

LINTONISM :

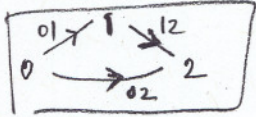
HALIFAX
JUNE 9, 2009

L1

(0) A On Fred Linton's theory of algebraic categories.
category-sketch

$$L = (|L|, Id_L, Tr_L) = (L, Id, Tr)$$

consist of a graph L , a subset $Id \subseteq hom(id, |L|)$
where id is the graph ,
and a subset $Tr \subseteq hom(tr, |L|)$,

where tr is the graph .

A model $L \xrightarrow{M} \mathbb{C}$, $\mathbb{C}$ a category, is

a graph-map $M : |L| \rightarrow \mathbb{C}$ such that

for each

$$\varepsilon \in Id, M(\varepsilon(00)) = 1_{M(\varepsilon(0))};$$

and for each

$$\tau \in Tr, M(\tau(02)) = M(\tau(12)) \circ M(\tau(01)).$$

Version: also, shapes for composites of 3 or more arrows.

(1) An. (abstract) Linton sketch

$$||L = (L, P)$$

is a category sketch L , and a subset $P \subset Arr(L)$

We also write P for the subgraph of L with (all the) objects of L , and arrows in P .

(2) A (concrete) Linton sketch over a category $\mathbb{S}$,

$$||L = (L, P, \pi)$$

↑
arbitrary category!

(L2)

is a Linkin sketch (L, P) , together
with a graph-map $\pi : P^{\text{op}} \rightarrow \mathcal{S}$,

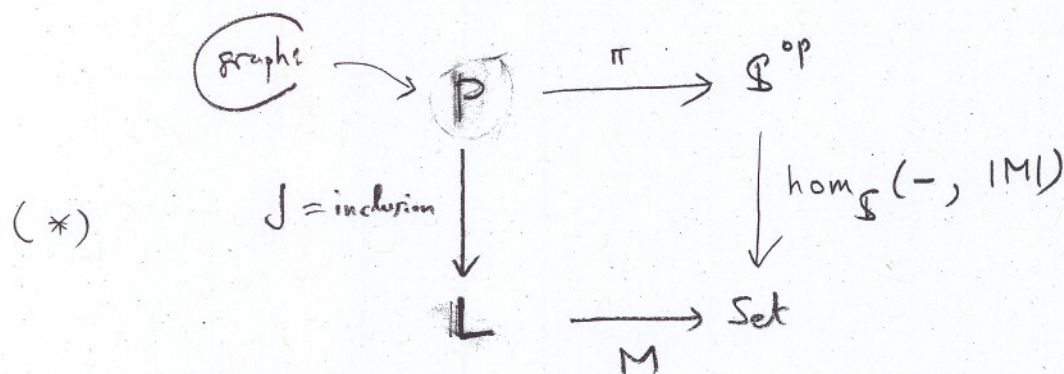
where P is the subgraph of L , consisting
of all the objects of L , and the arrows in P .

(3) Given a Linkin-sketch $\mathbb{L} = (L, P, \pi)$ over $\mathcal{S}$,
an $\mathbb{L}$ -algebra $M = (|M|, M)$ is an object $|M|$ of $\mathcal{S}$
and a Set-model $M : \mathbb{L} \rightarrow \text{Set}$ of the sketch $\mathbb{L}$

①

$\mathcal{S} \neq \text{Set}$
possibly!

satisfying the following commutativity of graph-maps:



NB: All categories
are assumed
to be locally small

recall:

Remarks: 1. J is the identity on objects.

write: for $R, S \in \text{Ob}(\mathcal{S}) \Rightarrow$

$S^R := \text{hom}_{\mathcal{S}}(R, S) : \text{a set!}$

projection

↑
arrows of P .

