§11 Regular cardinals

In what follows, κ , λ , μ , ν , ρ always denote cardinals.

A cardinal κ is said to be *regular* if κ is infinite, and the union of fewer than κ sets, each of whose cardinality is less than κ, is of cardinality less than κ. In symbols: κ is regular if κ is infinite, and

for any set I with $|I|<\kappa$ and any family $\langle A_i\rangle_{i\in I}$ of sets such that $|A_i|<\kappa$ for all $i\in I$, we have $|\bigcup_{i\in I}A_i|<\kappa$.

(Among finite cardinals, only 0 , 1 and 2 satisfy the displayed condition; it is not worth including these among the cardinals that we want to call "regular".)

To see two examples, we know that $\aleph_0$ is regular: this is just to say that the union of finitely many finite sets is finite. We have also seen that $\aleph_1$ is regular: the meaning of this is that the union of countably many countable sets is countable.

For future use, let us note the simple fact that

the ordinal-least-upper-bound of any set of cardinals is a cardinal; for any set I *, and cardinals* λ_i *for* $i\in I$ *,* $\operatorname{lub}_{i\in I}\lambda_i$ *is a cardinal.*

Indeed, if $\alpha=\operatorname{lub}_{i\in I}\lambda_i$, and $\beta<\alpha$, then for some $i\in I$, $\beta<\lambda_i$. If we had $\beta\sim\alpha$, then by $\beta<\lambda_i\le\alpha$, and Cantor-Bernstein, we would have $\beta\sim\lambda_i$, contradicting the facts that λ_i is a cardinal and $\beta<\lambda_i$. Thus, if $\beta<\alpha$, then $\beta\not\sim\alpha$, which means that α is a cardinal.

The condition of regularity can be stated in the following equivalent manner: the infinite cardinal κ is regular iff

(1) for each set I with cardinality less than κ, and for each family $\langle\alpha_i\rangle_{i\in I}$ of ordinals α_i , each less than κ, we have that $\operatorname{lub}_{i\in I}\alpha_i<\kappa$.

Indeed, assuming that κ is regular, and assuming that $|I|<\kappa$ and $\alpha_i<\kappa$ for all $i\in I$, we have that $|\alpha_i|\le\alpha_i<\kappa$, thus, $|\operatorname{lub}_{i\in I}\alpha_i|=|\bigcup_{i\in I}\alpha_i|<\kappa$; but, for any cardinal κ, and any ordinal α, $|\alpha|<\kappa$ implies that $\alpha<\kappa$ (why?); thus, $\operatorname{lub}_{i\in I}\alpha_i<\kappa$ follows.

Conversely, assume that (1) holds, and show that κ is regular. In particular, κ is a limit ordinal. Let $|I|<\kappa$ and $|A_i|<\kappa$ for each i. Let $\lambda_i=|A_i|$; λ_i is an ordinal less than κ. Hence, by (1), $\lambda \underset{\text{def}}{=} \operatorname{lub}_{i\in I}\lambda_i<\kappa$. As we noted above, λ is a cardinal. Let $\mu=\max(\lambda,|I|)$; since λ, $|I|$ are both cardinals less than κ, μ is a cardinal $<\kappa$. But then $|I|\le\mu$ and $|A_i|=\lambda_i\le\lambda\le\mu$. Hence, if μ is infinite, we have

$$|\bigcup_{i\in I}A_i|\le|\sum_{i\in I}A_i|\le\mu\cdot\mu\underset{\S 10}{\overset{\uparrow}{=}}\mu<\kappa;$$

and if μ is finite, then $\bigcup_{i\in I}A_i$ is finite, and so again

$$|\bigcup_{i\in I}A_i|<\kappa.$$

This shows that κ is regular.

In fact, we can formulate regularity in terms of ordinals, without (overtly) referring to cardinals. Let us say of a limit ordinal δ that it is *regular* if it satisfies

(1') for any ordinal β less than δ, and for each family $\langle\alpha_\iota\rangle_{\iota<\beta}$ of ordinals α_ι, each less than δ, we have that $\operatorname{lub}_{\iota<\beta}\alpha_\iota<\delta$.

Note that if κ is an infinite cardinal, then it is a limit ordinal, and if it satisfies (1), then it satisfies (1'), since $\beta<\kappa$ implies that $|\beta|<\kappa$. Conversely, I claim that if the limit ordinal δ satisfies (1'), then it is a regular cardinal. δ is infinite. Assume (1') for such a δ. Let $\beta<\delta$. If we had $\beta\sim\delta$, then we would have an indexing $\langle\alpha_\iota\rangle_{\iota<\beta}$ of *all* ordinals less than δ, and so, also using that δ is limit, $\operatorname{lub}_{\iota<\beta}\alpha_\iota=\operatorname{lsub}_{\iota<\beta}\alpha_\iota=\delta$, contradicting (1'). This shows that δ is a cardinal. But now looking at formulation (1), if we have an arbitrary set I with

$|I|<\delta$, we can index I by an ordinal $\beta<\delta$, and transform any instance of (1) into one of (1').

We have shown that an equivalent definition of "regular cardinal" is given by the condition (1'), together with the condition that δ is a limit ordinal.

We have a large class of regular cardinals. Let us write κ^+ for the least cardinal greater than κ. If $\kappa=n\in\omega$, then $\kappa^+=n+1$, and if $\kappa=\aleph_\alpha$, then $\kappa^+=\aleph_{\alpha+1}$. A *successor cardinal* is one of the form κ^+. The fact is that all infinite successor cardinals are regular. The reason is that if $|I|<\kappa^+$ and $|A_i|<\kappa^+$ for $i\in I$, then $|A_i|\le\kappa$, and $|I|\le\kappa$, thus

$$|\bigcup_{i\in I}A_i|\le\sum_{i\in I}|A_i|\le|I|\times\kappa\le\kappa\cdot\kappa\underset{\substack{\uparrow\\ \S 10}}{=}\kappa<\kappa.$$

On the other hand, e.g. $\aleph_\omega=\operatorname{lub}_{n<\omega}\aleph_n$ is not regular (we say *singular* for not regular). Namely, now $|\omega|=\omega=\aleph_0<\aleph_\omega$, and for each $n\in\omega$, $|\aleph_n|=\aleph_n<\aleph_\omega$, but $|\bigcup_{n\in\omega}\aleph_n|=|\aleph_\omega|=\aleph_\omega$; thus, the family $\langle\aleph_n\rangle_{n\in\omega}$ is a counter-example to the regularity condition for $\aleph_\omega$.

We have now seen that the cardinals $\aleph_0$ and $\aleph_{\alpha+1}$ are regular. We just saw that for $\alpha=\omega$, $\aleph_\omega$ is singular. Are there *limit cardinals*, that is, cardinals of the form $\aleph_\delta$ for a limit ordinal δ that are regular? One sees that if so, then we must have $\aleph_\delta=\delta$: the reason is that the set $\{\aleph_\alpha:\alpha<\delta\}$ is *cofinal* in $\aleph_\delta$: $\operatorname{lub}_{\alpha<\delta}\aleph_\alpha=\aleph_\delta$ (why?); therefore, if $\delta<\alpha$ were the case, we would have a contradiction to formulation (1') of the regularity of $\aleph_\delta$. However, if we take the *smallest* δ for which $\aleph_\delta=\delta$, $\kappa=\aleph_\delta$ turns out to be singular; thus, the above does not have a converse! Consider the following sequence:

$$\kappa_0=\aleph_0$$

$$\kappa_{n+1}=\aleph_{(\kappa_n)}\qquad(n<\omega)$$

defined recursively. We have $\kappa_0<\kappa_1$ since $0<\aleph_0$. By an easy induction, we show that $\kappa_n<\kappa_{n+1}$ for all $n<\omega$. Let us define $\lambda=\operatorname{lub}_{n<\omega}\kappa_n$. I claim that $\aleph_\lambda=\lambda$. We have

$\aleph_\lambda = \mathrm{lub}_{\alpha<\lambda}\, \aleph_\alpha = \mathrm{lub}_{n<\omega}\, \aleph_{\kappa_n}$ since $\lambda = \mathrm{lub}_{n<\omega}\, \kappa_n$ (**exercise**: show that if $\alpha_\beta \le \alpha_{\beta'}$, for $\beta \le \beta' < \gamma$, and $\gamma = \mathrm{lub}_{i\in I}\, \beta_i$, then $\mathrm{lub}_{i\in I}\, \alpha_{\beta_i} = \mathrm{lub}_{\beta<\gamma}\, \alpha_\beta$); but $\aleph_{\kappa_n} = \kappa_{n+1}$, and so $\aleph_\lambda = \mathrm{lub}_{n<\omega}\, \kappa_{n+1} = \lambda$. However, $\lambda = \aleph_\lambda$ is not regular: we have $\lambda = \mathrm{lsub}_{n<\omega}\, \kappa_n$ and $\omega < \lambda$.

It turns out that, using the usual axioms of set-theory, one cannot prove that there are regular limit cardinals. A regular limit cardinal is called a *weakly inaccessible* cardinal. A *strongly inaccessible*, or simply *inaccessible*, cardinal κ has, by definition, the additional property that for all $\lambda<\kappa$, we also have that $2^\lambda<\kappa$. It is easy to see that limit cardinals with this latter property are exactly the ones of the form beth_δ for a limit ordinal δ; these latter are called *strong limit cardinals*. Thus, an inaccessible cardinal is the same as a regular strong limit cardinal. In section 13, we will take a look at inaccessible cardinals -- despite the fact that we cannot prove their existence with the axioms so far listed. In fact, the existence of inaccessible cardinals will be seen as a reasonable *new* axiom of set theory.

