Egyptian fractions revisited

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Lower case Roman letters up to w, (not x, y,....!) : non-negative integers.

I allow p to mean $+\infty$; then $\frac{1}{p} = \frac{1}{\infty} = 0$.

x is any real such that $0 \le x < 1$.

1. Lemma Suppose $p \in \mathbb{N}$ and $\frac{1}{p} \le x < \frac{1}{p-1}$ and $\frac{1}{q} \le x - \frac{1}{p} < \frac{1}{q-1}$. Then

$$q \ge p(p-1)+1 .$$

Terminology: When $\frac{1}{p} \le x < \frac{1}{p-1}$, we say that $\frac{1}{p}$ is the *first* or *main part of* x ; we may write $p(x)$ for p, and $\pi_1(x)$ for $\frac{1}{p}$. The *second part* of x is the main part of $x - \pi_1(x)$. The lemma gives an upper estimate for the second part in terms of the first part.

We can define $p(x)$ by $p(x) = \lceil 1/x \rceil$, the *ceiling* of $1/x$: the smallest integer $\ge 1/x$. We allow $p=\infty$: $p(0)=\infty$. Of course, $\pi(0)=\frac{1}{\infty}=0$.

Proof of 1. Enough: $q \overset{?}{>} p(p-1)$, or equivalently, $\frac{1}{q} \overset{?}{<} \frac{1}{p(p-1)}$. Since $\frac{1}{q} \le x - \frac{1}{p}$, it is enough: $x - \frac{1}{p} \overset{?}{<} \frac{1}{p(p-1)}$. But

$$x - \frac{1}{p} < \frac{1}{p(p-1)}$$

$$\iff x < \frac{1}{p} + \frac{1}{p(p-1)} = \frac{1}{p-1} ,$$

and the last inequality is a true statement by assumption.

Write $\pi_1 = \pi_1(x) = \frac{1}{p_1} = \frac{1}{p_1(x)}$, where $\frac{1}{p_1}$ is the first part of x ; when $x=0$, then $p_1(0)=\infty$ and $\pi_1(0)=0$.

Define

$$\pi_{n+1}(x) = \pi_1(x - \sum_{k=1}^{n} \pi_k(x)) \ .$$

Also write $p_n(x)$ for the denominator of $\pi_n(x)$: $\pi_n(x) = \frac{1}{p_n(x)}$. By 1. Lemma, dropping x from the notation, we have the inequality

$$p_{n+1} \geq p_n(p_n-1)+1 \ . \tag{1}$$

With a fixed x , and dropping x from the notation, write $s_n=p_n-1$. By (1),

$$s_{n+1} \geq (s_n+1)s_n > s_n^2 \ ; \tag{2}$$

thus, $s_{n+m} \geq s_n^{(2^m)}$. $s_1=1$ is possible; but $s_2 \geq 2$; so $s_n \geq s_2^{(2^{n-2})} \geq 2^{(2^{n-2})}$ $(n \geq 2)$. We conclude

$$\frac{1}{s_n} = \frac{1}{p_n-1} \leq 2^{-(2^{n-2})} \qquad (n \geq 2) \ . \tag{3}$$

2. Corollary $\sum_{n=1}^{\infty} \pi_n(x)$ converges to x .

Proof By definition, for $\pi_{n+1}(x) = \frac{1}{p_{n+1}}$, we have that

$$\frac{1}{p_{n+1}} \leq x - \sum_{k=1}^{n} \pi_k(x) < \frac{1}{p_{n+1}-1} = \frac{1}{s_{n+1}} \leq \frac{1}{p_n \cdot (p_n-1)}$$

By (3), the assertion is clear.

As an example, for $x=\sqrt{2}-1$, the expansion given by 2. Corollary (which we will call the E-expansion) has the following beginning:

$$\begin{aligned} \sqrt{2} &= 1 + \frac{1}{3} + \frac{1}{13} + \frac{1}{253} + \frac{1}{218201} + \ldots \\ &= 1 + \frac{1}{3} + \frac{1}{13} + \frac{1}{11 \cdot 23} + \frac{1}{23 \cdot 53 \cdot 179} + \ldots \ . \end{aligned}$$

The error of approximating $\sqrt{2}$ with the sum shown is

$$< \frac{1}{218201\cdot 218200} < \frac{1}{4\cdot 10^{10}} .$$

For convenience, I also use $x_0=x$, $x_{n+1}=x_n-\pi(x_n)$. Thus, $x-\sum_{k=1}^{n}\pi_k(x) = x_n$. Also, $x_{n+1}=(x_n)_1$ (if I may say so ...).

Now, assume that x is rational, $x=\frac{a}{b}>0$. Let $\frac{1}{p} = \pi_1(x)$; $p\neq\infty$. Thus, $\frac{1}{p} \le \frac{a}{b} < \frac{1}{p-1}$; and $b\le ap$ and $a(p-1)<b$;

$$a(p-1) < b \le ap .$$

Thus, $0\le ap-b < ap-a(p-1) = a$.