write: $X, Y, \dots$ for objects of $P = \text{objects of } L$; $f, g, \dots$

$\hat{X} = \pi(X)$
 $\hat{f} = \pi(f)$

2. For object X of P , $(*)$ means:

$M(X) = |M|^{\hat{X}} = \text{set of arrows from } \hat{X} \text{ to } |M|$
 I write M^X for $|M|^{\hat{X}}$, M^f for $|M|^{\hat{f}}$, M^X : set of X -tuples in M .
 For arrows $X \xrightarrow{f} Y$ of P , $(*)$ means:

$$M(X) \xrightarrow{M(f)} M(Y) = M^{Y \hat{Y}} \xrightarrow{|M|^f} M^{X \hat{X}}$$

Composition: $(\hat{Y} \xrightarrow{g} |M|) \circ (\hat{X} \xrightarrow{f} \hat{Y} \xrightarrow{g} |M|) \xrightarrow{x = y \hat{f}}$
 [In short: a projection (operation, one in P) has a prescribed meaning: comparing tuple with the projection]
 3. In general: for arrows of L (operations)

$$X \xrightarrow{w} Y$$

$$M(X) \xrightarrow{\pi(w)} \pi(Y) \text{ is}$$

a function $M^X \xrightarrow{M(w)} M^Y$: 'arity function'

④ Let M, N be L -algebras (L : Linton-skeleton over $\mathcal{S}$)
 and let $h: |M| \rightarrow |N|$ be an arrow in $\mathcal{S}$.
 Then, for each $X \in L$, we have the induced map

$$\begin{aligned} h_X : M(X) &\longrightarrow N(X) \\ \text{DEF } \parallel &\quad \parallel \quad \parallel \\ \text{hom}_{\mathcal{S}}(\hat{X}, h) : \left\{ \begin{array}{l} \text{hom}_{\mathcal{S}}(\hat{X}, |M|) \longrightarrow \text{hom}_{\mathcal{S}}(\hat{X}, |N|) \\ x \longmapsto h \circ x \end{array} \right. \end{aligned}$$

A morphism $M \rightarrow N$ of L -algebras is an arrow $h: |M| \rightarrow |N|$ in $\mathcal{S}$ such that h induces a natural transformation

$$h: M \rightarrow N; h_x \text{ as defined above;}$$

of graph-to-category maps:

$$L \xrightarrow[\downarrow h]{M} \text{Set}.$$

$L\text{-Alg}$ is the category of L -algebras.

5) We have the forgetful functor

$$\begin{array}{ccc} |-| : L\text{-Alg} & \longrightarrow & \mathcal{S} \\ M & \xrightarrow{\quad} & |M| \\ h \downarrow & & \downarrow h \\ N & \xrightarrow{\quad} & |N| \end{array}$$

6) Let L and L' be Linden sketches over the same category $\mathcal{S}$. A morphism $\varphi: L \rightarrow L'$ is:

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & L' \\ J \uparrow & \circ & \uparrow J \\ P & \xrightarrow{\varphi} & P' \\ \pi \swarrow & \circ & \swarrow \pi \\ & \mathcal{S}^{\text{op}} & \end{array} \quad (\text{sketch-map})$$

$\varphi: \mathbb{L} \rightarrow \mathbb{L}'$ gives rise to

'reduct' functor:

$$\varphi^*: \mathbb{L}'\text{-Alg} \rightarrow \mathbb{L}\text{-Alg}$$

$$M' \mapsto \varphi^*(M') = M \quad (?)$$

$$\begin{aligned} \textcircled{6} \Rightarrow & \quad |M| \stackrel{\text{DEF}}{=} |M'|; \\ \textcircled{6} \Rightarrow & \quad M(X) \stackrel{\text{DEF}}{=} M'(\varphi X) \end{aligned}$$

($X \in \text{ob}(\mathbb{L})$)

Note: $M'(\varphi X) = \text{hom}_{\mathbb{S}}(\widehat{\varphi X}, |M'|) = \text{hom}_{\mathbb{S}}(\widehat{X}, |M|)$

since $\widehat{\varphi X} = \pi(\varphi X) = \pi(X) = \widehat{X}$.

$$\textcircled{6} \Rightarrow \begin{array}{ccc} X & M(X) & M'(\varphi X) \\ \omega \downarrow & \downarrow M\omega & \downarrow M'(\varphi\omega) \\ Y & M(Y) & M'(\varphi Y) \end{array} =$$

$\textcircled{7}$ A Linbn-theory is a Linbn sketch $\mathbb{L} = (\mathbb{L}, P)$

in which $\mathbb{L}$ is a category [every category $\mathbb{C}$ gives rise to a canonical category sketch $\widehat{\mathbb{C}}$; " $\mathbb{L}$ is a category" means $\mathbb{L} = \widehat{\mathbb{C}}$ for a category $\mathbb{C}$].