There is an important calculation of the cardinality of a particular kind of set involving regularity. We consider the following situation. We have $(B, \mathcal{F})$ where

(i) B is a set

and

(ii) $\mathcal{F}$ is a family $\mathcal{F} = \langle f_i \rangle_{i\in I}$ of *partial operations* on B, that is, each f_i is a function with domain contained in B^{J_i}, $\mathrm{dom}(f_i) \subset B^{J_i}$ for a particular "arity" J_i (an arbitrary set, given for each $i\in I$; usually, J_i is a natural number) and $\mathrm{range}(f_i) \subset B$. As a reminder,

$$B^J = \{f:\ f \text{ is a function, } \mathrm{dom}(f)=J,\ \mathrm{range}(f)\subset B\}\ .)$$

We are going to write $f{:}\subset A \longrightarrow B$ to mean that $f{:}\mathrm{dom}(f) \longrightarrow B$ and $\mathrm{dom}(f) \subset A$.

Thus, the partial J_i-ary operation f_i may be displayed as $f_i{:}\subset B^{J_i} \longrightarrow B$.

Examples. (A) The notion of *group* is a very important one in mathematics. A group is a set B together with a binary operation $\cdot : B^2 \longrightarrow B$, a unary operation $(\ )^{-1} : B \longrightarrow B$, and a 0-ary operation $e \in B$, where these data are required to satisfy the identities $(x\cdot y)\cdot z = x\cdot(y\cdot z)$, $e\cdot x = x\cdot e = x$, $x^{-1}\cdot x = x\cdot x^{-1} = e$. This fits our setup above, with $I = 3$, $J_0 = 2$, $f_0 = \cdot$, $J_1 = 1$, $f_1 = (\ )^{-1}$, $J_2 = 0$, $f_2 = e$ (note that a 0-ary operation $f : A^0 \to A$ is basically the same as a distinguished element of A : $A^0 = \{\emptyset\} = 1$, and to give a function $f : \{\emptyset\} \to A$ is the same as to give the element $e = f(\emptyset) \in A$). In this case, all operations are *total*, that is, $\mathrm{dom}(f_0) = B^2$, etc.

A similar example is provided by each of a number of other kinds of algebraic structure: lattice, Boolean algebra (these we will see in some detail later), ring, field, etc. (In fact, in the case of a field, division is a genuinely *partial* operation.)

(B) The notion of limit in analysis can be considered as a partial operation. If $\vec{x} \in \mathbb{R}^\omega$ is an infinite (ω-type) sequence of reals, $\vec{x} = \langle x_n \rangle_{n<\omega}$, then $\lim \vec{x} = \lim\limits_{n\to\infty} x_n$ may not exist, but if it does, it is uniquely determined. Let $\lim : \subset \mathbb{R}^\omega \longrightarrow \mathbb{R}$ be the partial operation whose domain is the set of those sequences of which the limit exists, and whose value, at such a sequence, is the limit of that sequence. In this case, the arity of $\lim$ is the infinite ordinal ω.

Lim can also be defined on $(\mathbb{R}^\mathbb{R})^\omega$. That is, if $\vec{f} \in (\mathbb{R}^\mathbb{R})^\omega$, i.e., if $\vec{f}$ is a sequence $\vec{f} = \langle f_n \rangle_{n<\omega}$ of functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ from the reals to the reals, then $\lim \vec{f}$ is the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which $f(x) = \lim\limits_{n\to\infty} f_n(x)$; $\lim \vec{f}$ is defined iff the limit $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$ ("pointwise limit").

Below, we will consider the systems $(\mathbb{R}, \lim)$, $(\mathbb{R}^\mathbb{R}, \lim)$ as examples of $(B, \mathcal{F})$ of the general situation; in both cases, $I = 1 = \{0\}$ and $f_0 = \lim$.

Let us return to a general "system" $(B, \mathcal{F})$ as above. Relative to $(B, \mathcal{F})$, a *closed set* is any subset X of B for which

$$\vec{x} \in X^{J_i} \cap \mathrm{dom}(f_i) \implies f(\vec{x}) \in X$$

whenever $i \in I$, $\vec{x} \in B^{J_i}$. For instance, if $(B.\mathcal{F})$ is a group, then a closed set is what is usually called a *subgroup* of the given group: a subset of the (underlying set of the) group which is closed under $\cdot$, $(\ )^{-1}$ and contains e. In the "limit" examples, closed sets are what are usually called *closed*: closed under limits (in the function-space example, this is only one possible kind of closedness; there are also other "topologies" on that space).

It is clear that the intersection of any family of closed sets is closed (**exercise**; the empty intersection is taken to be B itself). Therefore, for any subset A of B, we may take the intersection of all closed sets containing A, and call it the *closure*, or $\mathcal{F}$-*closure* of A; notation: $\bar{A}$, or $\bar{A}^{(\mathcal{F})}$.

In the example of type (A) (groups, Boolean algebras, ...), $\bar{A}$ is called the *subalgebra generated by* A; in the examples under (B), it is called the *closure* of the set A.

What we want to do is calculate the cardinality of $\bar{A}$, in terms of some cardinalities given with the situation. Let us denote:

$$\lambda \underset{\mathrm{def}}{=} \max(\aleph_0, |I|) ,$$

$$\mu \underset{\mathrm{def}}{=} \max(\aleph_0, |A|)$$

(that is, $\lambda = |I|$ unless I is finite, in which case $\lambda = \aleph_0$; similarly for μ),

and

$\kappa \underset{\mathrm{def}}{=}$ the least infinite *regular* cardinal for which

$\kappa > |J_i|$ for every $i \in I$.

We use the notation

$$\nu^{<\kappa} \stackrel{=}{\text{def}} \text{lub}\{\nu^{\rho} : \rho < \kappa \text{ , } \rho \text{ any cardinal}\}$$

We have, with these notations, the estimate formulated in the next

Theorem. With $(B, \mathcal{F})$ any system of partial operations, we have, for the cardinality $|\bar{A}|$ of the closure $\bar{A}$ of a subset A of B, that

$$|\bar{A}| \leq \max(\lambda, \mu)^{<\kappa} \qquad (2)$$

with κ, λ and μ as determined above.

Of course, we always have that $\mu \leq |\bar{A}|$ (why?); the two estimates determine $|\bar{A}|$ quite sharply (see also more on this below).

Before we turn to the proof of (2), we note three properties of the operation $\nu^{<\kappa}$. One is

$$\nu \leq \nu^{<\kappa} \quad (\text{for } \kappa \geq 2), \qquad (3)$$

which is obvious; the other is

$$\kappa \leq \nu^{<\kappa} \quad (\text{for } \nu \geq 2); \qquad (4)$$

the third is

$$(\nu^{<\kappa})^{\rho} = \nu^{<\kappa} \quad (\kappa \text{ infinite regular, } \rho<\kappa \text{ , } \nu\geq 2). \qquad (5)$$

For (4), note that $\rho \underset{\text{Cantor's}}{\underset{\uparrow}{<}} 2^{\rho} \le \nu^{\rho} \le \nu^{<\kappa}$ for every $\rho < \kappa$; but κ is the least cardinal $>$ all $\rho < \kappa$; it follows that $\kappa \le \nu^{<\kappa}$.

(5) is slightly more involved. We claim that

$$^{\rho}(\nu^{<\kappa}) = \bigcup \{^{\rho}(\nu^{\mu}) : \mu < \kappa\} ;$$

here, we have made a distinction between exponentiation "of sets" and that of "cardinals"; $^{\rho}\sigma$ means the set of all functions $\rho \to \sigma$; ν^{μ} is a cardinal (in particular, an ordinal), the result of cardinal exponentiation applied to the cardinal-arguments ν and μ. In this display, on the left side, we have the set of all ρ-type sequences of ordinals $< \nu^{<\kappa}$; on the right, we have a union of sets of ρ-type sequences.

The claim is clearly the same as to say that for any $\langle \alpha_{\xi} \rangle_{\xi<\rho} \in {}^{\rho}(\nu^{<\kappa})$, there is $\mu < \kappa$ such that $\langle \alpha_{\xi} \rangle_{\xi<\rho} \in {}^{\rho}(\nu^{\mu})$. But, for each $\xi<\rho$, $\alpha_{\xi} \in \nu^{\mu_{\xi}}$ for some $\mu_{\xi} < \kappa$, since $\nu^{<\kappa}$ is the union of the sets ν^{μ}, $\mu < \kappa$. Since κ is regular, $\rho < \kappa$ and each $\mu_{\xi} < \kappa$, we have that $\operatorname{lub}_{\xi<\rho} \mu_{\xi} < \kappa$ (see the second definition of regularity). With $\mu = \operatorname{lub}_{\xi<\rho} \mu_{\xi}$, μ is a cardinal, $\nu^{\mu_{\xi}} \le \nu^{\mu}$ for all $\xi < \rho$, and thus $\alpha_{\xi} \in \nu^{\mu}$ for all $\xi < \rho$. We have that $\langle \alpha_{\xi} \rangle_{\xi<\rho} \in {}^{\rho}(\nu^{\mu})$ as we wanted.

On the basis of the last claim, we have

$$\begin{aligned}(\nu^{<\kappa})^{\rho} &= |{}^{\rho}(\nu^{<\kappa})| = |\bigcup \{^{\rho}(\nu^{\mu}) : \mu < \kappa\}| \\ &\le \sum_{\mu<\kappa} (\nu^{\mu})^{\rho} = \sum_{\mu<\kappa} \nu^{\mu\cdot\rho} \underset{\mu\cdot\rho=max(\mu,\,\rho)<\kappa}{\underset{\uparrow}{\le}} \kappa\cdot\nu^{<\kappa} \underset{(4)}{\underset{\uparrow}{=}} \nu^{<\kappa}\end{aligned}$$

as required for (5) .