Look at $x-\frac{1}{p} = \frac{a}{b} - \frac{1}{p} = \frac{ap-b}{bp} = \frac{c}{bp}$ for $c=ap-b$; we have that $c<a$.

3. Proposition When x is rational, the series $\sum_{n=1}^{\infty}\pi_n(x)$ is finite ($\pi_n(x)=0$ for large enough n). In fact, if $x=\frac{a}{b}$, then the sum $\sum_{n=1}^{\infty}\pi_n(x)$ has at most a non-zero terms.

Proof According to the calculation just made, the numerator of the fraction x_1 is smaller than that of x if the latter is $\neq 0$; and the numerator of $x_{n+1}=(x_n)_1$ is smaller than that of x_n , if the latter is $\neq 0$. Thus, after at most a steps, where a is the numerator of x , the process of producing non-zero parts will stop.

Let us define the *E-length* of a rational fraction $x=\frac{a}{n}$, denoted $\ell_E(x)$, to be the least ℓ such that $x_\ell=0$, or equivalently, the largest ℓ such that $x_{\ell-1}\neq 0$. We have shown that $\ell_E(\frac{a}{n})\le a$. For many fractions $\frac{a}{n}$, we have $\ell_E(\frac{a}{n})=a$.

3* Observation Whenever $n\equiv 1 \pmod{a!}$, we have that $\ell_E(\frac{a}{n})=a$.

There are weaker congruence conditions that also suffice for $\ell_E(\frac{a}{n})=a$.

Proof of 3* By induction on a.

For $a=0$ and 1, the assertion is clearly true.

Let $a\geq 2$, and assume the truth of the assertion for $a-1$.

Since, in particular, $n\equiv 1 \pmod{a}$, we have, for some m,

$$n=(m-1)\cdot a+1=m\cdot a-(a-1).$$

We deduce

$$1 = m\cdot\frac{a}{n} - \frac{a-1}{n}$$

$$\frac{1}{m} = \frac{a}{n} - \frac{a-1}{m\cdot n}$$

$$\frac{a}{n} = \frac{1}{m} + \frac{a-1}{m\cdot n}.$$

Therefore, for $x=\frac{a}{n}$, $p_1=p_1(x)=m$ and $(x)_1=x_1=\frac{a-1}{m\cdot n}$. If I can show that the denominator $m\cdot n$ satisfies the condition of 3* with respect to the numerator $a-1$, I am done -- and I can as you'll see.

The condition $n\equiv 1 \pmod{a!}$ says that $n=N\cdot a!+1$ for some N. We have that our m above is

$$m=\frac{n-1}{a}+1=\frac{N\cdot a!+1-1}{a}+1=N\cdot(a-1)!+1,$$

hence

$$m\equiv 1 \pmod{(a-1)!}.$$

But our denominator is not m, but $m\cdot n$. Butbut,

$$m\cdot n\equiv m \pmod{a!}$$

and, a fortiori

$$m\cdot n\equiv m \pmod{(a-1)!}$$

since $n\equiv 1 \pmod{a!}$. Thus,

$$m\cdot n\equiv m\equiv 1 \pmod{(a-1!)}$$

which is what we wanted.

By an *Egyptian fraction* (E-fraction, for short) I mean a (possibly infinite) series of the form

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

where for all n ,

p_n is either an integer ≥ 2 or ∞ (in which case $\frac{1}{p_n} = \frac{1}{\infty} = 0$) ;

if $p_n=\infty$, then $p_{n+1}=p_{n+2}= \ldots \infty$;

and, if both p_n , p_{n+1} are finite, we have

$$\boxed{p_{n+1} \geq p_n(p_n-1)+1} \quad . \tag{4}$$

According to this definition,

$$\frac{1}{3} + \frac{1}{4} + 0 + 0 + 0 + \ldots$$

is not an E-fraction; an E-fraction

$$\frac{1}{3} + \frac{1}{p_2} + \ldots$$

must have $p_2 \geq 3 \cdot 2+1=7$. On the other hand,

$$\frac{1}{2} + \frac{1}{12} + 0 + 0 + \ldots$$

is an E-fraction, and its sum is the same as that of the non-example, namely $\frac{7}{12}$.

Unfortunately, the terminology is not accurate historically. Ancient Egyptian arithmetic used all *lax E-fractions*, that is, sums of *distinct* unit fractions as representations of (rational) fractional quantities.

In fact, on the historical record, we can perceive (see: Victor Katz, A History of Mathematics) bits and pieces of the following "historical algorithm" for producing a lax E-fraction representation for any $\frac{a}{b}$ (except that I have not convinced myself that the process (that I formulated myself, not by following historians) always terminates (non-termination may possibly occur only as a self-repeating loop)).