$\textcircled{8}$ Every category sketch $\mathbb{L}$ gives rise to a category $\mathbb{L}^+$ "freely generated by $\mathbb{L}$ ". Let $\mathbb{L} = (\mathbb{L}, P)$ be a Linbn sketch over $\mathbb{S}$. Then $\mathbb{L}^+ = (\mathbb{L}^+, P)$ is a Linbn theory over $\mathbb{S}$, and $\mathbb{L}\text{-Alg} \cong \mathbb{L}^+\text{-Alg}$.

7.1 Comparison with Lawvere algebraic theories.

A Lawvere algebraic theory is a category L

with $Ob(L) = \{0, 1, 2, \dots\} = \{n : n \in \mathbb{N}\}$

and specified product diagrams

$$\begin{array}{ccc} & n & \\ \pi_1 \swarrow & & \searrow \pi_n \\ \underline{1} & \dots & \underline{1} \end{array} \quad (*)_n$$

(for $n=0$: 0 is a terminal object).

For $\mathcal{S}$ a category with finite products, $L\text{-Alg}[\mathcal{S}]$

is the category of functors $L \rightarrow \mathcal{A}$ that turn

every diagram $(*)_n$ ($n \in \mathbb{N}$) into a product diagram.

With L a Lawvere algebraic theory, we let $\mathbb{L} = (L, P)$

be the Linton-theory whose underlying category is L ,

and where P is the subcategory generated by the objects
occurring in all the $(*)_n$.

For $\mathcal{S} = \text{Set}$, we define $\pi : P^{op} \rightarrow \text{Set}$

by $\pi(\underline{n}) = n$, and $(*)_n$ mapped to the coproduct diagrams

$$\begin{array}{ccc} & n & \\ i_1 \nearrow & & \nwarrow i_n \\ \underline{1} & \dots & \underline{1} \end{array}$$

With these choices

$$\mathbb{L}\text{-Alg}[\text{Set}] = \mathbb{L}\text{-Alg}[\text{Set}]$$

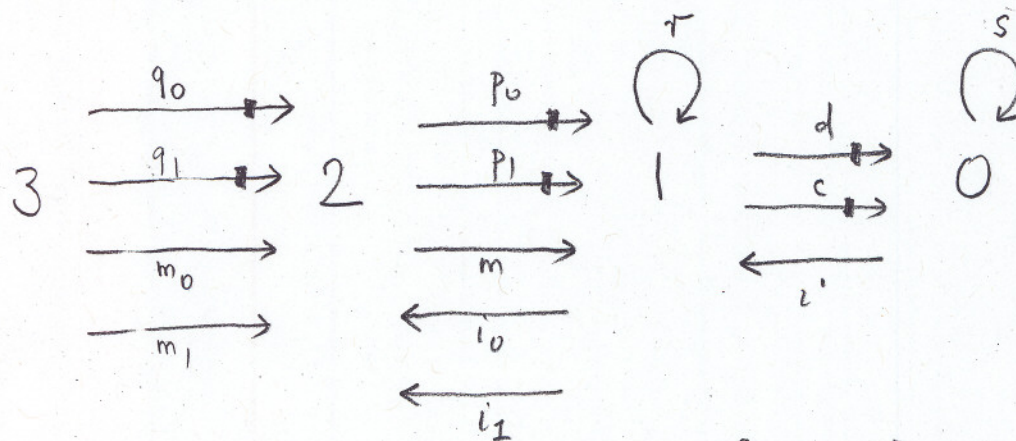
Note, however, that we do not, have a "Linden-interpretation"
in general,

of $\mathbb{L}\text{-Alg}[\mathbb{S}]$ for a general category $\mathbb{S}$ with finite products.

