred}(\mathcal{R} \times \mathcal{R})$, the set of binary predicates on $\mathcal{R}$ (for predicates in general, see p. 4.14), with

$$\| R =_{\mathcal{R}^{\mathcal{R}}} S \| = \| R =_{P(\mathcal{R}\times\mathcal{R})} S\|\cdot\|Func(R)\|\cdot\|Func(S)\| ,$$

where $\| R =_{P(\mathcal{R}\times\mathcal{R})} S\| = \|\forall s,t\in\mathcal{R}\ (Rst\longleftrightarrow Sst)\|$,
$\|Func(R)\| = \|\forall s\in\mathcal{R}\ \exists! t\in\mathcal{R}\ Rst\|$.

Let us say that the function $f:|\mathcal{R}|\longrightarrow|\mathcal{R}|$ is *extensional* if

$$\|s = t\| \leq \|f(s) = f(t)\| \qquad (1')$$

for all $s , t \in |\mathcal{R}|$ (this is the condition (5) on page 4.5). Let us define the full H-set Y by letting $|Y|$ be the set of all extensional $f:|\mathcal{R}|\longrightarrow|\mathcal{R}|$, and letting

$$\|f =_Y g\| = \|\forall s\in\mathcal{R}\ f(s) = g(s)\| = Int\{x\in T : f(s)(x) = g(s)(x)\} . (2)$$

For any $f \in |Y|$, we can define a predicate $\varphi(f)$ on $R \times R$ by putting $\varphi(f)(s,t) = \| t = f(s) \|$; since f is extensional, $\varphi(f)$ is a predicate ("extensional"; see p. 4.13), in fact, $\|Func(\varphi(f)\| = 1$ (**exercise**). The mapping $\varphi:|Y|\longrightarrow|\mathcal{R}^{\mathcal{R}}|$ so defined is extensional:

$$\| f =_Y g \| \leq \| \varphi(f) =_{\mathcal{R}^{\mathcal{R}}} \varphi(g) \| ;$$

hence, it defines a morphism, also denoted by $\varphi:Y\longrightarrow\mathcal{R}^{\mathcal{R}}$ (see p.4.5). We claim that φ is an isomorphism; we verify that it is an epimorphism.

So, let $R \in |\mathcal{R}^{\mathcal{R}}|$, and let C be any clopen set contained in $\|Func(R)\|$; we construct $f \in |Y|$ with

$$C \leq \|\varphi(f) =_{\mathcal{R}^{\mathcal{R}}} R\| . \qquad (3)$$

Let $s \in |\mathcal{R}|$. By $C \leq \|Func(R)\|$, we have $C \leq \bigvee_{t\in|\mathcal{R}|} \|Rst\|$ and $C \wedge \|Rst\| \wedge \|Rsu\| \leq \|t = u\|$ for all $t , u \in |\mathcal{R}|$. It follows that the function $r:T \longrightarrow \mathbb{R}$ defined by

$$r(x) = \begin{cases} t(x) & \text{for any (some) } t \in |\mathcal{R}| \text{ such that } x \in \|Rst\| \\ & \text{if } x \in C \,; \\ 0 & \text{if } x \in T - C \end{cases}$$

is well-defined and continuous (partly because C is clopen). We put $f(s) = r$. It is left as an **exercise** to show that f is indeed extensional, and thus $f \in |Y|$. Note that by the definition of r , we have that for all $s \in |\mathcal{R}|$, $C \leq \|R(s,r(s))\|$; it follows easily that (3) holds.

Since $\|Func(R)\|$ is the union of its clopen subsets, we conclude that

$$E_{\mathcal{R}^{\mathcal{R}}} R = \|Func(R)\| = \bigvee_{f \in |Y|} \|\varphi(f) =_{\mathcal{R}^{\mathcal{R}}} R\|$$

which shows that φ is an epi (surjective). The proof of the fact that φ is a mono is left as an exercise.

The above isomorphism enables us to identify $\mathcal{R}^{\mathcal{R}}$ as the H-set Y of all extensional $f:|\mathcal{R}| \longrightarrow |\mathcal{R}|$, with equality defined as in (2). The evaluation $e:\mathcal{R} \times \mathcal{R}^{\mathcal{R}} \longrightarrow \mathcal{R}$ (more precisely, the function representing it) is defined as expected: $e(x,f) = f(x)$.