To prove (2), we give a formula for the set $\bar{A}$ itself first. Define, by recursion on the ordinal

$\alpha < \kappa$, the sets A_α as follows:

$$A_0 = A$$

$$A_\alpha = \bigcup_{\beta<\alpha} A_\beta \quad \text{if } \alpha < \kappa \text{ is a limit ordinal; and}$$

$$A_{\beta+1} = A_\beta \cup \{f_i(\vec{a}) : i \in I\,,\ \vec{a} \in \mathrm{dom}(f_i) \cap A_\beta{}^{J_i}\}\ .$$

Clearly, $A_\beta \subset A_\alpha \subset B$ whenever $\beta < \alpha < \kappa$. We claim that

$$\bar{A} = \bigcup_{\alpha<\kappa} A_\alpha\ . \qquad (6)$$

For one thing, the right-hand-side is contained in the left-hand-side: by induction on $\alpha < \kappa$, we show that $A_\alpha \subset \bar{A}$. The induction-steps for $\alpha = 0$ and α limit are obvious. For $\alpha = \beta+1$, note that since $A_\beta \subset \bar{A}$ (induction hypothesis), and since $\bar{A}$ is closed, $f_i(\vec{a}) \in \bar{A}$ whenever $i \in I$ and $\vec{a} \in \mathrm{dom}(f_i) \cap A_\beta{}^{J_i}$; thus, by the definition of $A_{\beta+1}$, $A_{\beta+1} \subset \bar{A}$ as desired.

Secondly, we need to show that the union $\bigcup_{\alpha<\kappa} A_\alpha$ is closed; since $\bar{A}$ is the least closed set containing A, and the union contains A, it will follow that $\bar{A} \subset \bigcup_{\alpha<\kappa} A_\alpha$, which is the other containment we need.

To show that the union is closed, let $i \in I$, and $\vec{a} = \langle a_j \rangle_{j \in J_i} \in (\bigcup_{\alpha<\kappa} A_\alpha)^{J_i} \cap \mathrm{dom}(f_i)$; we want that $f_i(\vec{a}) \in \bigcup_{\alpha<\kappa} A_\alpha$. Since $a_j \in \bar{A}$, there is $\alpha_j < \kappa$ such that $a_j \in A_{\alpha_j}$. We now have the family $\langle \alpha_j \rangle_{j \in J_i}$ of ordinals less than κ. By the definition of κ, we have that $|J_i| < \kappa$; also, κ is regular. Therefore, by the second version of the definition of "regular",

$$\beta \underset{\mathrm{def}}{=} \mathrm{lub}\{\alpha_j : j \in J_i\} < \kappa$$

(second use of regularity of κ !).

Then, of course, $a_j \in A_\beta$ for all $j \in J_i$, since $A_{\alpha_j} \subset A_\beta$. Thus,

$\vec{a} \in \mathrm{dom}(f_i) \cap A_\beta^{J_i}$, and by the definition of $A_{\beta+1}$, $f_i(\vec{a}) \in A_{\beta+1}$, and as a consequence, $f_i(\vec{a}) \in \bigcup_{\alpha<\kappa} A_\alpha$.

Having the formula (6), it is easy to estimate $|\bar{A}|$. By induction on $\alpha < \kappa$, we show that $|A_\alpha| \le \max(\mu,\lambda)^{<\kappa}$. By (3), this is true for $\alpha = 0$. If the assertion holds for all $\beta < \alpha < \kappa$, and α is a limit ordinal, then

$$|A_\alpha| \le \sum_{\beta<\alpha} |A_\beta| \le \sum_{\beta<\alpha} \max(\mu,\lambda)^{<\kappa} = |\alpha| \cdot \max(\mu,\lambda)^{<\kappa}$$

$$\le \kappa \cdot \max(\mu,\lambda)^{<\kappa} \underset{\S 10}{\overset{\uparrow}{=}} \max(\kappa, \max(\mu,\lambda)^{<\kappa}) \underset{(4)}{\overset{\uparrow}{=}} \max(\mu,\lambda)^{<\kappa}.$$

Finally, for $\alpha = \beta+1$, the set $U \underset{\mathrm{def}}{=} \{f_i(\vec{a}) : i \in I,\ \vec{a} \in \mathrm{dom}(f_i) \cap A_\beta^{J_i}\}$ has cardinality

$$\le |I| \cdot (\underset{\text{ind. hyp}}{\overset{\uparrow}{\max}}(\lambda,\mu)^{<\kappa})^{|J_i|} \le \lambda \cdot (\max(\lambda,\mu)^{<\kappa})^{\underset{\rho \,=_{\mathrm{def}}\, |J_i|}{\overset{\uparrow}{\rho}}}$$

$$\underset{(5)}{\overset{\uparrow}{=}} \lambda \cdot \max(\lambda,\mu)^{<\kappa} = \max(\lambda, \max(\lambda,\mu)^{<\kappa}) \underset{(3)}{\overset{\uparrow}{=}} \max(\lambda,\mu)^{<\kappa}.$$

Since $A_\alpha = A_\beta \cup U$, $|A_\alpha| \le \max(\lambda,\mu)^{<\kappa} + \max(\lambda,\mu)^{<\kappa} = \max(\lambda,\mu)^{<\kappa}$ as desired. Our induction is complete.

Finally,

$$|\bar{A}| = |\bigcup_{\alpha<\kappa} A_\alpha| \le \sum_{\alpha<\kappa} |A_\alpha| \le \kappa \cdot \max(\lambda,\mu)^{<\kappa} =$$

$$\max(\kappa, \max(\lambda,\mu)^{<\kappa} \underset{(4)}{\overset{\uparrow}{=}} \max(\lambda,\mu)^{<\kappa}$$

as wanted for (2). **This completes the proof of the Theorem**.

If, in particular, $\lambda \le \mu$ and μ is of the form $\nu^{<\kappa}$, then $\max(\lambda,\mu)^{<\kappa} = \mu^{<\kappa} = (\nu^{<\kappa})^{<\kappa} = \sup_{\rho<\kappa} (\nu^{<\kappa})^{\rho} \underset{(5)}{\overset{\uparrow}{=}} \sup_{\rho<\kappa} \nu^{<\kappa} = \nu^{<\kappa} = \mu$. Thus, we obtain:

if $|A| = \nu^{<\kappa}$ *for some* $\nu \ge 2$, *and* $|I| \le |A|$ *, then* $|\bar{A}| = |A|$.

Note that for all infinite ν , $\nu^{<\aleph_0} = \sup_{n<\omega} \nu^n = \sup_{n<\omega} \nu = \nu$. Thus, if $\kappa = \aleph_0$, then for all infinite A , we have that $|A| = |A|^{<\kappa}$, and we conclude that

if $\kappa = \aleph_0$, A *is infinite, and* $|I| \le |A|$ *, then* $|\bar{A}| = |A|$.

Let us turn to the examples above. In the example of type (A), we have that $\kappa = \aleph_0$, because all operations are finitary, that is, the arities J_i are finite cardinals. Also, $\lambda=\aleph_0$. Thus, if $(B, \mathcal{F})$ is an infinite group (say), and $A \subset B$ is any infinite subset of B , then $\bar{A}$, the subgroup generated by A , is of the same cardinality as A . If $|B| = \nu$, then for any $\mu \le \nu$, we may choose a subset $A \subset B$ such that $|A| = \mu$ (why?); we thus obtain that

an infinite group has subgroups in each infinite cardinality less than or equal the cardinality of the group.

Of course, the same conclusion holds for Boolean algebras, etc.

In the examples under (B), we have $\kappa = \aleph_1$. Now, $\nu^{<\kappa} = \nu^{\aleph_0}$. Also, now

$\lambda = \max(\aleph_0, 1) = \aleph_0$. With $(\mathbb{R}, \lim)$, we get a trivial conclusion, namely that $|\bar{A}| \le \max(\aleph_0, |A|)^{\aleph_0} \le (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ (trivial since $\bar{A} \subset \mathbb{R}$, and $|\mathbb{R}| = 2^{\aleph_0}$). But, with $(\mathbb{R}^{\mathbb{R}}, \lim)$ we obtain, similarly, that $A \subset \mathbb{R}^{\mathbb{R}}$, $|A| \le 2^{\aleph_0}$ imply that $|\bar{A}| \le 2^{\aleph_0}$ as well, which is "non-trivial" now since $|\mathbb{R}^{\mathbb{R}}| = 2^{2^{\aleph_0}}$ (for the cardinalities related to $\mathbb{R}$, see later ...)

Here is another example. Let us fix a positive integer n. The class (in fact, set) $\mathcal{B}$ of *Borel* sub*sets* of $\mathbb{R}^n$ is the least subset X of $\mathcal{P}(\mathbb{R}^n)$ which contains all open sets, and which is closed under countable union and complementation:

whenever $A_i \in X$ for all $i \in \omega$, then $\bigcup_{i \in I} A_i \in X$;

$A \in X \Longrightarrow \mathbb{R}^n - A \in X$

(Borel sets are important in measure theory). We have that $|\mathcal{B}| = 2^{\aleph_0}$: the number of Borel sets is $2^{\aleph_0}$. (It immediately follows that there are non-Borel sets; why?) This result comes out of the above general theorem in the following way. Let A be the set of all open subsets of $\mathbb{R}^n$. Let $B = \mathbb{R}^n$, $I = \{0, 1\}$; $J_0 = \omega$; $J_1 = 1$; $f_0 : B^\omega \to B$ is the operation of countable union: $f_0(\langle A_i \rangle_{i \in \omega}) = \bigcup_{i \in I} A_i$; $f_1 : B \to B$ is complementation: $f_1(A) = B - A$. Then it is clear that $\mathcal{B}$ is nothing but $\bar{A}$, $\mathcal{B} = \bar{A}$, the closure of A in the given system of operations.