Let $b=2^k \cdot c$, where $c \equiv 1 \pmod 2$. Write the binary-rational $\frac{a}{2^k}$ in the form of a finite sum of *distinct* positive, negative, and zero powers of 2 (Yes, that's right: the Egyptians knew how to do this!) $\frac{a}{b}$ now appears as the sum of distinct terms of the form $\frac{2^k}{c}$ and $\frac{1}{2^k \cdot c}$ with non-negative integers k . In the next step, one decomposes the individual terms of the first kind (the second kind being already unit fractions). One deals with $\frac{2^k}{c}$, with $c=2d-1$, using (repeatedly) the formula

$$\frac{2 \cdot e}{2d-1} = \frac{e}{d} + \frac{e}{d \cdot c}$$

(Yes; the Egyptians were aware of this piece of the *proper* E-expansion!; see Observation 3*). One may have to repeat the process since equal unit fractions may appear in different parts of the sum.

Applied to $\frac{7}{12} = \frac{7}{4 \cdot 3}$, the historical algorithm gives

$$\frac{7}{4} = 1+\frac{3}{4} = 1 + \frac{2+1}{4} = 1 + \frac{1}{2} + \frac{1}{4}\,;$$

and, dividing with 3 ,

$$\frac{7}{12} = \frac{1}{3} + \frac{1}{6} + \frac{1}{12}\,.$$

This is a lax E-expansion, but not a proper one. For the Egyptian scribe, this could have been the officially accepted representation of $\frac{7}{12}$, and not $\frac{7}{12} = \frac{1}{3}+\frac{1}{4}$, nor $\frac{7}{12} = \frac{1}{2}+\frac{1}{12}$, -- but I don't know for sure.

For $\frac{11}{12}$, the historical algorithm gives:

$$\frac{11}{12} = \frac{11}{4 \cdot 3}\,; \quad \frac{11}{4} = \frac{8+2+1}{4} = 2 + \frac{1}{2} + \frac{1}{4}\,;$$
$$\frac{11}{12} = \frac{2}{3} + \frac{1}{6} + \frac{1}{12}\,;$$

and at this point the Egyptian scribe would have stopped since $\frac{2}{3}$ was accepted on its own right, it was not expanded as $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$. If he had expanded $\frac{2}{3}$, he would have gotten

$$\frac{11}{12} = \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}\,,$$

which *is* a proper E-expansion.

When we apply the "algorithm" to $\frac{9}{20}$, we get:

$$\frac{9}{20} = \frac{9}{2^2\cdot 5}\,;\quad \frac{9}{2^2} = 2 + \frac{1}{2^2}\,;\quad \frac{9}{20} = \frac{2}{5} + \frac{1}{2^2\cdot 5}\,;$$

$$5=2\cdot 3-1\,;\quad \frac{2}{5} = \frac{1}{3} + \frac{1}{3\cdot 5}\,;\quad \underline{\frac{9}{20} = \frac{1}{3} + \frac{1}{3\cdot 5} + \frac{1}{20}}\,.$$

This is not proper, since $\neg(20\geq 15\cdot 14+1)$. The proper E-expansion of $\frac{9}{20}$ is

$$\frac{9}{20} = \frac{1}{3} + \frac{1}{9} + \frac{1}{180}\,,$$

and that of $\frac{5}{11}$ is

$$\frac{5}{11} = \frac{1}{3} + \frac{1}{9} + \frac{1}{99}\,.$$

The historical algorithm gives

$$\frac{5}{11} = \frac{1}{3} + \frac{1}{11} + \frac{1}{33}\,.$$

The singling out of the particular sum-of-unit-fractions form we call "E fraction" is justified by the following lemma and the subsequent theorem.

Let p_1 be any real number at least 2 , and define

$$p_{n+1} = p_n(p_n-1)+1 \quad (n=1,\, 2,\, ...)\,. \tag{5}$$

4. Lemma Under (5), we have $\sum\limits_{n=1}^{\infty}\frac{1}{p_n} = \frac{1}{p_1-1}$.

Proof I claim that

$$\sum_{k=1}^{n-1}\frac{1}{p_k} = \frac{1}{p_1-1} - \frac{1}{p_n-1} \qquad (n=1,\, 2,\, ...\,) \tag{6}$$

For $n=1$,this is right. To pass from n to $n+1$, we need

$$\frac{1}{p_n} \stackrel{?}{=} \frac{1}{p_n-1} - \frac{1}{p_{n+1}-1}\,.$$

But $p_{n+1}-1 = p_n(p_n-1)$, $p_{n+1}-p_n = p_n(p_n-1)+1-p_n = p_n^2-2p_n+1 = (p_n-1)^2$; thus

$$\frac{1}{p_n-1} - \frac{1}{p_{n+1}-1} = \frac{p_{n+1}-p_n}{(p_n-1)(p_{n+1}-1)} = \frac{(p_n-1)^2}{(p_n-1)p_n(p_n-1)} = \frac{1}{p_n}$$

as desired.

