LINTONISM : HALIFAX JUNE 9, 2009

On Fred Linton's theory of algebraic categories.

(0) A **category-sketch**

$$\mathbb{L} = (|\mathbb{L}|, \mathrm{Id}_{\mathbb{L}}, \mathrm{Tr}_{\mathbb{L}}) = (L, \mathrm{Id}, \mathrm{Tr})$$

consists of a graph L, a subset $\mathrm{Id} \subseteq \hom(\mathrm{id}, |L|)$

where id is the graph $\boxed{\begin{tikzcd} 0 \arrow[loop above, "00"] \end{tikzcd}}$

and a subset $\mathrm{Tr} \subseteq \hom(\mathrm{tr}, |L|)$,

where tr is the graph
$$\boxed{\begin{tikzcd} & 1 \arrow[dr, "12"] & \\ 0 \arrow[ur, "01"] \arrow[rr, "02"'] & & 2 \end{tikzcd}}.$$

A **model** $L \xrightarrow{M} \mathbb{C}$, $\mathbb{C}$ a category, is

a graph-map $M : |L| \longrightarrow \mathbb{C}$ such that

for each
$$\varepsilon \in \mathrm{Id}, \quad M(\varepsilon(00)) = 1_{M(\varepsilon(0))};$$

and for each
$$\tau \in \mathrm{Tr}, \quad M(\tau(02)) = M(\tau(12)) \circ M(\tau(01)).$$

Version: also, shapes for composites of 3 or more arrows.

(1) An (abstract) **Linton sketch**

$$\mathbb{L} = (L, P)$$

is a category sketch L, and a subset $P \subset \mathrm{Arr}(L)$.

We also write P for the subgraph of L with (all the) objects of L, and arrows in P.

$J : P \to L$ denotes the inclusion; J is the identity on objects.

(2) A (concrete) **Linton sketch over** a category $\mathbb{S}$,

$$\mathbb{L} = (L, P, \pi)$$

arbitrary category!

is a Linton sketch (L, P), together with a graph-map $\pi : P^{op} \longrightarrow \mathbb{S}$, where $\mathbb{P}$ is the subgraph of L, consisting of all the objects of L, and the arrows in P.

(3) Given a Linton-sketch $\mathbb{L} = (L, P, \pi)$ over $\mathbb{S}$, an $\mathbb{L}$-*algebra* $M = (|M|, M)$ is an *object* $|M|$ of $\mathbb{S}$ and a Set-model $M : \mathbb{L} \to Set$ of the sketch S

(!)
$\mathbb{S} \neq Set$ possibly!

satisfying the following commutativity of graph-maps:

graphs

$$(*)\qquad \begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "J = \text{inclusion}"'] & \mathbb{S}^{op} \arrow[d, "\hom_{\mathbb{S}}(-, |M|)"] \\ \mathbb{L} \arrow[r, "M"'] & Set \end{tikzcd}$$

NB: All categories are assumed to be locally small

recall:

Remarks: 1. J is the identity on objects.

write: for $R, S \in Ob(\mathbb{S}) \Rightarrow$

$$S^R .=. \hom_{\mathbb{S}}(R, S) : \text{a } set!$$

projections

arrows of P.

Write: $X, Y, \dots$ for objects of P = objects of L; $p, q, \dots$

$\hat{X} = \pi(X)$

$\hat{p} = \pi(p)$

2. For objects X of P, (*) means:

$$M(X) = |M|^{\hat{X}} = \text{set of arrows from } \hat{X} \text{ to } |M|$$

(M^X for $|M|^{\hat{X}}$, M^f for $|M|^{\hat{f}}$; M^X: set of X-tuples in M.