Let us mention again that in [Scott II], the functions from the reals to the reals are *defined* in a way corresponding to our last form for $\mathcal{R}^{\mathcal{R}}$.

Let $f \in |\mathcal{R}^{\mathcal{R}}|$. The extensionality of f , (1′) above translates into

$$\text{int}\{x \in T : s(x) = t(x)\} \subset \text{int}\{x \in T : f(s)(x) = f(t)(x)\} ,$$

or, what is the same,

$$\text{int}\{x \in T : s(x) = t(x)\} \subset \{x \in T : f(s)(x) = f(t)(x)\} ,$$

or even,

$$\mathrm{cl}(\mathrm{int}\{x\in T : s(x) = t(x)\}) \subset \{x\in T : f(s)(x) = f(t)(x)\} , \qquad (4)$$

since the right-hand-side set is closed, as a consequence of the functions $f(s)$, $f(t)$ being continuous (cl refers to closure).

We claim that the stronger fact

Claim 1.

$$\{x\in T : s(x) = t(x)\} \subset \{x\in T : f(s)(x) = f(t)(x)\}$$

i.e.

$$s(x) = t(x) \implies f(s)(x) = f(t)(x) \qquad (5)$$

is also true. To show this, fix s , $t \in |\mathfrak{R}|$ and $x \in T$, and assume that $s(x) = t(x)$. Let us construct open sets U and V in T such that

$$\mathrm{cl}(U) \cap \mathrm{cl}(V) = \{x\} \quad \text{and} \quad s|U , t|V \text{ are bounded.}$$

To do so, we look at T as the space of irrationals; we find distinct x_n ($n \in \omega$) in T such that $x_n \underset{n\to\infty}{\longrightarrow} x$; we take an open interval S_n around x_n , for each n , such that the $\mathrm{cl}(S_n)$ are pairwise disjoint, $x \notin \mathrm{cl}(S_n)$, and the lengths of the S_n tend to zero; by the continuity of the functions s , t , we can choose (decrease if necessary) the S_n so that both s and t are bounded on $\bigcup_{n\in\omega} \mathrm{cl}(S_n)$; finally, we put $U = \bigcup_{k\in\omega} S_{2k}$, $V = \bigcup_{k\in\omega} S_{2k+1}$; $\mathrm{cl}(U) = \bigcup_{k\in\omega} \mathrm{cl}(S_{2k}) \cup \{x\}$, and similarly for $\mathrm{cl}(V)$, thus U and V satisfy the requirements.

Now, we define a function u on $\mathrm{cl}(U) \cup \mathrm{cl}(V)$ so that $u(x) = s(x) = t(x)$, and $u(y) = s(y)$ for $y \in \mathrm{cl}(U) - \{x\}$, $u(y) = t(y)$ for $y \in \mathrm{cl}(V) - \{x\}$. The function u is continuous at each

$y \in cl(U) \cup cl(V)$, as is easily seen by looking at the cases $y = x$, $y \in cl(U) - \{x\}$, $y \in cl(V) - \{x\}$ separately. By the Tietze extension theorem (any real valued continuous bounded function from a closed subset of a normal space can be extended to a continuous real valued function to the whole space; T is certainly normal; it is even completely metrizable), there is $u \in |\mathcal{R}|$ extending the previous u . Now, we apply (4) to s and u , as well as t and u , in place of s and t . We have that $x \in cl(int\{y \in T : s(y) = u(y)\} \supset cl(U)$, hence $f(s)(x) = f(u)(x)$. Similarly, $f(t)(x) = f(u)(x)$, and (5) follows.

[] claim 1

The relation says that $f(t)(x)$ depends only on the value of t at x , not otherwise on t ; this fact allows us to make a "type-reduction" in the description of the elements $f \in \mathcal{R}^{\mathcal{R}}$; these are, at present, functions $f:\mathbb{R}^T \longrightarrow \mathbb{R}^T$; we can represent f by a function $\hat{f}:T\times\mathbb{R} \longrightarrow \mathbb{R}$, as follows. Let $\hat{f}(x,a) = (f(s_a))(x)$ where s_a is the constant function $s_a:T \longrightarrow \mathbb{R}$ with value a . For any $t \in |\mathcal{R}|$ and $x \in T$, by applying (5) to $s_{t(x)}$ and t , we get

$$f(t)(x) = \hat{f}(x,t(x)) \quad . \tag{6}$$

Claim 2. For any $f \in |\mathcal{R}^{\mathcal{R}}|$, the function $\hat{f}:T\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfying (6) is continuous (as a function on the product space $T \times \mathbb{R}$).