By (3) (whose assumption (2), with $s_n=p_n-1$, holds with equality for ≤ by the definition (4)), (6) implies the assertion of the lemma.

5. Theorem **(i)** Every E-fraction denotes (converges to) a value $x\in[0,1)$.

(ii) Every real number $x\in[0,1)$ has an Egyptian fraction expansion (E-expansion, for short): $x = \sum_{n=1}^{\infty}\frac{1}{p_n}$ for an E-fraction $\sum_{n=1}^{\infty}\frac{1}{p_n}$.

Moreover, we have that the partial sums approximate the sum x in a controlled manner; in particular,

$$0 \le x - \sum_{n=1}^{N}\frac{1}{p_n} < \frac{1}{p_N-1} ,$$

where we have the estimate (4) holding for all n. Furthermore, $\frac{1}{p_N-1}$ tends to zero double-exponentially: see (3).

(iii) Every rational $x\in[0,1)\cap\mathbb{Q}$ has a finite E-expansion ($x = \sum_{n=1}^{\infty}\frac{1}{p_n}$ in which, with a suitable N, $p_n=\infty$ for $n\ge N$).

(iv) Suppose $x\in[0,1)$ has two *different* E-expansions

$$x = \sum_{n=1}^{\infty}\frac{1}{p_n} = \sum_{n=1}^{\infty}\frac{1}{q_n} . \tag{7}$$

Let N be the *first* integer for which $p_N\neq q_N$; then either $p_N>q_N$, or $q_N>p_N$; assume, e.g., the first alternative. Thus,

1) $p_n=q_n$ for $1\le n<N$, and $p_N>q_N$.

The assertion is that (7) and 1) can happen together *only if*

2) $\quad p_{n+1} = p_n(p_n-)+1 \qquad$ for all $n=N$, $N+1$, ... ;

3) $\quad q_N = p_N-1$;

and

4) $\quad q_{N+1}=q_{N+2}=... = \infty$.

In particular, x must be rational.

In other words, the E.-fractions $\sum_{n=1}^{\infty}\frac{1}{p_n}$, $\sum_{n=1}^{\infty}\frac{1}{q_n}$ have the same value if and only if
either they are identical ($p_n=q_n$ for all n),
or there is $N=1, 2, ...$ such that conditions 1), 2), 3) and 4) hold,
or the same with the two E.-fractions interchanged.

Remark By 4. Lemma, 2) implies that $\sum_{n=N}^{\infty}\frac{1}{p_n} = \frac{1}{p_N-1}$ (apply lemma to $\{\tilde{p}_k\}$ defined by $\tilde{p}_k = p_{k+N-1}$). Thus, under the conditions 1), 2), 3) and 4), we do have

$$x = \sum_{n=1}^{\infty}\frac{1}{p_n} = \sum_{n=1}^{\infty}\frac{1}{q_n} = \sum_{n=1}^{N-1}\frac{1}{p_n} + \frac{1}{q_N} ;$$

in particular, x is rational.

(v) Every *rational* $x\in[0, 1)$ has a *unique finite* E-expansion.

(vi) Every *irrational* $x\in[0, 1)$ has a *unique* (necessarily infinite) E-expansion.
The irrational x's in $[0, 1)$ are in a bijective correspondence with the E-fractions $\sum_{n=1}^{\infty}\frac{1}{p_n}$ for which each p_n is finite and $p_{n+1} > p_n(p_n-)+1$ (strict inequality) holds *for infinitely many* $n=1, 2, ...$.

(vii) Scholium Let us say that the E. fraction $\sum_{n=1}^{\infty}\frac{1}{p_n}$ is *strict* if *either* it is finite (eventually $p_n=\infty$), *or* there are infinitely many n such that

$$p_{n+1} > p_n(p_n-1)+1 .$$

The strict E-fractions are in a bijective correspondence with the reals in the interval $[0, 1)$; in this correspondence, an E-fraction is mapped to its sum. Moreover, the finite E-fractions are mapped onto the set of all rationals in $[0, 1)$.

(viii) For two strict E-expansions $x=\sum_{n=1}^{\infty}\frac{1}{p_n}$ and $y=\sum_{n=1}^{\infty}\frac{1}{q_n}$, we have $x<y$ if and only if the two expansions are not the same, and for the first n where $p_n \neq q_n$, we have $p_n<q_n$.