For arrows $X \xrightarrow{f} Y$ of P, (*) means:

$$M(X) \xrightarrow{M(f)} M(Y) = M^Y \xrightarrow{|M|^f} M^X$$

composition: $(\hat{Y} \xrightarrow{y} |M|) \longmapsto (\hat{X} \xrightarrow{\hat{f}} \hat{Y} \xrightarrow{y} |M|)$, $x = y\hat{f}$

[In short: a projection (special operation, one in P) has a prescribed meaning: composing tuples with the projection]

3. In general: for arrows of L ("operations")

$$X \xrightarrow{\omega} Y$$

$$M(X) \xrightarrow{M(\omega)} M(Y) \quad \text{is}$$

a function $M^X \xrightarrow{M(\omega)} M^Y$: 'arbitrary function'

4. Let M, N be $\mathbb{L}$-algebras ($\mathbb{L}$: Linton-sketch over $\mathbb{S}$)

and let $h : |M| \longrightarrow |N|$ be an arrow in $\mathbb{S}$.

Then, for each $X \in \mathbb{L}$, we have the induced map

$$\begin{array}{rcccc} h_X & : & M(X) & \longrightarrow & N(X) \\ \text{DEF} \parallel & & \parallel & & \parallel \\ \hom_{\mathbb{S}}(\hat{X}, h) & : & \hom_{\mathbb{S}}(\hat{X}, |M|) & \longrightarrow & \hom_{\mathbb{S}}(\hat{X}, |N|) \\ & & x & \longmapsto & h \circ x \end{array}$$

A *morphism* $M \to N$ of $\mathbb{L}$-algebras is an arrow $h : |M| \to |N|$ in $\mathcal{B}$ such that h induces a natural transformation

$$h : M \to N \quad ; \quad h_X \text{ as defined above;}$$

of graph-to-category maps:

$$\begin{tikzcd} L \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\downarrow h"] & Set \end{tikzcd} .$$

$\mathbb{L}$-Alg is the category of $\mathbb{L}$-algebras.

(5) We have the forgetful functor

$$\begin{aligned} |-| : \mathbb{L}\text{-Alg} &\longrightarrow \mathcal{B} \end{aligned}$$

$$\begin{tikzcd} M \arrow[r, mapsto] \arrow[d, "h"'] & {|M|} \arrow[d, "h"] \\ N \arrow[r, mapsto] & {|N|} \end{tikzcd}$$

(6) Let $\mathbb{L}$ and $\mathbb{L}'$ be Linton sketches over the same category $\mathcal{B}$. A *morphism* $\varphi : \mathbb{L} \to \mathbb{L}'$ is:

$$\begin{tikzcd} L \arrow[r, "\varphi"] & L' \\ P \arrow[u, "J"] \arrow[r, "\varphi"] \arrow[dr, "\pi"'] \arrow[ur, phantom, "\circ"] & P' \arrow[u, "J"'] \arrow[d, "\pi"] \\ & \mathcal{B}^{op} \end{tikzcd} \quad \text{(sketch-map)}$$

$\varphi : \mathbb{L} \longrightarrow \mathbb{L}'$ gives rise to

'reduct' functor:

$$\varphi^* : \mathbb{L}'\text{-Alg} \longrightarrow \mathbb{L}\text{-Alg}$$

$$M' \longmapsto \varphi^*(M') = M \quad (?)$$

$$|M| \underset{DEF}{=} |M'|;$$

$$M(X) \underset{DEF}{=} M'(\varphi X)$$

$(X \in ob(L))$

Note: $M'(\varphi X) = hom_{\mathcal{S}}(\widehat{\varphi X}, |M'|) = hom_{\mathcal{S}}(\hat{X}, |M|)$

since $\widehat{\varphi X} = \pi(\varphi X) = \pi(X) = \hat{X}$.

$$\begin{array}{ccccc} X & & M(X) & & M'(\varphi X) \\ \downarrow{\scriptstyle w} & : & \downarrow{\scriptstyle Mw} & = & \downarrow{\scriptstyle M'(\varphi w)} \\ Y & & M(Y) & & M'(\varphi Y) \end{array}$$

(7) A Linton-theory is a Linton sketch $\mathbb{L} = (L, P)$ in which L is a category [every category $\mathbb{C}$ gives rise to a canonical category sketch $\tilde{\mathbb{C}}$; "L is a category" means $L = \tilde{\mathbb{C}}$ for a category $\mathbb{C}$].