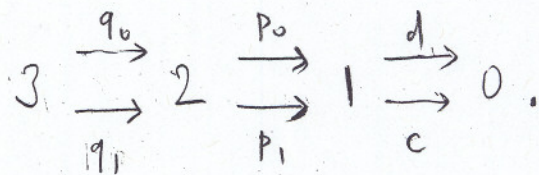
⑧ ABSTRACT LINTON SKETCH FOR MONOIDS / CATEGORIES (example)

LS

OBJECTS AND ARROWS:



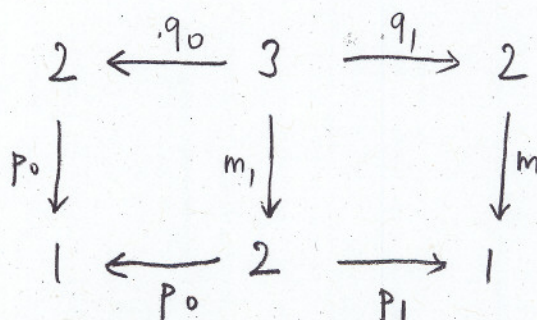
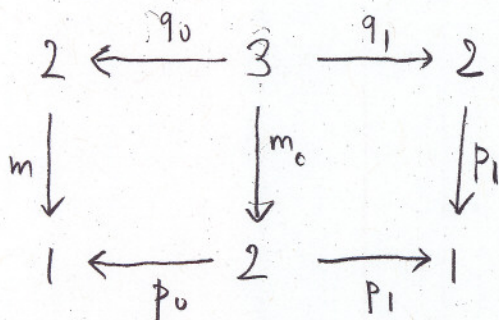
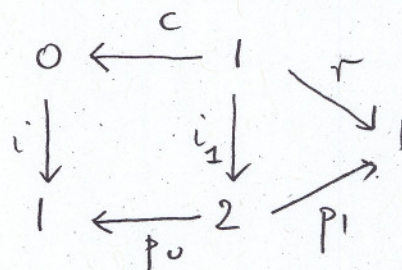
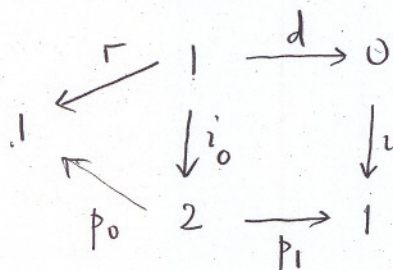
PROJECTIONS : marked as $\rightarrow$:



IDENTITIES : r, s .

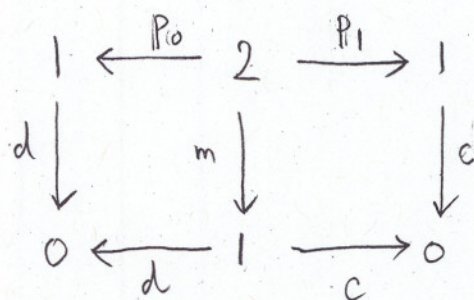
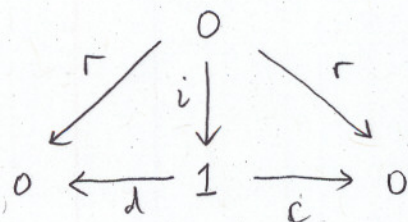
COMMUTATIVITIES :

"DEFINITIONS OF i_0, i_1, m_0, m_1 " [for future reference]:



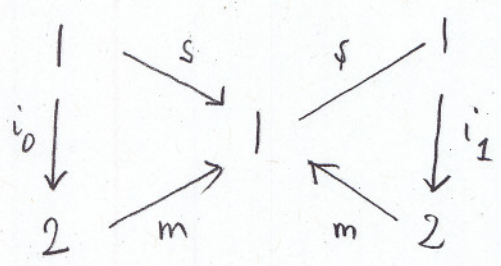
"laws":

"Domain-codomain laws":