Proof of 5. Theorem

(i): Consider the E-fraction $\sum_{n=1}^{\infty}\frac{1}{p_n}$. Let $q_1=p_1$, and $q_{n+1}=q_n(q_n-1)+1$. Then, obviously, $p_n \geq q_n$, since this holds for $n=1$, and, by induction,

$$p_{n+1} \underset{\substack{\uparrow \\ E.\ fr.}}{\geq} p_n(p_n-1)+1 \underset{\substack{\uparrow \\ ind.\ hyp.}}{\geq} q_n(q_n-1)+1 \underset{\substack{\uparrow \\ def.\ of\ q_{n+1}}}{=} q_{n+1} .$$

Therefore, $$\sum_{n=1}^{\infty}\frac{1}{p_n} \leq \sum_{n=1}^{\infty}\frac{1}{q_n} \underset{\substack{\uparrow \\ 4.\ Lemma}}{=} \frac{1}{q_1-1} \leq 1 .$$

(ii): This is contained in 2. Corollary and 1. Lemma (item (1)); the "moreover" part i seen by 4. Lemma; for this, see also below.

(iii): This is contained in 3. Proposition, and 1. Lemma (item (1)).

(iv): Let N be defined as in the statement; we have 1). By canceling the term $\sum_{n=1}^{N-1}\frac{1}{p_n} = \sum_{n=1}^{N-1}\frac{1}{q_n}$ in (7), we have

$$\sum_{n=N}^{\infty}\frac{1}{p_n} = \sum_{n=N}^{\infty}\frac{1}{q_n} \tag{8}$$

Recursively, let us define

$\hat{p}_N=p_N$ and $\hat{p}_{n+1} = \hat{p}_n(\hat{p}_n-1)+1$ for all $n=N,\ N+1,\ldots$.

We have $p_n \geq \hat{p}_n\ (n \geq N)$: this holds for $n=N$, and, by induction,

$$p_{n+1} \underset{\textit{E.-condition}}{\geq} p_n(p_n-1)+1 \underset{\textit{ind. hyp.}}{\geq} \hat{p}_n(\hat{p}_n-1)+1 \underset{\textit{definition}}{=} \hat{p}_{n+1} \;.$$

Therefore,

$$\sum_{n=N}^{\infty}\frac{1}{p_n} \overset{\mathbf{1}}{\leq} \sum_{n=N}^{\infty}\frac{1}{\hat{p}_n} \underset{\substack{\text{4. } \textit{Lemma} \\ (\textit{for } \tilde{p}_k=\hat{p}_{k+N-1})}}{=} \frac{1}{\hat{p}_N-1} \underset{\hat{p}_N-1=p_N-1\geq q_N}{\overset{\mathbf{2}}{\leq}} \frac{1}{q_N} \overset{\mathbf{3}}{\leq} \sum_{n=N}^{\infty}\frac{1}{q_n} \;. \qquad (9)$$

But the first and the last terms in line (9) are equal (see (8)). All inequalities in (9) must, therefore, be equalities. Equality at **1** means that 2) holds (since, also, $p_n \geq \hat{p}_n$ for $n \geq N$); at **2** that 3) holds; at **3** that 4) holds. The proof is complete.

(v): 3. Proposition says that every rational $x \in [0,1)$ has a finite E.-expansion. By (iv), a finite E. fraction cannot have the same sum as any other, non-identical, finite E. fraction.

(vi): By (ii) and (iv): (iv) contains the statement that a non-finite E. fraction cannot have the same sum as another, non-identical, non-finite E. fraction.

Let us summarize how we get the E-expansion of a number $x \in [0,1)$.

(For an arbitrary real y , we take x to be the fractional part of y : $x=\{y\}=y-\lfloor y\rfloor$; that is, we represent y as the sum of an integer and a fractional number (one in $[0,1)$); we do not deal with the representation of integers here.)

Given x , we recursively produce quantities x_n for $n=0, 1, \ldots$ and extended integers $p_n \geq 2$ $(n=1, 2, \ldots)$ ($p_n \in \mathbb{N}-\{0,1\} \cup \{\infty\}$) as follows:

$$x_0 = x$$

$$p_{n+1} = \lceil 1/x_n \rceil \qquad (n=0, 1, \ldots) \qquad \text{(ceiling)}$$

$$x_{n+1} = x_n - \frac{1}{p_{n+1}} \qquad (n=0, 1, \ldots) \;.$$

(We also write $p_n(x)$ for p_n .)

The desired result of the operation is the infinite sequence $\vec{p}(x) \overset{\text{DEF}}{=} (p_1, p_2, p_3, \ldots)$. The main fact is that the mapping

$$\begin{array}{ccc} [0.1) & \longrightarrow & (\mathbb{N}^{\infty})^{\mathbb{N}^+} \\ x & \longmapsto & \vec{p}(x) \end{array}$$

is one-to-one, and its image $\mathtt{SE}$ consists of the *strict E-sequences* $\vec{p} = (p_1, p_2, p_3, \ldots)$, that is, those $\vec{p}$ for which inequality (4) holds, and, in addition, if all p_n are finite, the strict version of (4) hold for infinitely many n . Moreover, the inverse of the same mapping is

$$\begin{array}{ccc} \mathtt{SE} & \longrightarrow & [0, 1) \\ \vec{p} & \longmapsto & \sum_{n=1}^{\infty} \frac{1}{p_n} \end{array}$$

For a comparison, let us look at the *Babylonian number system*: the familiar place-value system (Yes; there is excellent evidence that the Babylonians were aware of the principles of this system; they used the system with base $b=60$.)