Every category sketch L gives rise to a category L^+ "freely generated by L". Let $\mathbb{L} = (L, P)$ be a Linton sketch over $\mathcal{S}$. Then $\mathbb{L}^+ = (L^+, P)$ is a Linton theory over $\mathcal{S}$, and $\mathbb{L}\text{-Alg} \cong \mathbb{L}^+\text{-Alg}$.

(7.1) Comparison with Lawvere algebraic theories.

A Lawvere algebraic theory is a category L with $Ob(L) = \{\underline{0}, \underline{1}, \underline{2}, \dots\} = \{\underline{n} : n \in \mathbb{N}\}$ and specified <u>product</u> diagrams

$$\begin{tikzcd} & \underline{n} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_n"] & \\ \underline{1} & \cdots & \underline{1} \end{tikzcd} \qquad (*)_n$$

(for $n=0$: $\underline{0}$ is a <u>terminal</u> object).

For $\mathbb{S}$ a category with finite products, $L\text{-}Alg[\mathbb{S}]$ is the category of functors $L \to \mathbb{A}$ that turn every diagram $(*)_n$ $(n \in \mathbb{N})$ into a product diagram.

With L a Lawvere algebraic theory, we let $\mathbb{L} = (L, P)$ be the Linton-theory whose underlying category is L, and where P is the subcategory generated by the (objects and arrows in all the $(*)_n$.

For $\mathbb{S} = Set$, we define $\pi : P^{op} \longrightarrow Set$ by $\pi(\underline{n}) = n$, and $(*_n)$ mapped to the coproduct diagrams

$$\begin{tikzcd} & n & \\ 1 \arrow[ur, "i_1"] & \cdots & 1 \arrow[ul, "i_n"'] \end{tikzcd}$$

With these choices

$$\mathbb{L}\text{-}\mathrm{Alg}[\mathrm{Set}] = L\text{-}\mathrm{Alg}[\mathrm{Set}]$$

Note, however, that we do not, in general, have a "Linton-interpretation" of $L\text{-}\mathrm{Alg}[\mathbb{S}]$ for a general category $\mathbb{S}$ with finite products.

8 ABSTRACT LINTON SKETCH FOR MONOIDS / CATEGORIES

(example)

L5

OBJECTS AND ARROWS:

$$\begin{tikzcd}[column sep=huge]
3 \arrow[r, shift left=6, "q_0"] \arrow[r, shift left=2, "q_1"] \arrow[r, shift right=2, "m_0"'] \arrow[r, shift right=6, "m_1"'] & 2 \arrow[r, shift left=6, "p_0"] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "m"'] & 1 \arrow[loop above, "r"] \arrow[l, shift left=6, "i_0"] \arrow[l, shift left=10, "i_1"] \arrow[r, shift left=4, "d"] \arrow[r, "c"] & 0 \arrow[loop above, "s"] \arrow[l, shift left=4, "i"]
\end{tikzcd}$$

PROJECTIONS : marked as $\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;$! $\quad 3 \underset{q_1}{\overset{q_0}{\rightrightarrows}} 2 \underset{p_1}{\overset{p_0}{\rightrightarrows}} 1 \underset{c}{\overset{d}{\rightrightarrows}} 0$.

IDENTITIES : r, s.

COMMUTATIVITIES :

"DEFINITIONS OF i_0, i_1, m_0, m_1" [for future reference]:

$$\begin{tikzcd}
& 1 \arrow[dl, "r"'] \arrow[r, "d"] \arrow[d, "i_0"] & 0 \arrow[d, "i"] \\
1 & 2 \arrow[l, "p_0"] \arrow[r, "p_1"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[d, "i"'] & 1 \arrow[l, "c"'] \arrow[d, "i_1"] \arrow[dr, "r"] & \\
1 & 2 \arrow[l, "p_0"] \arrow[r, "p_1"'] & 1
\end{tikzcd}$$

$$\begin{tikzcd}
2 \arrow[d, "m"'] & 3 \arrow[l, "q_0"'] \arrow[r, "q_1"] \arrow[d, "m_0"] & 2 \arrow[d, "p_1"] \\
1 & 2 \arrow[l, "p_0"] \arrow[r, "p_1"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[d, "p_0"'] & 3 \arrow[l, "q_0"'] \arrow[r, "q_1"] \arrow[d, "m_1"] & 2 \arrow[d, "m"] \\
1 & 2 \arrow[l, "p_0"] \arrow[r, "p_1"'] & 1
\end{tikzcd}$$