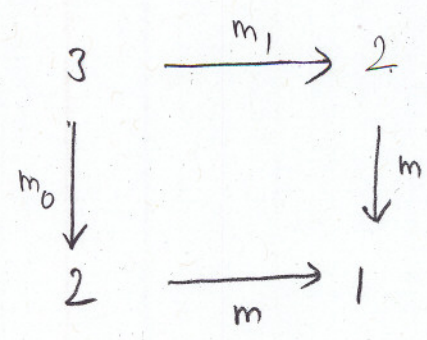


"Laws":

"unit laws":



"associative law":



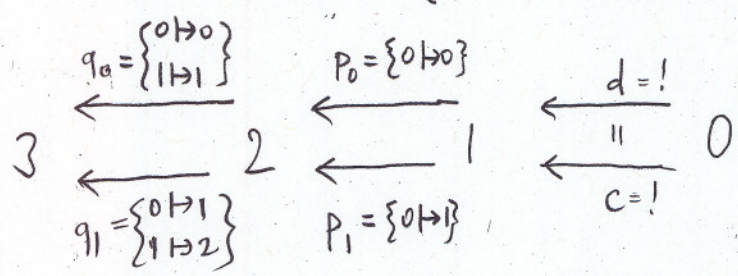
9) CONCRETE L-sketches: (examples; continued):

1. OVER $\mathcal{S} = \text{SET}$:

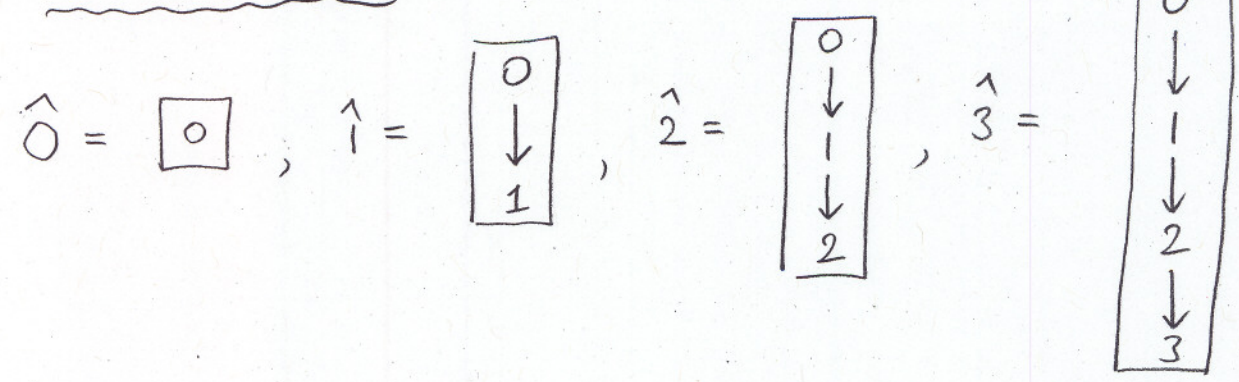
values of $\hat{\pi}_j := v(\hat{\pi}_j)$ on objects:

$$\hat{0} = 0 (= \emptyset), \hat{1} = 1 (= \{0\}), \hat{2} = 2, \hat{3} = 3.$$

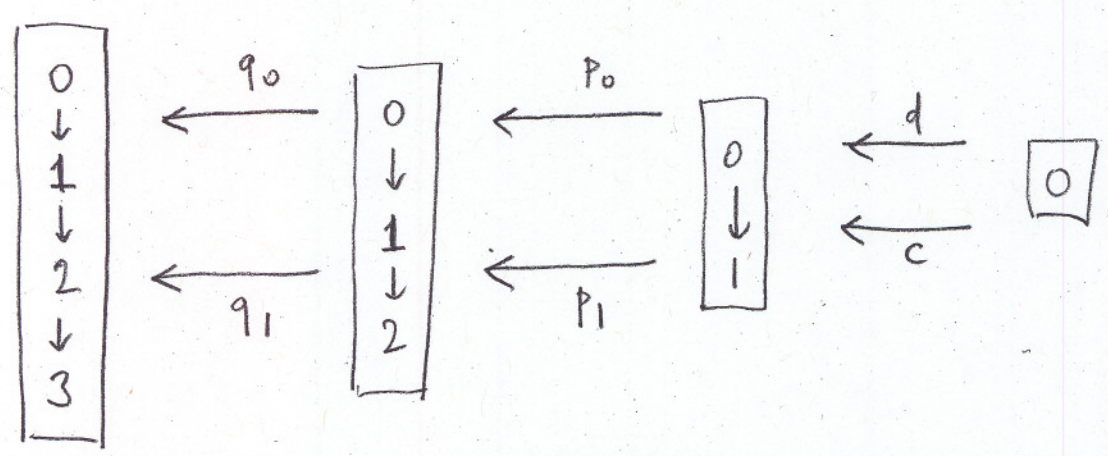
on arrows: ('hats' omitted):