We fix the integer $b\geq 2$.

Given $x\in [0, 1)$, we do the following recursion:

$$\begin{array}{lll} x_0 = x & & \\ d_{n+1} = \lfloor x_n \cdot b^{n+1} \rfloor & (n=0, 1, \ldots) & (\mathit{floor}) \\ x_{n+1} = x_n - \dfrac{d_{n+1}}{b^{n+1}} & (n=0, 1, \ldots) & \end{array}$$

We obtain the infinite sequence $\vec{d}=\vec{d}(x)=(a_1, a_2, \ldots)$ of integers. The mapping

$$\begin{array}{ccc} [0, 1) & \longrightarrow & \mathbb{N}^{\mathbb{N}^+} \\ x & \longmapsto & \vec{d}(x) \end{array}$$

has its image $\mathtt{SB}$ the set of *strict base-b fractions*, those integer sequences $\vec{d}=(d_1, d_2, \ldots)$, or infinite series $\sum_{n=1}^{\infty} \frac{d_n}{b^n}$, such that the *digits*

1) d_n are in the range $0\leq d_n<b$,

and

2) for infinitely many n , $d_n \neq b-1$.

The inverse function is given as

$$\begin{array}{ccc} \mathrm{SB} & \longrightarrow & [0,1) \\ \vec{d} & \longmapsto & \displaystyle\sum_{n=1}^{\infty} \frac{d_n}{b^n} \end{array}$$

Furthermore, if we drop condition 2), and consider the set B of all $\vec{d} \in \mathbb{N}^{\mathbb{N}^+}$ for which 1) holds, together with the function

$$\begin{array}{ccc} \mathrm{B} & \xrightarrow{\Sigma} & [0,1) \\ \vec{d} & \longmapsto & \displaystyle\sum_{n=1}^{\infty} \frac{d_n}{b^n} \end{array} \quad ,$$

then the fiber $\Sigma^{-1}(x)$ has more than one element only when x is a so-called b-adic rational number, a sum of negative-exponent powers of b , and in that case, $\Sigma^{-1}(x)$ has precisely two elements.

All this is, of course, extremely familiar. We also see that the basic facts for the E-expansions (listed above and in 5. Theorem) are in close analogy with those for B-expansions. The analog of 4. Lemma is the identity

$$\sum_{n=1}^{\infty} \frac{b-1}{b^n} = 1 \quad .$$

We have a third ancient number system, the Greek one: *anthypharesis*, or *continued fractions*; now I'll call them *G-fractions*. Once again, we restrict attention to expansions of fractional numbers, $x \in [0,1)$. Here is the recursive definition:

$$\begin{array}{lll} x_0 = x & & \\ a_n = \lfloor 1/x_n \rfloor & (n=1, 2, \ldots) & (\mathit{floor}) \\ x_{n+1} = \dfrac{1}{x_n} - a_n \quad . & (n=1, 2, \ldots) \; . & \end{array}$$

a_n is a positive integer, unless $x_n=0$. It should be understood that when $x_n=0$, then $a_n=1/0=\lfloor 1/0 \rfloor=\infty$, and, by definition, all x_N for $N \geq n$ are 0 , and all a_N for $N \geq n$ are ∞ .

The third line of the definition rewrites as

$$x_n = \cfrac{1}{a_n + \cfrac{1}{x_{n+1}}} \; .$$

Thus,

$$x = x_0 = \cfrac{1}{a_1 + \cfrac{1}{x_1}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_2}}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{x_3}}}} = \; .\; .\; .\; . \qquad (10)$$

Therefore, if any x_n becomes zero, we get that x is a rational expression of the integers $a_1, a_2, \ldots$, and therefore x is rational number. We will see that, conversely, if x is rational, then there is n such that $x_n = x_{n+1} = \ldots = 0$.

By the *convergents* of the G-expansion we mean the finite continued fractions obtained by omitting the "tails", the x_k , from the above expressions:

$$\gamma_1 = \frac{1}{a_1}$$

$$\gamma_2 = \cfrac{1}{a_1 + \cfrac{1}{a_2}}$$

$$\gamma_3 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}} \; ,$$

etc. When $a_n = \infty$, we have $\gamma_n = \gamma_{n-1}$, and thus $\gamma_N = \gamma_{n-1}$ for all $N \geq n-1$.