"Laws":

"Domain-codomain laws":

$$
\begin{tikzcd}
 & 0 \arrow[dl, "r"'] \arrow[d, "i"] \arrow[dr, "r"] & \\
0 & 1 \arrow[l, "d"] \arrow[r, "c"'] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
1 \arrow[d, "d"'] & 2 \arrow[l, "p_0"'] \arrow[r, "p_1"] \arrow[d, "m"] & 1 \arrow[d, "c"] \\
0 & 1 \arrow[l, "d"] \arrow[r, "c"'] & 0
\end{tikzcd}
$$

"LAWS":

"unit laws":

$$\begin{tikzcd}
1 \arrow[dd, "i_0"'] \arrow[dr, "s"] & & 1 \arrow[dl, "s"'] \arrow[dd, "i_1"] \\
& 1 & \\
2 \arrow[ur, "m"'] & & 2 \arrow[ul, "m"]
\end{tikzcd}$$

"associative law":

$$\begin{tikzcd}
3 \arrow[r, "m_1"] \arrow[d, "m_0"'] & 2 \arrow[d, "m"] \\
2 \arrow[r, "m"'] & 1
\end{tikzcd}$$

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(9) CONCRETE L-sketches: (examples; continued):

1. OVER $\mathcal{S} = \underline{SET}$:

values of [illegible] $\widehat{(-)}$: on objects:

$\hat{0} = 0\ (= \emptyset)$, $\hat{1} = 1\ (= \{0\})$, $\hat{2} = 2$, $\hat{3} = 3$.

on arrows: ('hats' omitted):

$$\begin{tikzcd}
3 & 2 \arrow[l, shift right=2, "q_0 = \left\{\begin{smallmatrix}0\mapsto 0\\ 1\mapsto 1\end{smallmatrix}\right\}"'] \arrow[l, shift left=2, "q_1 = \left\{\begin{smallmatrix}0\mapsto 1\\ 1\mapsto 2\end{smallmatrix}\right\}"] & 1 \arrow[l, shift right=2, "p_0 = \{0\mapsto 0\}"'] \arrow[l, shift left=2, "p_1 = \{0\mapsto 1\}"] & 0 \arrow[l, shift right=2, "d = !"'] \arrow[l, shift left=2, "c = !"]
\end{tikzcd}$$

2. OVER GRAPH :

$$\hat{0} = \boxed{0}\ , \quad \hat{1} = \boxed{\begin{matrix}0\\ \downarrow\\ 1\end{matrix}}\ , \quad \hat{2} = \boxed{\begin{matrix}0\\ \downarrow\\ 1\\ \downarrow\\ 2\end{matrix}}\ , \quad \hat{3} = \boxed{\begin{matrix}0\\ \downarrow\\ 1\\ \downarrow\\ 2\\ \downarrow\\ 3\end{matrix}}$$

effect of π on arrows: "obvious":

$$\begin{tikzcd}
\boxed{\begin{matrix}0\\ \downarrow\\ 1\\ \downarrow\\ 2\\ \downarrow\\ 3\end{matrix}} & \boxed{\begin{matrix}0\\ \downarrow\\ 1\\ \downarrow\\ 2\end{matrix}} \arrow[l, shift right, "q_0"'] \arrow[l, shift left, "q_1"] & \boxed{\begin{matrix}0\\ \downarrow\\ 1\end{matrix}} \arrow[l, shift right, "p_0"'] \arrow[l, shift left, "p_1"] & \boxed{0} \arrow[l, shift right, "d"'] \arrow[l, shift left, "c"]
\end{tikzcd}$$

— ∘ —

For $\mathbb{L}$-over-Set : $\mathbb{L}$-Alg $\simeq$ Monoid

For $\mathbb{L}$-over-Graph : $\mathbb{L}$-Alg $\simeq$ Cat

= the category of categories.