Clearly, $\lambda = \aleph_0$. We know that $|A| = 2^{\aleph_0}$; thus, $\mu = 2^{\aleph_0}$. $\kappa = \aleph_1$, the least regular cardinal above $|J_0| = \aleph_0$. Therefore,

$$|\mathcal{B}| \le \max(\aleph_0, 2^{\aleph_0})^{<\aleph_1} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.$$

The most obvious feature of the theorem is that in the estimate for the cardinality of the closure $\bar{A}$, the cardinality of the set B does not play any role. And indeed, B can be of any size, in fact, even a proper class. The version of the theorem when B is a (possibly proper) class, rather than just a set, is almost the same as the original; the definition of the class B^J remains the same; now, each f_i is possibly a proper class; the only change is that now we cannot talk about the collection of the operations f_i $(i\in I)$ in the same sense as before; but we take a class $\mathcal{F}$ whose elements are of the form (i, a) for $i\in I$, with I a set, and stipulate that $f_i \overset{=}{\text{def}} \{a: (i, a)\in\mathcal{F}\}$ is a Function whose domain is a subclass of B^{J_i}. The proof of the theorem remains esssentially the same.

Finally, we refine the distinction between regular and singular cardinals by introducing the *cofinality* of a cardinal κ; as it will turn out, a cardinal is regular if and only if its cofinality is itself.

We formulate the basic notion in terms of ordinals. For every ordinal α, we have that $\alpha=\operatorname{lsub}_{\beta<\alpha}\beta$; α is the least strict upper bound of all ordinals less than α. It may happen that α can be written in the form

$$\alpha = \operatorname{lsub}_{\xi<\gamma}\beta_\xi \tag{7}$$

for a system of ordinals β_ξ indexed by an ordinal γ *smaller* than α; the least ordinal γ for which there is a representation of the form (1) of α is called the *cofinality* of α, and it is denoted $\mathrm{cf}(\alpha)$; we have just seen that $\mathrm{cf}(\alpha)\le\alpha$.

When $\alpha=\beta+1$ is a successor ordinal, then we can take $\gamma=1$ and $\beta_0=\beta$; this shows that $\mathrm{cf}(\alpha)=1$; the cofinality of all successor ordinals is 1.

Let us note that if (7) holds, then we can change the β_ξ to some β'_ξ such that the sequence $\langle\beta'_\xi\rangle_{\xi<\gamma}$ is (not necessarily strictly) increasing: $\xi<\xi'<\gamma$ implies that $\beta'_\xi\le\beta'_{\xi'}$; and (7) still holds, with some $\gamma'\le\gamma$:

$$\alpha = \operatorname{lsub}_{\xi<\gamma'}\beta'_\xi\ . \tag{8}$$

To this end, we define $\beta'_\xi=\operatorname{lub}_{\zeta\le\xi}\beta_\zeta$ for all $\xi<\gamma$. It is clear that the β'_ξ are increasing, and also that $\beta'_\xi \leq \alpha$ for all $\xi < \gamma$. If in fact $\beta'_\xi < \alpha$ for all $\xi < \gamma$, then

$$\alpha = \underset{\xi<\gamma}{\mathtt{lsub}}\, \beta_\xi \leq \underset{\xi<\gamma}{\mathtt{lsub}}\, \beta'_\xi \leq \alpha$$

and so (8) holds with $\gamma' = \gamma$. If, on the other hand, there is $\xi < \gamma$ such that $\beta'_\xi = \alpha$, then we let γ' be the least such ξ; then $\beta'_\xi < \alpha$ for each $\xi < \gamma'$, and thus

$$\alpha = \underset{\xi<\gamma'}{\mathtt{lub}}\, \beta_\xi \leq \underset{\xi<\gamma'}{\mathtt{lsub}}\, \beta'_\xi \leq \alpha,$$

and (8) again holds. Note that if, in addition to the above, $\gamma = \mathtt{cf}(\alpha)$, then γ' is necessarily equal to γ, since $\gamma' < \gamma$ and (8) would contradict the "least-ness" of γ. Thus, if $\gamma = \mathtt{cf}(\alpha)$, the we have a representation (7) of α in which the β_ξ are increasing.

The characterization (1') of regular cardinals shows that

(9) *for a limit ordinal* α, $\mathtt{cf}(\alpha) = \alpha$ *implies that* α *is a cardinal, in fact a regular cardinal.*

In fact, more generally,

(10) *for a limit ordinal* α, $\mathtt{cf}(\alpha)$ *is always a regular cardinal.*

This follows from the identity

$$\mathtt{cf}(\mathtt{cf}(\alpha)) = \mathtt{cf}(\alpha) \tag{11}$$

that we prove as follows. Let $\gamma = \mathtt{cf}(\alpha)$ and $\delta = \mathtt{cf}(\gamma)$. We have equalities

$$\alpha = \underset{\xi<\gamma}{\mathtt{lsub}}(\beta_\xi) \qquad \text{and} \qquad \gamma = \underset{\zeta<\delta}{\mathtt{lsub}}(\gamma_\zeta)\ ;$$

and as we said, we may assume that $\beta_\xi \leq \beta_{\xi'}$ whenever $\xi < \xi' < \gamma$. But then

$$\alpha = \underset{\zeta<\delta}{\mathtt{lsub}}(\beta_{\gamma_\zeta}) :$$

indeed, clearly, α is a strict upper bound of all the β_{γ_ζ} ($\zeta<\delta$); and if $\alpha'<\alpha$, then there is $\xi<\gamma$ such that $\alpha'\leq\beta_\xi$ (why?); and then there is $\zeta<\delta$ such that $\xi\leq\gamma_\zeta$, thus $\beta_\xi\leq\beta_{\gamma_\zeta}$, and so $\alpha'\leq\beta_{\gamma_\zeta}$ -- which shows that α is a the *least* strict upper bound of all the β_{γ_ζ} ($\zeta<\delta$). But then, by the definition of $\gamma=\mathrm{cf}(\alpha)$, it follows that $\gamma\leq\delta$; that is, (9) holds.

Now, (10) follows from (11) and (9).

Exercise: Let $\gamma=\mathrm{cf}(\alpha)$. Then there exists a representation of α in the form of (7) in which the β_ξ are *strictly* increasing: $\beta_\xi<\beta_{\xi'}$, whenever $\xi<\xi'<\gamma$.

We have defined the cofinality of any ordinal, and in particular, of any cardinal; and we saw that it is always a regular cardinal. For instance, if α is a countable limit ordinal, then $\mathrm{cf}(\alpha)=\omega$; this is because $\mathrm{cf}(\alpha)\leq\alpha$, and $\mathrm{cf}(\alpha)$ is an infinite cardinal, and there is only one such, namely $\omega=\aleph_0$. What we have thus seen is that every countable limit ordinal α can be written in the form $\alpha=\mathrm{lsub}_{n<\omega}\,\alpha_n$, and we may also assume that the α_n here are strictly increasing.

For an ordinal α with $|\alpha|=\aleph_1$, the possibilities for $\mathrm{cf}(\alpha)$ are three: it can be 1 (when α is a successor), it can be $\aleph_0$, and it can be $\aleph_1$. Obviously, $\mathrm{cf}(\omega_1+1)=1$, $\mathrm{cf}(\omega_1+\omega)=\aleph_0$ and $\mathrm{cf}(\omega_1+\omega_1)=\aleph_1$ (we may write ω_1 for $\aleph_1$ when "we use it as an ordinal"; similarly for ω_α).

Exercise: show that if α is a limit ordinal, then $\mathrm{cf}(\aleph_\alpha)=\mathrm{cf}(\alpha)$.

Thus, we have $\mathrm{cf}(\aleph_\omega)=\mathrm{cf}(\omega)=\aleph_0$, $\mathrm{cf}(\aleph_{\omega_1})=\mathrm{cf}(\omega_1)=\aleph_1$, and $\mathrm{cf}(\aleph_{\aleph_{\omega_1}})=\mathrm{cf}(\aleph_{\omega_1})=\aleph_1$.

§12 Models of the axioms of set theory

Let us list the axioms of set theory (X, Y, ... are variables for classes, x, y, ... are variables for sets):

Every set is a class.

Every element of a class is a set.

Axiom of Extensionality: $\forall x(x\in X \longleftrightarrow x\in Y) \longrightarrow X=Y$.

Axiom Schema of Class Comprehension: for any meaningful property $P(x)$ of sets x, there is a (unique) class X such that $\forall x(x\in X \longleftrightarrow P(x))$. X is denoted by $\{x:P(x)\}$.

Axiom of Regularity: all sets are pure: $\forall x\forall X([\forall y(y\subset X\rightarrow y\in X)]\longrightarrow x\in X)$.

Axiom of Empty Set (AES)**:** the class $\emptyset \overset{=}{\text{def}} \{x:x\neq x\}$ is a set.

Axiom of Pair Set (APS)**:** for any sets x, y, the class $\{x, y\} \overset{=}{\text{def}} \{z:z=x \text{ or } z=y\}$ is a set.

Axiom of Subset (AS)**:** any subclass of a set is a set: $X\subset x$ implies that X is a set.

Axiom of Union Set (AUS)**:** for any set x, $\bigcup x \overset{=}{\text{def}} \{y:\exists z.y\in z \ \& \ z\in x\}$ is a set.

Axiom of Power Set (APoS)**:** for any set x, $\mathcal{P}(x) \overset{=}{\text{def}} \{y: y\subset x\}$ is a set.

Axiom of Replacement (AR)**:** for any Function F, if $\mathrm{Dom}(F)$ is a set, so is $\mathrm{Range}(F)$.