2. OVER GRAPH :



effect of π on arrows: "obvious":



For $\mathbb{L}$ -over-Set : $\mathbb{L}\text{-Alg} \simeq \text{Monoid}$
 For $\mathbb{L}$ -over-Graph : $\mathbb{L}\text{-Alg} \simeq \text{Cat}$
 = the category of categories.

Explanations:

for $\mathbb{L}$ -over-Set:

$M \in \mathbb{L}\text{-Alg}$ is : a set $|M| = M$;

plus:

for $X = 0, 1, 2, 3$, sets $M(X)$ as follows:

$$M(0) = \text{hom}_{\text{Set}}(0, M) = \{!\} \cong 1$$

↑
initial

$$M(1) = \text{hom}_{\text{Set}}(1, M) \cong M$$

$$M(2) \cong M \times M$$

$$M(3) \cong M \times M \times M$$

The arrows in P : become product projections:

$$M \times M \times M \xrightarrow{Mq_0} M \times M \xrightarrow{Mp_0} M \xrightarrow{Md} 1$$

$$\quad \quad \quad \xrightarrow{Mq_1} \quad \quad \quad \xrightarrow{Mp_1} \quad \quad \quad \xrightarrow{Mc}$$

[e.g.: for q_1 : $3 \xrightarrow{q_1} 2$ in P ;

$$\left\{ \begin{array}{l} 2 \xrightarrow{\hat{q}_1} 3 \text{ in Set} \\ 0 \mapsto 1 \\ 1 \mapsto 2 \end{array} \right.$$

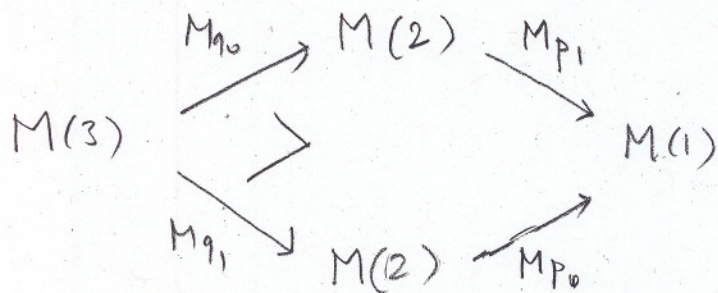
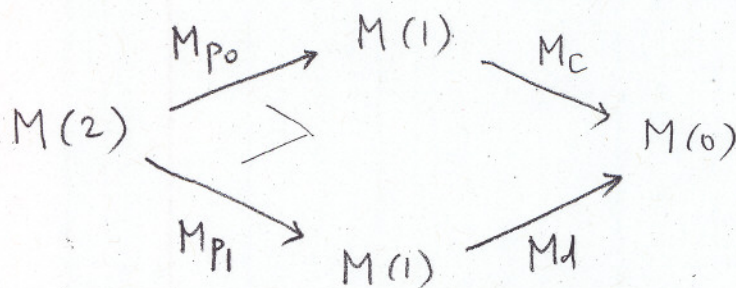
Composition with $\hat{q}_1$

$$M \times M \times M \cong \text{hom}(3, M) \xrightarrow{\text{hom}(\hat{q}_1, M)} \text{hom}(2, M)$$