Let's look at the expression γ_n as algebraic expressions (rational functions) of the variables $a_1, a_2, \ldots$. For any rational function γ of $a_1, a_2, \ldots$, let $\tilde{\gamma}$ denote the *shift* of γ , the expression in which all the a_i are simultaneously replaced by a_{i+1} : if $\gamma = \gamma(a_1, a_2, \ldots)$, then $\tilde{\gamma} = \gamma(a_2, a_3, \ldots)$. The point of this notation is that, obviously,

$$\gamma_{n+1} = \frac{1}{a_1 + \tilde{\gamma}_n} \; .$$

Thus, if $\gamma_n = \dfrac{p_n}{q_n}$ with polynomials p_n , q_n of $a_1, a_2, \ldots$, then

$$\gamma_{n+1} = \frac{1}{a_1 + \tilde{\gamma}_n} = \frac{1}{a_1 + \frac{\tilde{p}_n}{\tilde{q}_n}} = \frac{\tilde{q}_n}{\tilde{p}_n + a_1 \tilde{q}_n} \ .$$

Therefore, if we define

$$p_{n+1} = \tilde{q}_n \qquad (n=1,\ 2,\ \ldots) \qquad (11)$$

$$q_{n+1} = \tilde{p}_n + a_1 \tilde{q}_n \ , \qquad (n=1,\ 2,\ \ldots) \qquad (12)$$

we get $\gamma_{n+1} = \frac{p_{n+1}}{q_{n+1}}$.

With (10) and (11) as recursive clauses, and

$$p_1 = 1$$
$$q_1 = a_1$$

as starting clauses, we define the $(a_1, a_2, \ldots)$-polynomials, (and, for integers $a_1, a_2, \ldots$ the integers p_n , q_n); and we call $\gamma_n = \frac{p_n}{q_n}$ the nth *convergent* of the G-expansion at hand.

We derive a recursive formula for p_n and q_n without the shift operation, one which, on the other hand, expresses each term by *two* previous terms, rather than just one as in the original definition.

We put

$$p_{-1} = 1 \ , \qquad p_0 = 0$$
$$q_{-1} = 0 \ , \quad q_0 = 1 \ .$$

Note that $p_1 = \tilde{q}_0 = 1$, and $q_1 = \tilde{p}_0 + a_1 \tilde{q}_0 = a_1$; that is, (11) and (12) hold for $n=0$ too. We put $\gamma_0 = \frac{p_0}{q_0} = 0$.

We **claim** that, for all $n=1, 2, \ldots,$ we have

$$\begin{aligned} p_n &= a_n \cdot p_{n-1} + p_{n-2} \\ q_n &= a_n \cdot q_{n-1} + q_{n-2} \end{aligned} \qquad (n=1,\ 2,\ \ldots) \ . \qquad (13)$$

Indeed, for $n=1$, this holds. Assuming them for $n\geq 1$, for $n+1$, we have:

$$p_{n+1} = \tilde{q}_n = (a_n \cdot q_{n-1}+q_{n-2})^{\sim} = a_{n+1} \cdot \tilde{q}_{n-1}+\tilde{q}_{n-2} \underset{(11)}{\underset{\uparrow}{=}} a_{n+1} \cdot p_n+p_{n-1}$$

and

$$q_{n+1} \underset{(12)}{\underset{\uparrow}{=}} \tilde{p}_n+a_1\tilde{q}_n \underset{ind\, hyp\ and\ previous\ line}{\underset{\uparrow}{=}} (a_n \cdot p_{n-1}+p_{n-2})^{\sim} + a_1(a_{n+1} \cdot p_n+p_{n-1}) =$$

$$= a_{n+1} \cdot \tilde{p}_{n-1}+\tilde{p}_{n-2} + a_1(a_{n+1} \cdot p_n + p_{n-1}) =$$

$$= a_{n+1} \cdot (\tilde{p}_{n-1}+a_1 \cdot p_n) + \tilde{p}_{n-2}+a_1 \cdot p_{n-1} =$$

$$\underset{(11):p_n=\tilde{q}_{n-1}\ ,\ \ p_{n-1}=\tilde{q}_{n-2}}{\underset{\uparrow}{=}} a_{n+1} \cdot (\tilde{p}_{n-1}+a_1 \cdot \tilde{q}_{n-1}) + \tilde{p}_{n-2}+a_1 \cdot \tilde{q}_{n-2}$$

$$\underset{(12)\ for\ n\ and\ n-1}{\underset{\uparrow}{=}} a_{n+1} \cdot q_n + q_{n-1} \quad ,$$

which proves (13).

(13) shows that $q_{n+1}\geq q_n+q_{n-1}$; thus, q_n grows (at least) exponentially (Fibonacci ...).

So far, we were deriving identities that are true for arbitrary $a_1, a_2, \ldots$. Now, let us take $x\in[0, 1)$ as before, derive the (generalized) integers $a_1, a_2, \ldots$ from x as above, and define p_n, q_n and γ_n as above with these values of $a_1, a_2, \ldots$.

Note the inequalities

$$\gamma_0 < \gamma_2 < \gamma_4 < \gamma_6 < \ldots < x < \ldots \gamma_5 < \gamma_3 < \gamma_1 \tag{14}$$

holding for the subscripts n for which $x_n>0$; if there is any n such that $x_n=0$, we get $\gamma_n=\gamma_{n+1}= \;=x$.