Axiom of Infinity (AI)**:** the class

$\mathbb{N} \overset{=}{\text{def}} \{x:\forall X([\emptyset\in X \ \& \ \forall y(y\in X\rightarrow \mathrm{S}(y)\in X)]\longrightarrow x\in X\}$ is a set [here, $\mathrm{S}(x) = \bigcup \{x, \{x\}\}$].

(Global) Axiom of Choice ((G)AC)**:** there is a Function C with $\mathrm{Dom}(C)=\mathbf{V}-\{\emptyset\}$ such that for all $x\in\mathrm{Dom}(C)$, $C(x)\in x$.

The axiom system we described above is called the *Morse-Kelley (MK) class-set theory*. To be sure, our formulation is, in one respect, less precise than the official version of MK: the term "meaningful" in the formulation of class-comprehension should be made more specific to read "formulated in first-order logic"; we will say more about this below. On the other hand, there are two other, related, formal axiomatizations of set theory: *Zermelo-Fraenkel (ZF) set theory*, and *Godel-Bernays (GB) class-set theory*. As these terms indicate, in ZF, the only kind of entity is "set"; in GB, as in MK, we have both sets and classes. The formulations of both ZF and GB depend more sensitively on the concepts of first order logic than that of MK. Although MK and GB may seem related, on account of both being a theory of sets *and* classes, and ZF may seem separate since it only talks (directly) about sets, as a matter of fact, ZF and GB are very closely related, being essentially of the same deductive power, whereas MK is *stronger* than the previous two.

Henceforth, we assume the axioms of (class-)set theory, in the formulation given above (that is, MK).

We want to investigate what subclasses, and preferably, sub*sets*, of $\mathbf{V}$, the universe of (all) pure sets, may serve as *models of* all or part of the axioms of set theory. We take a subclass A of $\mathbf{V}$; the intention is that A should play the role of $\mathbf{V}$. We re-interpret the notion of "set" as to mean: being an element of A. To be a class should mean: being a subclass of A. Now, the very first axiom: "every set is a class", requires that $\forall x(x\in A\Longrightarrow x\subset A)$, which is the condition that we called the *transitivity* of A.

Henceforth, we assume that A is a transitive class.

For the "new" classes, the classes in the sense of the model given by A, one being an element of the other should retain its original meaning, the original $\in$-relation.

In summary, what we investigate is this. Given a transitive class A, and given certain axioms of set theory, we ask whether it is the case that, under the interpretation in which " a is a set " means " $a\in A$ ", " a is a class" means " $a\subset A$ ", and " a belongs to b " means " $a\in b$ ", the axioms are true. The model we are looking at consists of the *subclasses* of A; the elements of A, the "sets" in the model, are particular subclasses of A. Furthermore, we want to investigate the said questions *within* our (MK) class-set theory, that is, using those axioms freely in the total universe of sets and classes.

Let us write $\mathcal{SC}(A)$ for the totality of all subclasses of A; the "*model*" we are talking has $\mathcal{SC}(A)$ as *its domain of individuals*. To be sure, when A is a proper class, $\mathcal{SC}(A)$ is *not* a class ; its "elements" are sometimes proper classes. Thus, it is not possible to talk directly about $\mathcal{SC}(A)$ within our class-set theory. Still, there is no problem with the meaning, *within MK class-set theory*, of saying, about any particular statement referring to classes and sets, that it is *true* in the model $\mathcal{SC}(A)$. When the statement contains a quantifier $\forall X$ on classes, the understanding is that this should be interpreted as $\forall X(X\subset A\longrightarrow ...)$; and the quantifier $\exists X$ should be read, when meant in the sense of the model provided by A, as $\exists X(X\subset A\ \&\ ...)$. Note that $\forall X(X\in\mathcal{SC}(A)\longrightarrow ...)$ is intuitively the same as $\forall X(X\subset A\longrightarrow ...)$, since $X\in\mathcal{SC}(A)$ is intuitively the same as $X\subset A$, even though $X\in\mathcal{SC}(A)$ is not something meaningful within MK.

On the other hand, when A is a *set* (and this is in fact the case we are mostly interested in), then there is no problem of the kind described above at all. Now, $\mathcal{SC}(A)$ is the same as the set $\mathcal{P}(A)$; and we can write $X\in\mathcal{P}(A)$ just as well as $X\subset A$.

If a particular axiom is true in the model, we say the class $\mathcal{SC}(A)$ *is a model* ***of*** the axiom; if so, we write $\mathcal{SC}(A)\models\Phi$ when Φ stands for the axiom.

The first few observations tell us that, under the stated (meager) conditions, $\mathcal{SC}(A)$ is automatically a model of the axioms "Every element of a class is a set", Extensionality, Class Comprehension, and Regularity.

The first, "Every element of a class is a set", when interpreted in the model $\mathcal{SC}(A)$, says that "every element of a subclass of X is an element of X", which is certainly true.

The second, Extensionality, says, in $\mathcal{SC}(A)$, that

(1) "if X, Y are subclasses of A, and if for all $x\in A$, $x\in X$ iff $x\in Y$, then $X=Y$".

This holds since each of $x\in X\subset A$ or $x\in Y\subset A$ implies that $x\in A$, and thus "if for all $x \in A$, $x \in X$ iff $x \in Y$ "

is the same as

"if for all x , $x \in X$ iff $x \in Y$ ";

and so, (1) becomes an instance of Extensionality understood in the original universe of sets and classes.

The third, Class Comprehension, says that

for any meaningful property $P(x)$ of sets x *in* A , there is a (unique) subclass X of A such that $\forall x \in A . (x \in X \longleftrightarrow P(x))$.

But, given P , we can consider the property Q of sets in general defined by

$$Q(x) \iff P(x) \ \& \ x \in A \ .$$

Consider

$$X \underset{\text{def}}{=} \{x : Q(x)\} = \{x : x \in A \ \& \ P(x)\} = \{x \in A : P(x)\}$$

given by Class Comprehension in the original universe. By its definition, $X \subset A$, therefore, X is a "class in the sense of the model A ". But also,

$$x \in A \implies (x \in X \longleftrightarrow P(x)) ,$$

which shows what we want.

This simple proof is, alas, misleading. The root of the trouble is that we do not really have a clear idea what a "meaningful property" is. Later on, we will learn about *first order logic* in explicit detail, and interpret "meaningful property" by "a property expressible by means of first order logic in terms of the concepts of set, class and membership". In fact, although we do not yet have theoretical definitions concerning first order logic, we do have a working knowledge about it. When interpreting Class Comprehension in the model $\mathcal{SC}(A)$, we will encounter the situation illustrated by the following example.

Consider the property $P(x) \equiv$ " x is a natural number" . This is given by

$$P(x) \equiv \forall X([\emptyset \in X \,\&\, \forall y(y \in X \longrightarrow S(y) \in X)] \longrightarrow x \in X) .$$

P is not quite expressed in terms of the primitives "set", "class", and "membership"; but this can be easily remedied, by using

$$u = \emptyset \iff \forall v.\, v \notin u$$

and

$$z = S(y) \iff \forall w(w \in z \longleftrightarrow (w \in y \text{ or } w = y)) .$$

The phrase $\emptyset \in X$ can therefore be replaced by $\exists u((\forall v.\, v \notin u) \,\&\, u \in X)$:

$$\emptyset \in X \iff \exists u((\forall v.\, v \notin u) \,\&\, u \in X) ;$$

similarly,

$$S(y) \in X \iff \exists z\underset{4}{(}\forall w(w \in z \longleftrightarrow (w \in y \text{ or } w = y)) \,\&\, z \in X\underset{4}{)} .$$

Using these definitions, we have

$$P(x) \equiv \forall X\underset{1}{(}\underset{2}{[}\exists u((\forall v.\, v \notin u) \,\&\, u \in X) \,\&$$

$$\forall y\underset{3}{(}y \in X \longrightarrow \exists z\underset{4}{(}\forall w(w \in z \longleftrightarrow (w \in y \text{ or } w = y)) \,\&\, z \in X\underset{4}{)}\underset{3}{)}\underset{2}{]} \longrightarrow x \in X\underset{1}{)} . \qquad (1')$$

P now *is* expressed in terms of the said primitives. Now, consider the case that we want to take

the class of natural numbers in the model A .

This may be said to be given by the property $Q(x) \equiv x \in A \,\&\, P(x)$, the one we used above.

But there is another interpretation, one that is the really intended one. Namely, what we really want is, more explicitly,

the class of sets in A *that are natural numbers in sense of the model* A .

How can we describe the property

$R(x) \equiv x$ is a natural number *in the sense of* the model A ? (2)

This has a natural answer: interpret each of the quantifiers

$$\forall X, \exists u, \forall v, \forall y, \exists z, \forall w$$

as quantification over ""class" in the sense of the model $\mathcal{SC}(A)$ " (in the case of $\forall X$), and ""set" in the sense of the model $\mathcal{SC}(A)$ " (in the case of the rest). The result is that we should replace the above quantifiers by the respective expressions

$$\forall X{\subset}A,\ \exists u{\in}A,\ \forall v{\in}A,\ \forall y{\in}A,\ \exists z{\in}A,\ \forall w{\in}A\ .$$

Of course, writing

$\forall X{\subset}A$ means $\forall X.X{\subset}A \longrightarrow$ ...;
$\exists u{\in}A$ means $\exists u.u{\in}A$ & ...;
$\forall v{\in}A$ means $\forall v.v{\in}A \longrightarrow$ ... ;

etc. The result of the said replacement is the rather large expression

$$R(x) \equiv \forall X{\subset}A\underset{1}{(}\underset{2}{[}\exists u{\in}A((\forall v{\in}A.v{\notin}u)\ \&\ u{\in}X)$$

$$\&\ \forall y{\in}A\underset{3}{(}y{\in}X \longrightarrow \exists z{\in}A\underset{4}{(}\forall w{\in}A(w{\in}z \longleftrightarrow (w{\in}y \text{ or } w{=}y))\ \&\ z{\in}X\underset{4}{)}\underset{3}{)}\underset{2}{]} \longrightarrow x{\in}X\underset{1}{)} \qquad (3)$$

Let us write $P^{(A)}(x)$ (although we should really write $P^{(\mathcal{SC}(A))}(x)$ ...) for this property $R(x)$, and call it the *relativization of* P *to* the model $\mathcal{SC}(A)$.