Proof of Claim 2. Suppose not. This means: there are x , $x_n \in T$, a , $a_n \in \mathbb{R}$ and a positive ϵ such that $|x - x_n|$ (again, we consider T as the space of irrationals), $|a - a_n|$ both tend to zero with $n \longrightarrow \infty$, but

$$|\hat{f}(x_n,a_n) - \hat{f}(x,a)| > \epsilon \tag{7}$$

for all n . Since the function $f(s_b)$ is continuous for any b , we can slightly move, if necessary, each x_n so that the x_n become pairwise distinct, in addition to the above properties. Now, we can define the function t on the closed set $\{x_n : n \in \omega\} \cup \{x\}$ by putting $t(x_n) = a_n$, $t(x) = a$; since a_n converges to a , t is continuous; by Tietze, we can extend t to $t : T \longrightarrow \mathbb{R}$. By (6),

$$f(t)(x) = \hat{f}(x,a) \ , \ f(t)(x_n) = \hat{f}(x,a_n) \ ; \qquad (8)$$

but $f(t)$ is a continuous function, and hence $f(t)(x) = \lim_{n \to \infty} f(t)(x_n)$; this is in contradiction with (8) and (7).

[] Claim 2

Theorem. The following statement is true in $Sh(\mathbb{N}^{\mathbb{N}})$:

$$\forall f \in \mathcal{R}^{\mathcal{R}} \ \forall q,r \in Q \ \forall \epsilon \in Q \ (\epsilon > 0 \longrightarrow \exists \delta \in Q \ (\delta > 0 \ \wedge \\ \forall s,t \in \mathcal{R} \ ((s,t \in [q,r] \wedge |s - t| < \delta) \longrightarrow |f(s) - f(t)| < \epsilon)))$$

Proof. We have to show that, for any $f \in |\mathcal{R}^{\mathcal{R}}|$, and q , r , $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$, we have that

$$\|\exists \delta \in Q \ (\bar{\delta} > 0 \wedge \forall s,t \in \mathcal{R} \ ((s,t \in [\bar{q},\bar{r}] \wedge |s - t| < \bar{\delta}) \longrightarrow \\ |f(s) - f(t)| < \bar{\epsilon}))\| \ = \ 1$$

(the bars indicate constant functions with the appropriate values). This means

$$\bigvee_{\substack{\delta \in \mathbb{Q} \\ \delta > 0}} \ \bigwedge_{\substack{s \in |\mathcal{R}| \\ t \in |\mathcal{R}|}} ((\|s,t \in [\bar{q},\bar{r}]\| \wedge \||s - t| < \bar{\delta}) \longrightarrow \||f(s) - f(t)| < \bar{\epsilon}\|) = 1.$$

We have

$$(\|s,t \in [\bar{q},\bar{r}]\| \wedge \||s - t| < \bar{\delta}) = \text{Int}(S^{\delta}_{s,t})$$

for $S^{\delta}_{s,t} = \{x \in T : s(x), t(x) \in [q,r] \text{ and } |s(x) - t(x)| < \delta\}$; we also have

$$U_{s,t} \underset{\text{def}}{=} \||f(s) - f(t)| < \bar{\epsilon}\| = \{x \in T : |f(s)(x) - f(t)(x)| < \epsilon\}$$
$$= \{x \in T : |\hat{f}(x,s(x)) - \hat{f}(x,t(x))| < \epsilon\} ,$$

by also using (6).

Thus, we have to show:

$$\bigcup_{\substack{\delta \in \mathbb{Q} \\ \delta > 0}} \text{Int} \bigcap_{\substack{s \in |\mathcal{R}| \\ t \in |\mathcal{R}|}} (\text{Int}(S^{\delta}_{s,t}) \xrightarrow{i} U_{s,t}) = T$$

where $\xrightarrow{i}$ is the (intuitionistic) operation of implication in the cHa H : $V \xrightarrow{i} W = \text{Int}(V \xrightarrow{c} W)$, with $V \xrightarrow{c} W = \{x \in T : x \in V \Rightarrow x \in W\}$. It is immediate to see that $\text{Int}(S) \xrightarrow{i} U = \text{Int}(\text{Int}(S) \xrightarrow{c} U) = \text{Int}(S \xrightarrow{c} U)$; thus we get that for the set