The inequalities can be seen directly from (10) and the definitions of the γ_n.

The main point is the equality

$$p_{n+1}q_n - q_{n+1}p_n = (-1)^n \qquad (n=0, 1, \ldots) .$$

For $n=0$, this becomes $1\cdot 1 - a_1\cdot 0 = 1 = (-1)^0$, thus true. By induction,

$$p_{n+1}q_n - q_{n+1}p_n \underset{\substack{\uparrow \\ (13)}}{=} (a_n\cdot p_n+p_{n-1})\cdot q_n - (a_n\cdot q_n+q_{n-1})\cdot p_n =$$

$$= -(p_nq_{n-1} - q_np_{n-1}) = -(-1)^{n-1} = (-1)^n .$$

Therefore, for $n=0, 1, \ldots$, we have

$$\gamma_{n+1}-\gamma_n = \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{p_{n+1}q_n - q_{n+1}p_n}{q_nq_{n+1}} = (-1)^n\cdot\frac{1}{q_nq_{n+1}} . \quad (15)$$

From (14) and $q_n \xrightarrow[n\to\infty]{} \infty$, it follows that $\lim\limits_{n\to\infty} \gamma_n=x$.

The G-expansion is also related to unit fractions: it gives x as an alternating sum of unit fractions.

Because:

$$\gamma_n = \gamma_0 + (\gamma_1-\gamma_0) + (\gamma_2-\gamma_1) + \ldots + (\gamma_n-\gamma_{n-1}) =$$

$$\gamma_0 + \frac{1}{q_0\cdot q_1} - \frac{1}{q_1\cdot q_2} + \frac{1}{q_2\cdot q_3} - \ldots + (-1)^n\cdot\frac{1}{q_n\cdot q_{n-1}} ,$$

and thus x is the sum of the absolutely convergent alternating series

$$x = \sum_{n=0}^{\infty} (-1)^n\cdot\frac{1}{q_n\cdot q_{n-1}} ,$$

and x is the difference of two positive series

$$= \sum_{n=0}^{\infty} \frac{1}{q_{2n}\cdot q_{2n-1}} - \sum_{n=0}^{\infty} \frac{1}{q_{2n+1}\cdot q_{2n}} .$$

Thus, the Greek number system gives any $x\in[0, 1)$ as the difference of two lax E-expansions; finite ones to be sure for rational x .

Let us (re)turn to the issue of efficiently calculating many terms of the E-expansion of $\sqrt{N}$ for a non-square integer N . What I am going to say generalizes to essentially any (irrational real) algebraic number. In particular, Newton's method of approximation and the effectively proved

Liouville's theorem for algebraic numbers are used.

I should say that, although the anthyphairesis of $\sqrt{N}$ is very nicely calculable (it is eventually periodic), the approximation it gives is not fast enough for our purposes.

Put

$$a_0 = \lceil \sqrt{N} \rceil \qquad \text{(ceiling)}$$

$$a_{n+1} = a_n - \frac{a_n^2 - N}{2a_n}$$

We will have

$$\sqrt{N} \ldots < a_{n+1} < a_n < \ldots < a_0$$

and, for $b_n \stackrel{\text{DEF}}{=} a_n^2 - N$,

$$b_{n+1} < \frac{b_n^2}{4N} \quad ,$$

in particcular

$$b_n < \frac{b^{(2^n)}}{(4N)^{2^n-1}}$$

where $b=b_0=a^2-N < 1$. Very fast convergence! For $e_n=a_n-\sqrt{N}$,

$$0 < e_n < \frac{b_n}{2\sqrt{N}-1} \quad .$$

(I have taught this in MATH338).

On the other hand, we have that, with $c=\lceil 2\sqrt{N}+1 \rceil$,

$$\left|\sqrt{N} - \frac{p}{q}\right| > \frac{1}{c \cdot q^2}$$

always (this we call Lioyville's inequality). Indeed, if $\left|\sqrt{N}-\frac{p}{q}\right| > 1$, there is nothing to prove. Assume $\left|\sqrt{N}-\frac{p}{q}\right| < 1$. Then, clearly, $\frac{p}{q} < \sqrt{N}+1$, amd we have

$$\frac{1}{q^2} \underset{\substack{\uparrow \\ \text{since } \sqrt{N} \text{ is irrational}}}{\leq} \frac{|Nq^2-p^2|}{q^2} = |N - \frac{p^2}{q^2}| = |\sqrt{N}+\frac{p}{q}|\,|\sqrt{N}-\frac{p}{q}| < (2\sqrt{N}+1)|\sqrt{N}-\frac{p}{q}| <$$

$$c \cdot |\sqrt{N}-\frac{p}{q}|$$

and the assertion is proved.

Let $x=\sqrt{N}-\lfloor\sqrt{N}\rfloor$.

[This part is under development.]