Unfortunately, the notation $P^{(A)}(x)$ is, still, slightly misleading; $P^{(A)}(x)$ is given only when the *expression* for P in first order logic is given; what if the same property P can be given by two different expressions which, when undergoing relativization, give *non-equivalent* expressions for the relativizations? When dealing with logic in earnest, we will have to face this problem; for the time being, we ignore it.

We have concluded that the property R in (2) is $P^{(A)}(x)$. Thus, when we want to see that, *in the sense of the model* $\mathcal{SC}(A)$, the class of natural numbers exists, what we have to take is the class

$$\{x\in A: P^{(A)}(x)\}$$

rather than

$$\{x\in A: P(x)\} .$$

Of course, there is no problem with the existence of the class in question *in* the model $\mathcal{SC}(A)$; the said expression denotes, again, a subclass of A .

Finally, the validity of the fourth of the four axioms mentioned above, Regularity, is left as an **exercise** to verify.

Next, we discuss the meaning of each of the next seven axioms: those of Empty Set, Pair Set, Subset, Union Set, Power Set, Replacement and Infinity in the full model based on the (transitive) class A . Let us call these, to have a simple term, the *set-existence* axioms.

Note that each set-existence axiom asserts that a certain class, formed as a comprehension-term on the basis of some data satisfying some conditions, is a set. In other words, in each case, we have a property $P(w)$, a property P for an undetermined set w , P formulated in terms of some given data; the assertion is that $\{w:P(w)\}$ is a set. To be sure, P is not always formulated using only the primitives of set theory ("class", "set" and "membership"), but we can remedy this as was illustrated above. Now, according to what we said above, the corresponding set-existence axiom, " $\{w:P(w)\}$ is a set", when interpreted *in* *the sense of the model* $\mathcal{SC}(A)$, is to mean that, *provided the data are in the model* A , we have that

$$\{w{\in}A\colon P^{(A)}(w)\} \text{ is an element of } A\,, \tag{4}$$

where we referred to the relativization $P^{(A)}(w)$ of P described above. We are to see what this last statement (4) means, for each of the seven set-existence axioms. In many cases, we will establish that

(5) for all $w{\in}A$, $P(w) \Longleftrightarrow P^{(A)}(w)$.

When (5) holds, we say that the property P is *absolute* (*with respect to relativization to* A). Under the condition that P is absolute, (4) becomes

(6) $\{w{\in}A\colon P(w)\} \in A$.

A quality of P that is *stronger* than (5), but which, nevertheless, is often present, is that

(7) for *all* $w{\in}\mathbf{V}$, $P(w) \Longleftrightarrow P^{(A)}(w)$.

In this case,

(8) $\{w{\in}A\colon P^{(A)}(w)\} = \{w\colon P(w)\}$,

and (4) becomes

(9) $\{w\colon P(w)\} \in A$.

Now, to the individual set-existence axioms.

AES : now, $P(x)$ is $x{\neq}x$. This contains no quantifiers; thus $P^{(A)}(x)$ is just the same as $P(x)$. Therefore, (7) trivially holds. Hence, to say that $\mathcal{SC}(A){\models}$AES is to say that $\{w\colon P(w)\}$, that is, $\emptyset$, is an element of A . Since A is transitive, it is easy to see that this always holds provided A is non-empty. We summarize:

$$\mathcal{SC}(A)\vDash\text{AES} \iff \emptyset \in A\ .$$

APS : we are given $x, y\in A$; the property $P(z)$ is $z=x$ or $z=y$. The same argument applies as in the previous case to show that (7) holds. Thus, $\mathcal{SC}(A)\vDash$APS means that, under the conditions that $x, y\in A$, (9) holds. The conclusion is that

$$\mathcal{SC}(A)\vDash\text{APS} \iff A \text{ is closed under taking pair-sets:}$$
$$x, y\in A \Longrightarrow \{x, y\}\in A\ .$$

AS : we are given a set $x\in A$, and a class $X\subset A$ such that $X\subset x$; the property $P(w)$ is $w\in X$; note that " X is a set" is the same as " $\{w: w\in X\}$ is a set". The same argument applies as in the previous two cases to show that (7) holds. Conclusion:

$$\mathcal{SC}(A)\vDash\text{AS} \iff A \text{ is closed under taking subsets of its elements:}$$
$$x\in A\ ,\ y\subset x \Longrightarrow y\in A\ .$$

AUS : we are given $x\in A$; $P(y)$ is $\exists z.y\in z\ \&\ z\in x$. Therefore, $P^{(A)}(y)$ is $\exists z\in A.y\in z\ \&\ z\in x$. I claim that (7) holds. The fact that $P^{(A)}(y)$ implies $P(y)$ is tautologous. Given that $P(y)$, that is, $y\in z\ \&\ z\in x$ for a suitable z , since $x\in A$ and A is transitive, we infer that $z\in A$; this shows $P^{(A)}(y)$. In conclusion:

$$\mathcal{SC}(A)\vDash\text{AUS} \iff A \text{ is closed under taking union-sets:}$$
$$x\in A \Longrightarrow \bigcup x \in A\ .$$

APoS : We are given $x\in A$; $P(y)$ is $y\subset x \equiv \forall z.(z\in y\longrightarrow z\in x)$; $P^{(A)}(y)$ is $\forall z\in A.(z\in y\longrightarrow z\in x)$. For any $y\in A$, $P(y)$ if and only if $P^{(A)}(y)$; this is immediate from the transitivity of A . We have shown (5), the fact that the subset-relation is absolute. (Note, however, that we have *not* shown (7).) The term in (6),

$\{y \in A: P(y)\} = \{y \in A: y \subset x)\} = \mathcal{P}(x) \cap A$. We have

$$\mathcal{SC}(A) \models \text{APS} \iff x \in A \Longrightarrow \mathcal{P}(x) \cap A \in A .$$

Note, on the other hand, that under the condition that $A \models$AS , that is, $x \in A$, $y \subset x \Longrightarrow y \in A$, we have that $\mathcal{P}(x) \cap A = \mathcal{P}(x)$; thus,

if $\mathcal{SC}(A) \models$AS , then
$\mathcal{SC}(A) \models$APS $\iff$ A is closed under taking the power-set of its elements;
$$x \in A \Longrightarrow \mathcal{P}(x) \in A .$$

Also note that if $\mathcal{P}(x) \in A$, then $\mathcal{P}(x) \subset A$ (transitivity), hence, $\mathcal{P}(x) \cap A = \mathcal{P}(x)$;

if for all $x \in A$, $\mathcal{P}(x) \in A$, then $\mathcal{SC}(A) \models$APS .

AR : As a preliminary remark, let us note that for $F \subset A$, that is, for F a class in the model, to say that F is a Function in the sense of the model is the same as to say that F is a Function: the concept of Function is absolute. This will be seen by inspecting the equivalence

F is a Function $\iff$

$$\forall x \in F . \exists y \in x . \exists z \in x . \underset{1}{[} \forall w \in x (w=y \text{ or } w=z))$$
$$\& \exists u \in y \underset{2}{[} (\forall t \in y . t=u) \ \& \ u \in z \ \& \ \exists v \in z . \forall s \in z . (s=u \text{ or } s=v) \underset{2}{]} \underset{1}{]}$$

&

$$\forall x \in F . \forall y \in x . \forall z \in x \underset{1}{[} \forall w \in x (w=y \text{ or } w=z)) \longrightarrow$$
$$\forall u \in y \underset{2}{[} \forall t \in y . t=u \longrightarrow \underset{3}{[} \forall v \in z \underset{4}{[} \forall s \in z . (s=u \text{ or } s=v) \longrightarrow$$
$$\forall x' \in F . \forall y' \in x . \forall z' \in x . \underset{5}{[} \forall w' \in x' (w=y' \text{ or } w=z')) \longrightarrow$$
$$\underset{6}{[} \forall t' \in y' . t'=u \longrightarrow \underset{7}{[} \forall v' \in z' [\forall s' \in z' (s'=u \text{ or } s'=v') \longrightarrow v=v' \underset{7}{]}\underset{6}{]}\underset{5}{]}\underset{4}{]}\underset{3}{]}\underset{2}{]}\underset{1}{]}$$

(the first part says that every x in F is an ordered pair $(u, v) = \{y=\{u\}, z=\{u, v\}\}$; the second part says that if $x=(u, v) \in F$ and $x'=(u, v') \in F$, then $v=v'$).

Indeed, since each quantifier in the expression is *bounded*, i.e., of the form $\forall a{\in}b$ or $\exists a{\in}b$, and $F{\subset}A$, the expression, when *relativized* to the transitive class A, will not change its meaning: $\forall x{\in}F.Q(x)$ relativized to $\mathcal{SC}(A)$ is $\forall x{\in}A[x{\in}F\longrightarrow Q(x)]$; and

$$\forall x{\in}A[x{\in}F\longrightarrow Q(x)] \equiv \forall x[x{\in}F\longrightarrow Q(x)] \equiv \forall x{\in}F.Q(x)$$

since $F{\subset}A$; similarly, $\forall y{\in}x.Q(x,y)$ relativized to $\mathcal{SC}(A)$ is $\forall y{\in}A[y{\in}x\longrightarrow Q(x,y)]$, and

$$\forall y{\in}A[y{\in}x\longrightarrow Q(x,y)] \equiv \forall y[y{\in}x\longrightarrow Q(x,y)] \equiv \forall y{\in}x.Q(x,y)$$

since $x{\in}A$ and so $x{\subset}A$.