$$P^{\delta}_{s,t} = \{x \in T : (s(x), t(x) \in [q,r] \wedge |s(x) - t(x)| < \delta) \Rightarrow$$
$$|\hat{f}(x,s(x)) - \hat{f}(x,t(x))| < \epsilon\} ,$$

we want to show

$$\bigcup_{\substack{\delta \in \mathbb{Q} \\ \delta > 0}} \text{Int} \bigcap_{\substack{s \in |\mathcal{R}| \\ t \in |\mathcal{R}|}} \text{Int}(P^{\delta}_{s,t}) = T .$$

Now, notice that s and t occur in $P^{\delta}_{s,t}$ only through their values at x . Let, for $a , b \in \mathbb{R}$,

$$P^{\delta}_{a,b} = \{x \in T : (a,b \in [q,r] \wedge |a - b| < \delta) \Rightarrow |\hat{f}(x,a)) - \hat{f}(x,b)| < \epsilon |\}.$$

It clearly suffices to show that

$$\bigcup_{\substack{\delta \in \mathbb{Q} \\ \delta > 0}} \mathrm{Int} \bigcap_{\substack{a \in \mathbb{R} \\ b \in \mathbb{R}}} \mathrm{Int}(P^{\delta}_{a,b}) = T . \tag{8'}$$

Let us fix $\delta \in \mathbb{Q}$, $\delta > 0$ .Let

$C_\delta = \{(a,b) \in \mathbb{R}^2 : a,b \in [q,r] \wedge |a - b| \leqslant \delta\}$; C_δ is a compact subset of $\mathbb{R}^2$. , and define for $x \in T$

$$\epsilon(x) = \sup_{(a,b) \in C_\delta} |\hat{f}(x,a)) - \hat{f}(x,b)| .$$

We claim that $\epsilon(x)$ is a continuous function of x . Suppose otherwise; then there is a positive e and a sequence x_n tending to x such that

$$|\epsilon(x_n) - \epsilon(x)| > e \quad \text{for all} \quad n . \tag{9}$$

Writing $g(x,c)$ for $|\hat{f}(x,a)) - \hat{f}(x,b)|$ with $c = (a,b)$, $\epsilon(y) = g(y,c_y)$ for some $c_y \in C$ (since C is compact; sup = max); the sequence c_{x_n} has a limit point in C , again by compactness, thus without loss of generality, c_{x_n} tends to some $c \in C$; since $g(y,d)$ is a continuous function in (y,d) simultaneously, we get that $\epsilon(x_n) = g(x_n,c_{x_n})$ tends to $g(x,c)$. Now, by definition, $g(x,c) \leq \epsilon(x)$, and by (9),

$$g(x,c) \leq \epsilon(x) - e \tag{10}$$

and

$$\epsilon(x_n) \leq \epsilon(x) - \frac{e}{2} \tag{11}$$

for all $n > n_0$, with some n_0.

But $\epsilon(x) = g(x,c_x)$, and by continuity, we can find $n > n_0$ such that

$$|g(x,c_x) - g(x_n,c_x)| = |\epsilon(x) - g(x_n,c_x)| < \frac{e}{4} . \qquad (12)$$

(11) and (12) imply $\epsilon(x_n) < g(x_n,c_x)$, contradiction to the definition of $\epsilon(x_n)$.

Now, let us write $\epsilon(x,\delta)$ for $\epsilon(x)$, to show the dependence on δ. For a fixed x, $\hat{f}(x,a)$ is uniformly continuous for $a \in [q,r]$; it follows that, with a fixed x, $\epsilon(x,\delta)$ tends to zero with δ tending to zero.

Let $x \in T$, choose $\delta > 0$ such that $\epsilon(x,\delta) < \epsilon$, and, by the continuity of $\epsilon(x,\delta)$ in x, let U be an open neighborhood of x such that $\epsilon(y,\delta) < \epsilon$ for $y \in U$. Reading the definition of the set $P^{\delta}_{a,b}$, we see that $U \subset P^{\delta}_{a,b}$ for all $a, b \in \mathbb{R}$. But this means that U is a subset of the left-hand side of (8'). Since we have found an open neighborhood of every point in T contained in that left-hand side, that must be equal to the total space T.

[] Theorem