Explanations:

for $\mathbb{L}$-over-Set:

$M \in \mathbb{L}\text{-Alg}$ is: a set $|M| = M$;

plus:

for $X = 0, 1, 2, 3$, sets $M(X)$ as follows:

$$M(0) = \hom_{Set}(0, M) = \{!\} \cong 1$$

$\uparrow$ initial

$$M(1) = \hom_{Set}(1, M) \cong M$$

$$M(2) \cong M \times M$$

$$M(3) \cong M \times M \times M$$

The arrows in P: become product projections:

$$\begin{tikzcd} M\times M\times M \arrow[r, shift left, "Mq_0"] \arrow[r, shift right, "Mq_1"'] & M\times M \arrow[r, shift left, "Mp_0"] \arrow[r, shift right, "Mp_1"'] & M \arrow[r, shift left, "Md"] \arrow[r, shift right, "Mc"'] \arrow[r, phantom, "="] & 1 \end{tikzcd}$$

[e.g.: for q_1: $3 \xrightarrow[q_1]{} 2$ in P;

$$\left\{ \begin{array}{l} 2 \xrightarrow[\hat{q}_1]{} 3 \text{ in Set} \\ 0 \mapsto 1 \\ 1 \mapsto 2 \end{array} \right.$$

composition with $\hat{q}_1$

$$M\times M\times M \cong \hom(3, M) \xrightarrow[\hom(\hat{q}_1, M)]{} \hom(2, M)$$

$$\left.\begin{array}{l} 3 \longrightarrow M \\ 0 \mapsto x_0 \\ 1 \mapsto x_1 \\ 2 \mapsto x_2 \end{array}\right\} \longmapsto \begin{array}{l} 2 \longrightarrow 3 \longrightarrow M \\ 0 \longmapsto x_1 \\ 1 \longmapsto x_2 \end{array}$$

$$(x_0, x_1, x_2) \longmapsto (x_1, x_2)$$]

For $\mathbb{L}$-over-graph:

$M \in \mathbb{L}\text{-Alg}$ is : a graph $|M| = M$

plus

for $X = 0, 1, 2, 3$, graphs $M(X)$ as follows

$$M(0) = \hom_{\text{Graph}}(\hat{0}, M) \cong \mathrm{Ob}(M)$$

$$M(1) = \hom(\hat{1}, M) \cong \mathrm{Arr}(M)$$

$$M(2) = \hom(\hat{2}, M) \cong \{\text{composable pairs of arrows in } M\}$$

$$M(3) = \hom(\hat{3}, M) \cong \{\text{composable triples of arrows in } M\}$$

For $f \in P$, $M(f)$ is a "projection" again

$$M(3) \underset{Mq_1}{\overset{Mq_0}{\rightrightarrows}} M(2) \underset{Mp_1}{\overset{Mp_0}{\rightrightarrows}} M(1) \underset{Mc}{\overset{Md}{\rightrightarrows}} M(0)$$

E.g.,

$$\begin{cases} M(1) \xrightarrow{Md} M(0) \\ \left(a \xrightarrow{e} b\right) \longmapsto a \end{cases}$$

$$\begin{cases} M(3) \xrightarrow{Mq_1} M(2) \\ (a,b,c) = \left(\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet\right) \longmapsto \left(\bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet\right) = (b,c) \end{cases}$$