Next, consider the fact that

$$\begin{aligned} v{\in}\mathrm{Range}(F) \iff{}& \exists x{\in}F.\exists y{\in}x.\exists z{\in}x.[_1(\forall w{\in}x(w{=}y \text{ or } w{=}z))\\ &\& \exists u{\in}y[_2(\forall t{\in}y.t{=}u) \,\&\, u{\in}z \,\&\, \forall s{\in}z.(s{=}u \text{ or } s{=}v)]_2]_1 \end{aligned}$$

Writing $P(v)$ for the right-hand-side, we see that (7), the stronger version of absoluteness holds. We may write that $\mathrm{Range}^{(A)}(F)$, the Range of F in the sense of the model $\mathcal{SC}(A)$, is $\mathrm{Range}(F)$; $\mathrm{Range}^{(A)}(F) = \mathrm{Range}(F)$. A similar inspection will show that $\mathrm{Dom}^{(A)}(F) = \mathrm{Dom}(F)$. Now, AR in the model means that

$$F{\subset}A \;\&\; F \text{ is a Function}^{(A)} \;\&\; \mathrm{Dom}^{(A)}(F){\in}A \implies \mathrm{Range}^{(A)}(F){\in}A\,.$$

Under the hypothesis that $F{\subset}A$, we can drop the superscript A from all three positions, and we obtain that

$\mathcal{SC}(A){\models}\text{AR}$ if and only if
for all classes $F{\subset}A$,
F is a Function & $\mathrm{Dom}(F){\in}A \implies \mathrm{Range}(F){\in}A$.

If, in addition, $\mathcal{SC}(A){\models}\text{APS}$, that is, A is closed under taking pair-sets, then we have that

$x, y \in A$ imply that $(x, y) \in A$, and therefore, $X \subset A$, $Y \subset A$ imply that $X \times Y \subset A$. Thus, if F is a Function, and $\mathrm{Dom}(F) \subset A$, $\mathrm{Range}(F) \subset A$, then $F \subset \mathrm{Dom}(F) \times \mathrm{Range}(F) \subset A$. This shows that in the last equivalence, we can drop the condition "$F \subset A$", and we obtain:

if $\mathcal{SC}(A) \vDash$APS, then
$\mathcal{SC}(A) \vDash$AR if and only if
for all classes F,
F is a Function & $\mathrm{Dom}(F) \in A \implies \mathrm{Range}(F) \in A$.

AI : We will indicate the proof of the fact that

if $\mathcal{SC}(A) \vDash$AES, APS and AUS, then $\mathcal{SC}(A) \vDash$AI iff $\mathbb{N} \in A$.

Assume that $\mathcal{SC}(A) \vDash$AES, APS and AUS. According to what we showed above, this means that $\emptyset \in A$, and $x, y \in A$ implies that $\{x, y\}, \bigcup x \in A$.

We first claim that we have $\mathbb{N} \subset A$. In fact, we have the stronger statement that $\mathbf{V}_\omega \subset A$. This is left as an **exercise** to prove.

We show that $\mathbb{N}^{(A)} = \mathbb{N}$; that is, for the property $P(x)$ that defines the class $\mathbb{N}$ as $\{x : P(x)\}$, we have the relation (7), and as a consequence, (8). Recall that $P(x)$ was described in (1'), and $P^{(A)}(x)$ in (3).

The proof of the inclusion $\mathbb{N} \subset \mathbb{N}^{(A)}$, that is, $x \in A$ & $P(x) \implies P^{(A)}(x)$, is left as an **exercise**. For the converse inclusion $\mathbb{N}^{(A)} \subset \mathbb{N}$, the main point is that X in the quantifier $\forall X \subset A$ in (3) can be instantiated by $X = \mathbb{N}$, since $\mathbb{N} \subset A$; therefore, if $x \in A$ is such that $x \in \mathbb{N}^{(A)}$, that is, $P^{(A)}(x)$, then the statement $(\ldots)_1$ after the quantifier $\forall X \subset A$ in (3) is true for $X = \mathbb{N}$; the facts $\emptyset \in A$ and $y \in A \Rightarrow \mathrm{S}(y) \in A$ can be applied to show that $[\ldots]_2$ in (3) true (this expresses that $X = \mathbb{N}$ satisfies $\emptyset \in X$ & $\forall y (y \in X \longrightarrow \mathrm{S}(y) \in X)$ in the sense of $\mathcal{SC}(A)$); now, $x \in \mathbb{N}$ follows; this shows $\mathbb{N}^{(A)} \subset \mathbb{N}$.

Finally, we turn to the axiom of choice.

GAC : We show that if $\mathcal{SC}(A)\models$APS , then $\mathcal{SC}(A)\models$GAC (of course, here it is assumed that A is a transitive class, and that GAC holds in the universe). In fact, if C is a *global choice function*, one whose existence is asserted by GAC as read in the universe, then $C\cap A$ is a global choice function in the model $\mathcal{SC}(A)$. Indeed, by the concept of Function being absolute (see the discussion of AR above), we have that $C\cap A$ is a Function in the sense of $\mathcal{SC}(A)$. By two uses of the fact that $\mathcal{SC}(A)\models$APS , it follows that $a\in A-\{\emptyset\}$ and $b\in a$ implies that $(a,b)\in A$. Hence, it is immediate that $\mathrm{Dom}^{(A)}(C\cap A)=\mathrm{Dom}(C\cap A)=A-\{\emptyset\}$. Of course, when $a\in\mathrm{Dom}(C\cap A)$, then $(C\cap A)(a)=C(a)\in a$. This proves what we claimed.

Let us draw some conclusions from our analyses. We see that the condition for each of the seven set-existence axioms is related to a *closure condition*. Let us elaborate.

Let us call a predicate $P(x,X)$ of a set-variable x and a class-variable X a *closure-predicate* if it is monotone in its second variable:

for all x , X and Y ,

$$P(x,X)\ \&\ X\subset Y \implies P(x,Y)\ .$$

A class X is said to *satisfy the closure condition associated with* the closure predicate $P=P(\cdot,\cdot)$, or, X *is closed under* P , or again, X *is* P*-closed*, if

for all x ,

$$P(x,X) \implies x\in X\ .$$

The main fact is that, for a closure-predicate P ,

the intersection of any number of classes closed under P is again closed under P ; if $Q(X)$ is a predicate on classes, and

$$\forall X(Q(X) \longrightarrow X \text{ is } P\text{-closed}) \implies \{x: \forall X(Q(X) \longrightarrow x \in X)\} \text{ is } P\text{-closed}\}.$$

The proof of this is easy, but very important; it is left as an **exercise**. (At this point, I mention that the present formulation of the notion of "closure predicate", with the last-stated property, was given by Peter Green.)

In particular, the intersection of *all* P-closed classes is P-closed.

Let us note the easily seen fact that if P_1 and P_2 are closure-predicates, then $P(x, X)$ defined as $P(x, X) \equiv P_1(x, X) \,\&\, P_2(x, X)$ is again a closure-predicate.

Now, let us define

$$\begin{aligned}
P_{AE}(x, X) &\equiv x=\emptyset \\
P_{APS}(x, X) &\equiv \exists y \in X. \exists z \in X.\ x=\{y, z\} \\
P_{AUS}(x, X) &\equiv \exists y \in X.\ x= \bigcup y \\
P_{AS}(x, X) &\equiv \exists y \in X.\ x \subset y \\
P_{APoS}(x, X) &\equiv \exists y \in X.\ x=\mathcal{P}(y) \\
P_{AR}(x, X) &\equiv \exists F.(F \text{ is a Function } \& \operatorname{Dom}(F) \in X \,\&\, x=\operatorname{Range}(F)) \\
P_{AI}(x, X) &\equiv x=\mathbb{N}\ .
\end{aligned}$$

Inspection shows that each of these predicates is a closure-predicate.

We have

$$\begin{aligned}
&X \text{ is } P_{AES}\text{-closed} \iff \emptyset \in X\ ; \\
&X \text{ is } P_{APS}\text{-closed} \iff \forall y. \forall z(y, z \in X \longrightarrow \{y, z\} \in X)\ ; \\
&X \text{ is } P_{AUS}\text{-closed} \iff \forall y(y \in X \longrightarrow \bigcup y \in X)\ ; \\
&X \text{ is } P_{AS}\text{-closed} \iff \forall y. \forall x((y \in X \,\&\, x \subset y) \longrightarrow x \in X)\ ; \\
&X \text{ is } P_{APoS}\text{-closed} \iff \forall y(y \in X \longrightarrow \mathcal{P}(y) \in X)\ ; \\
&X \text{ is } P_{AR}\text{-closed} \iff \\
&\qquad\qquad \forall F(F \text{ is a Function } \& \operatorname{Dom}(F) \in X \longrightarrow \operatorname{Range}(F) \in X)\ ; \\
&X \text{ is } P_{AI}\text{-closed} \iff \mathbb{N} \in X\ .
\end{aligned}$$

Above, we saw that, for any transitive class A, being P_{AES}-closed is equivalent to $\mathcal{SC}(A) \models \text{AES}$; similarly for the axioms APS, AUS and AS, in relation with the closure-predicates P_{APS}, P_{AUS} and P_{AS}, respectively. The situation with APoS is different; the condition on X is

$$\forall y(y \in X \longrightarrow \mathcal{P}(y) \cap X \in X)$$

is *not* a closure condition (**exercise**). However, we saw that, under the condition that A is P_{AS}-closed, $\mathcal{SC}(A) \vDash \text{APS}$ iff A is P_{APS}-closed.