$$\left. \begin{array}{l} 3 \rightarrow M \\ 0 \mapsto x_0 \\ 1 \mapsto x_1 \\ 2 \mapsto x_2 \end{array} \right\} \mapsto \left. \begin{array}{l} 2 \rightarrow 3 \rightarrow M \\ 0 \mapsto x_1 \\ 1 \mapsto x_2 \end{array} \right\}$$

$$(x_0, x_1, x_2) \mapsto (x_1, x_2)$$

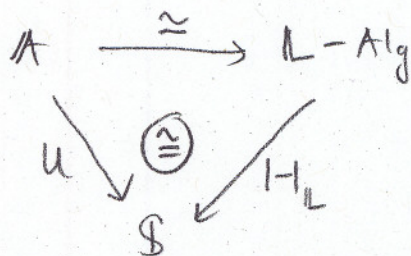
The following are pullbacks in $\mathbf{Set}$:



$=: =$

11. Definition $\mathcal{A}, \mathcal{B}$ categories, $\mathcal{A} \xrightarrow{U} \mathcal{B}$ functor.

U is Linton-algebraic if there is a Linton sketch (equivalently: Linton theory) $\mathbb{L}$ over $\mathcal{B}$, possibly large, such that $\mathcal{A} \cong \mathbb{L}\text{-Alg}$, and more precisely:



12. Theorem [Linton]. Supp if $\mathcal{A} \xrightarrow{U} \mathcal{B}$ is monadic, then: it is Linton-algebraic.

Then $\mathcal{A} \cong \mathbb{L}\text{-Alg}$ for some Linton sketch $\mathbb{L}$ over $\mathcal{B}$.

The proof is by the following construction

L II

(13) Given any functor $A \xrightarrow{U} \mathcal{S}$, we

construct a (large) Linton theory $\mathbb{L} = (L, P, \pi)$ over $\mathcal{S}$ as follows.

$$\mathbb{L}[U] =$$

Given $X \in \mathcal{S}$, write $U^X \stackrel{\text{def}}{=} \text{hom}_{\mathcal{S}}(X, -) \circ U: A \rightarrow \text{Set}$

$$A \xrightarrow{U} \mathcal{S} \xrightarrow{\text{hom}_{\mathcal{S}}(X, -)} \text{Set}$$

Define $\mathbb{L}$: an object of $\mathbb{L}$ is an object of $\mathcal{S}$.

an arrow

$$X \xrightarrow[\omega]{\circlearrowright} Y$$

is: natural transformation

an operation of arity X
& co-arity Y

$$\begin{array}{ccc} A & \xrightarrow{U^X} & \text{Set} \\ \omega \downarrow & & \\ A & \xrightarrow{U^Y} & \text{Set} \end{array}$$

The arrows in P are: for $Y \xrightarrow{e} X$ in $\mathcal{S}$:

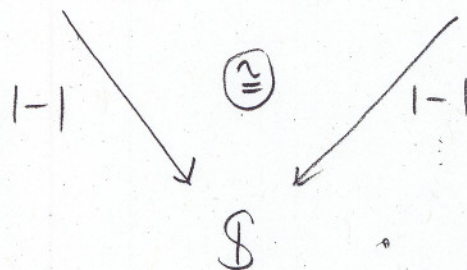
$$X \xrightarrow[\omega^e]{\circlearrowright} Y \quad :: \quad U^X \xrightarrow[\omega^e]{\circlearrowright} U^Y$$

which, at $A \in A$, is composition with e :

$$\text{hom}(X, UA) \longrightarrow \text{hom}(Y, UA)$$

$$\omega^e = (-) \circ e$$

14 Thm [Lien] , Assume U has a left adjoint F .

$$\mathbb{L}[u] - \text{Alg} \xrightarrow{\simeq} \mathbb{T} - \text{Alg}$$


The proof is completely explicit: both functors

$$L[u] - A^L g \quad \begin{matrix} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{matrix} \quad \Pi - A^I g$$

are defined explicitly.

06. The proof starts out by showing that the Linton Neog

$IL(U) = (L, P)$ has L equal the Kleisli category of the monad IL .