The following are pullbacks in Set:

$$\begin{tikzcd} & M(1) \arrow[dr, "Mc"] & \\ M(2) \arrow[ur, "Mp_0"] \arrow[dr, "Mp_1"'] \arrow[rr, phantom, ">"] & & M(0) \\ & M(1) \arrow[ur, "Md"'] & \end{tikzcd}$$

$$\begin{tikzcd} & M(2) \arrow[dr, "Mp_1"] & \\ M(3) \arrow[ur, "Mq_0"] \arrow[dr, "Mq_1"'] \arrow[rr, phantom, ">"] & & M(1) \\ & M(2) \arrow[ur, "Mp_0"'] & \end{tikzcd}$$

$$=:=$$

11. Definition $\mathbb{A}$, $\mathbb{S}$ categories, $\mathbb{A} \xrightarrow{U} \mathbb{S}$ functor.

U is Linton-algebraic if there is a Linton sketch (equivalently: Linton theory) $\mathbb{L}$ over $\mathbb{S}$, possibly large, such that $\mathbb{A} \simeq \mathbb{L}\text{-}\mathrm{Alg}$, and more precisely:

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\simeq"] \arrow[dr, "U"'] & \arrow[d, phantom, "\cong"] & \mathbb{L}\text{-}\mathrm{Alg} \arrow[dl, "|{-}|_{\mathbb{L}}"] \\ & \mathbb{S} & \end{tikzcd}$$

12. Theorem [Linton]. If $\mathbb{A} \xrightarrow{U} \mathbb{S}$ is monadic, then it is Linton-algebraic.

The proof is by the following construction

(13) Given any functor $\mathbb{A} \xrightarrow{U} \mathbb{S}$, we construct a (large) Linton theory $\mathbb{L} = (L, P, \pi)$ over $\mathbb{S}$ as follows.

$\mathbb{L}[U] =$

Given $X \in \mathbb{S}$, write $U^X \underset{\text{def}}{=} \hom_{\mathbb{S}}(X,-) \circ U : \mathbb{A} \to Set$

$$\mathbb{A} \xrightarrow{U} \mathbb{S} \xrightarrow[\hom_{\mathbb{S}}(X-)]{} Set$$

Define $\mathbb{L}$: an object of $\mathbb{L}$ is an object of $\mathbb{S}$.

an arrow

$X \underset{\omega}{\multimap} Y$ is : natural transformation

$\boxed{\text{an operation of arity } X \text{ \& co-arity } Y}$

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "U^X"] \arrow[r, bend right, "U^Y"'] \arrow[r, phantom, "\omega\Downarrow"] & Set \end{tikzcd}$$

The arrows in P are : for $Y \xrightarrow{e} X$ in $\mathbb{S}$:

$$X \underset{U^e}{\multimap} Y \ :: \ U^X \xrightarrow[U^e]{} U^Y$$

which, at $A \in \mathbb{A}$, is composition with e:

$$\hom(X, UA) \xrightarrow{} \hom(Y, UA)$$
$$U^e = (-)\circ e$$

$\boxed{14}$ <u>Thm [Linton]</u>. Assume U has a left adjoint F.

Let $\mathbb{T} = (T = UF, \eta, \mu)$ be the monad on $\mathcal{S}$ induced by the adjunction $F \dashv U$, $\mathbb{T}$-Alg is the category of (Eilenberg–Moore) algebras of $\mathbb{T}$. Then we have an equivalence

$$\begin{tikzcd}
\mathbb{L}[U]\text{-Alg} \arrow[rr, "\simeq"] \arrow[dr, "|-|"'] & \arrow[d, phantom, "\cong"] & \mathbb{T}\text{-Alg} \arrow[dl, "|-|"] \\
& \mathcal{S} &
\end{tikzcd}$$

The proof is completely explicit: both functors

$$\mathbb{L}[U]\text{-Alg} \underset{\simeq}{\overset{\longleftarrow}{\longrightarrow}} \mathbb{T}\text{-Alg}$$

are defined explicitly.

The proof starts out by showing that the Linton theory $\mathbb{L}(U) = (L, P)$ has L equal the Kleisli category of the monad $\mathbb{L}$.

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Clearly, $\boxed{15}$ implies $\boxed{12}$

$\boxed{14.1}$ Addendum to Linton's theorem $\boxed{12}$.

Suppose: $\mathcal{S}$ is κ-accessible

$\mathbb{T} = (T, \eta, \mu)$ monad on $\mathcal{S}$

with T κ-accessible functor

Then (by Accessible Categories by M and BOB PARÉ)

$\mathbb{T}\text{-Alg} \xrightarrow[\text{forget}]{U} \mathcal{S}$ are κ-accessible category and functor.

Moreover:

$$\begin{tikzcd} \mathbb{L}_\kappa[U]\text{-Alg} \arrow[rr, "\simeq"] \arrow[dr] \arrow[rr, phantom, shift right=6ex, "\cong"] & & \mathbb{T}\text{-Alg} \arrow[dl] \\ & \mathcal{S} & \end{tikzcd}$$

where $\mathbb{L}_\kappa[U]$ is the small Linton theory, a full subcategory of $\mathbb{L}[U]$ on the κ-presentable objects of $\mathcal{S}$.

$\boxed{15}$ Proposition Let $\mathbb{L}$ be a Linton sketch over $\mathcal{S}$;

$$U = |-| : \mathbb{L}\text{-Alg} \longrightarrow \mathcal{S} \quad : \text{forgetful}.$$

$\boxed{(1)}$ U creates limits [in the precise sense of Mac Lane: see CWM, p 108].

Beck's monadicity cond's
$\boxed{(2)}$ U reflects isomorphisms.

$\boxed{(3)}$ Suppose $M \underset{k}{\overset{h}{\rightrightarrows}} N$ in $\mathbb{L}$-Alg, and that

$$UM \underset{Uk}{\overset{Uh}{\rightrightarrows}} UN$$ has an absolute coequalizer in $\mathcal{S}$.

Then $M \underset{k}{\overset{h}{\rightrightarrows}} N$ has a coequalizer in $\mathbb{L}$-Alg preserved by U.

$\boxed{(4)}$ If $\mathcal{S}$ is accessible [κ-accessible, and for each $X \in \mathrm{Ob}(\mathbb{L})$, $\hat{X} = \pi(X)$ is κ-presentable], and $\mathbb{L}$ is small, then $\mathbb{L}$-Alg and U are accessible [κ-accessible ?] as well.

$\boxed{(5)}$ If $\mathcal{S}$ is locally presentable, then so is $\mathbb{L}$-Alg; U has a left adjoint, and in fact, <u>U is monadic</u>.

<u>proofs</u>: (1) & (2): straightforward

(3): similar to Mac Lane's proof (p 152; Chapter VI, section 8) of the monadicity of the category of the models (algebras) of an equational theory $\langle \Omega, E \rangle$ in $\underset{\sim}{\mathrm{Set}}$.

(4): By the Limit Theorem of M-Bob PARÉ Accessible Categories (AC).

(5): By (1), $\mathbb{L}$-Alg is small complete

By ? AC, and (4), $\mathbb{L}$-Alg is locally presentable

By ? AC, and (4), U has a left adjoint.

By (2), (3), U is monadic.

16 **Scholium.**

Let $\mathbb{S}$ be a locally presentable category, $\mathbb{A} \xrightarrow[U]{} \mathbb{S}$ a functor. The following are equivalent

(1) There is an accessible monad $\mathbb{T}$ on $\mathbb{S}$ such that

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\simeq"] \arrow[dr, "U"'] & \arrow[d, phantom, "\cong"] & \mathbb{T}\text{-Alg} \arrow[dl, "\text{forget}"] \\ & \mathbb{S} & \end{tikzcd}$$

(2) There is small Linton theory $\mathbb{L}$ over $\mathbb{S}$ such that

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\simeq"] \arrow[dr, "U"'] & \arrow[d, phantom, "\cong"] & \mathbb{L}\text{-Alg} \arrow[dl, "\text{forget}"] \\ & \mathbb{S} & \end{tikzcd}$$

Can put prefix κ- at appropriate places. In Linton theory: this means: TX is κ-presentable for every $X \in \mathbb{L}$.