Clearly, $\boxed{15}$ implies $\boxed{12}$

[14.1]

Addendum to Linton's theorem [12]

[L 12.1]

Suppose: $\mathcal{S}$ is κ -accessible

$\mathbb{T} = (T, \eta, \mu)$ monad on $\mathcal{S}$

with T κ -accessible functor

Then (by Accessible Categories by M and BOB PARE)

$\mathbb{T}\text{-Alg} \xrightarrow[\text{forget}]{U} \mathcal{S}$ are κ -accessible category and functor.

Moreover:

$$\begin{array}{ccc} \mathbb{L}_{\kappa}[U]\text{-Alg} & \xrightarrow{\cong} & \mathbb{T}\text{-Alg} \\ & \searrow \quad \swarrow & \\ & \mathcal{S} & \end{array} \quad (\cong)$$

where $\mathbb{L}_{\kappa}[U]$ is the small Linton theory,

a full subcategory of $\mathbb{L}[U]$ on the κ -presentable objects of $\mathcal{S}$.

[15] Proposition Let $\mathbb{L}$ be a Linbn sketch over $\mathbb{S}$,
 $U = |-| : \mathbb{L}\text{-Alg} \rightarrow \mathbb{S} : \text{forgetful.}$

(1) U creates limits [in the precise sense of Mac Lane:
 see CWM, p 108].

Beck's monadicity cond's { (2) U reflects isomorphisms.
 (3) Suppose $M \begin{smallmatrix} \xrightarrow{h} \\ \xrightarrow{k} \end{smallmatrix} N$ in $\mathbb{L}\text{-Alg}$, and that

$UM \begin{smallmatrix} \xrightarrow{Uh} \\ \xrightarrow{Uk} \end{smallmatrix} UN$ has an absolute coequalizer in $\mathbb{S}$.

Then $M \begin{smallmatrix} \xrightarrow{h} \\ \xrightarrow{k} \end{smallmatrix} N$ has a coequalizer in $\mathbb{L}\text{-Alg}$
 preserved by U .

(4) If $\mathbb{S}$ is accessible [κ -accessible], and
 for each $X \in \text{Ob}(\mathbb{L})$, $\hat{X} = \pi(X)$ is κ -presentable], and $\mathbb{L}$ is
 small, then $\mathbb{L}\text{-Alg}$ and U are accessible [κ -accessible?]
 as well.

(5) If $\mathbb{S}$ is locally presentable, then so is $\mathbb{L}\text{-Alg}$;
 U has a left adjoint, and in fact, U is monadic.

proofs: (1) & (2): straightforward

(3): similar to Mac Lane's proof [p 152; Chapter VI, section 8]
 of the monadicity of the category of the models (algebras) of
 an equational theory $\langle \Omega, E \rangle$ in Set.

(4): By the Limit Theorem of M-Bo-PARE
Accessible Categories (AC).

(5): By (1), $\mathbb{L}\text{-Alg}$ is small complete

By ? AC, and (4), $\mathbb{L}\text{-Alg}$ is locally presentable

By ? AC, and (4), $\mathbb{U}$ has a left adjoint.

By (2), (3), $\mathbb{U}$ is monadic.

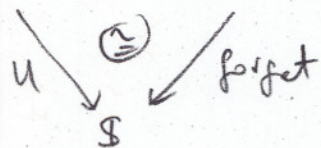
[16] Scholium.

Let $\mathcal{S}$ be a locally presentable category,

$\mathbb{A} \xrightarrow{\mathbb{U}} \mathcal{S}$ a functor. The following are equivalent

(1) There is an accessible monad $\mathbb{T}$ on $\mathcal{S}$ such

that $\mathbb{A} \xrightarrow{\mathbb{U}} \mathbb{T}\text{-Alg}$



(2) There is small Linton theory $\mathbb{L}$ over $\mathcal{S}$ such that

$\mathbb{A} \xrightarrow{\mathbb{U}} \mathbb{L}\text{-Alg}$



Can put prefix κ - at appropriate places. In Linton theory: this means: $\mathbb{T}X$ is κ -presentable for every $X \in \mathbb{L}$.