The rest of the axioms are similarly clarified in terms of closure conditions. Always assuming that A is transitive, we have that

If A is P_{APS}-closed, then $\mathcal{SC}(A) \vDash \text{AR}$ iff A is P_{AR}-closed;

if A is P_{AES}- and P_{APS}-closed, then $\mathcal{SC}(A) \vDash \text{AI}$ iff A is P_{AI}-closed.

Note being transitive is also a closure condition:

$$X \text{ is transitive} \iff X \text{ is } P_{TR}\text{-closed}$$

where

$$P_{TR}(x, X) \equiv \exists y(y \in X \,\&\, x \in y) .$$

Let us denote the conjunction of the eight closure-predicates

$$P_{TR}, P_{AES}, P_{APS}, P_{AUS}, P_{AS}, P_{APoS}, P_{AR} \text{ and } P_{AI}$$

by P_{SE}, for "Set Existence" (P_{TR} is taken as a conjunct in P_{SE} too); P_{SE} is the *set-existence* closure predicate.

We conclude the following

Proposition. For any class A , $SC(A)$ is a transitive model of Morse-Kelly class-set theory if and only if A is closed under (the) set-existence (closure-predicate P_{SE}).

It is worth remembering that, on the basis of our analyses, we have similar statements for a variety of combinations of the axioms of set theory.

§13 Inaccessible cardinals

In the last section, we characrerized the classes A that give rise to a model $\mathcal{SC}(A)$ of Morse-Kelley class-set theory by closure under the combined set-existence predicate P_{SE}. Our first main goal in this section is to relate the same condition to the notion of inaccessible cardinal.

(1) Theorem For any class A, $\mathcal{SC}(A)$ is a model of all axioms of MK class-set theory if and only if either $A=\mathbf{V}$ or $A=\mathbf{V}_\theta$ for an inaccessible cardinal θ.

By the last section, an equivalent statement is this:

(1') Theorem For any class A, A is closed under P_{SE} if and only if either $A=\mathbf{V}$ or $A=\mathbf{V}_\theta$ for an inaccessible cardinal θ.

To establish the Theorem in its second form, we prove some preliminary facts.

(2) Lemma. Suppose that α is an ordinal, and for each $\beta<\alpha$, B_β is a set such that $B_\beta\subset B_{\beta'}$ and $|B_\beta| < |B_{\beta'}|$ for all $\beta<\beta'<\alpha$. Then

$$|\bigcup_{\beta<\alpha} B_\beta| = \operatorname{lub}_{\beta<\alpha} |B_\beta| . \qquad (3)$$

Proof. We prove the assertion by transfinite induction on the ordinal α. Suppose that $\alpha\in\mathbf{Ord}$, and for all $\alpha'<\alpha$, the assertion with α' replacing α holds.

If $\alpha=0$, then both sides of the equation are equal to 0.

Assume that $\alpha=\alpha'+1$. This case is also trivial; on both sides, we have "maxima". In more detail: $\operatorname{lub}_{\beta<\alpha} |B_\beta| = |B_{\alpha'}|$; on the other hand, $\bigcup_{\beta<\alpha} B_\beta = B_{\alpha'}$; and so

$$|\bigcup_{\beta<\alpha} B_\beta| = |B_{\alpha'}| = \operatorname{lub}_{\beta<\alpha} |B_\beta| \ .$$

For the rest of the proof, let α be a limit ordinal; let $\langle B_\beta\rangle_{\beta<\alpha}$ be a system of sets as stated in the hypotheses of the Lemma. Let $\kappa_\beta = |B_\beta| \ (\beta<\alpha)$. We have $\kappa_\beta<\kappa_{\beta'}$ for $\beta<\beta'<\alpha$.

Let us define the sets $C_\beta \overset{\mathrm{def}}{=} \bigcup_{\gamma<\beta} B_\gamma$ for each $\beta<\alpha$. Note that $C_0=\emptyset$ and $C_{\beta+1}=B_\beta$ for all $\beta<\alpha$ ($\beta<\alpha$ implies that $\beta+1<\alpha$). Clearly, we have

$$C_\beta \subset C_{\beta'} \ \text{ for } \ \beta<\beta'<\alpha \ . \tag{4'}$$

Another fact is that for a limit ordinal $\beta<\alpha$ (if there is any such) , we have that

$$\bigcup_{\gamma<\beta} C_\gamma = \bigcup_{\gamma+1<\beta} C_{\gamma+1} = \bigcup_{\gamma+1<\beta} B_\gamma = \bigcup_{\gamma<\beta} B_\gamma = C_\beta \ ; \tag{4}$$

For (4') and (4), we say that the system $\langle C_\beta\rangle_{\beta<\alpha}$ is *continuous*.

Now, let β any ordinal $<\alpha$, and let $\lambda_\beta \overset{\mathrm{def}}{=} |C_\beta|$. When $\beta=\gamma+1$, then $\lambda_\beta = |B_\gamma| = \kappa_\gamma$. Now, let $\beta<\alpha$ be limit. Since the restriction of the original system $\langle B_\gamma\rangle_{\gamma<\alpha}$ to ordinals $\gamma<\beta$, the system $\langle B_\gamma\rangle_{\gamma<\beta}$, satisfies the same assumptions as the original, by the induction hypothesis applies and we can conclude that, for a limit ordinal $\beta<\alpha$,

$$\lambda_\beta = |C_\beta| = |\bigcup_{\gamma<\beta} C_\gamma| = \operatorname{lub}_{\gamma<\beta} \kappa_\gamma = \bigcup_{\gamma<\beta} \kappa_\gamma = \bigcup_{\gamma<\beta} \kappa_{\gamma+1} = \bigcup_{\gamma<\beta} \lambda_\gamma \ . \tag{5}$$

On the cardinals λ_β now we have the inequalities $\lambda_\beta<\lambda_{\beta'}$ for $\beta<\beta'<\alpha$, and also the continuity relation (5). Finally, note that

$$\operatorname{lub}_{\beta<\alpha} \lambda_\beta = \operatorname{lub}_{\beta<\alpha} \lambda_{\beta+1} = \operatorname{lub}_{\beta<\alpha} \kappa_\beta \ ,$$

and

$$\bigcup_{\beta<\alpha} C_\beta = \bigcup_{\beta<\alpha} C_{\beta+1} = \bigcup_{\beta<\alpha} B_\beta$$

Therefore, if we can show that

$$?: \qquad |\bigcup_{\beta<\alpha} C_\beta| = \mathrm{lub}_{\beta<\alpha}\, \lambda_\beta\,, \tag{6}$$

the desired equality (3) will follow. What we have done so far was to reduce the general case of the lemma for α a limit ordinal to the case when the system of sets involved is continuous.

Let us extend the definition of the sets C_β and the cardinals λ_β to $\beta=\alpha$ by $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$, and $\lambda_\alpha = \mathrm{lub}_{\beta<\alpha}\, \lambda_\beta = \bigcup_{\beta<\alpha} \lambda_\beta$. Thus, (4) and (5) hold for $\beta=\alpha$ too.

By recursion on $\beta\le\alpha$, we define a bijection $f_\beta : \lambda_\beta \xrightarrow{\cong} C_\beta$, such that, in addition, we also have that for $\beta<\beta'<\alpha$, $f_\beta \subset f_{\beta'}$ (that is, $f_\beta(\xi) = f_{\beta'}(\xi)$ for $\xi\in\mathrm{dom}(f_\beta)$).

The fact that $\lambda_0 = |C_0|$ gives a bijection $f_0 : \lambda_0 \xrightarrow{\cong} C_0$. (In fact, $f_0=\emptyset$.)

Suppose that we have defined f_β, and $\beta+1<\alpha$; we define $f_{\beta+1}$. I claim that $|C_{\beta+1}-C_\beta| = \lambda_{\beta+1}$. Indeed, if $|C_{\beta+1}-C_\beta| = \mu$, then

$$\lambda_\beta+\mu = |C_\beta| + |C_{\beta+1}-C_\beta| = |C_{\beta+1}| = \lambda_{\beta+1}\,.$$

But this is possible only if $\mu=\lambda_{\beta+1}$: if we had $\mu\le\lambda_\beta$, then we would have $\lambda_\beta+\mu=\lambda_\beta\ne\lambda_{\beta+1}$; and if $\mu\ge\lambda_\beta$, then $\lambda_{\beta+1}=\lambda_\beta+\mu=\mu$; this shows that the claim is valid.

By an identical argument, $|\lambda_{\beta+1}-\lambda_\beta| = \lambda_{\beta+1}$.

Therefore, we have that the sets $C_{\beta+1}-C_\beta$ and $\lambda_{\beta+1}-\lambda_\beta$ are equinumerous, and thus we can find a bijection $g:(\lambda_{\beta+1}-\lambda_\beta) \xrightarrow{\cong} (C_{\beta+1}-C_\beta)$. Finally, put $f_{\beta+1}=f_\beta\cup g$; that is, for $\xi\in\lambda_\beta$, $f_{\beta+1}(\xi)=f_\beta(\xi)$, and for $x\in\lambda_\beta-\lambda_{\beta+1}$, $f_{\beta+1}(\xi)=g(\xi)$. Clearly, $f_{\beta+1} : \lambda_{\beta+1} \xrightarrow{\cong} C_{\beta+1}$, and $f_\beta\subset f_{\beta+1}$, and as a consequence, $f_\gamma\subset f_{\beta+1}$ for all $\gamma<\beta+1$.

When $\beta\le\alpha$ is a limit ordinal, we define $f_\beta = \bigcup_{\gamma<\beta} f_\gamma$. Since we have that $f_\gamma\subset f_{\gamma'}$ for all $\gamma<\gamma'<\beta$, and each of the f_γ is a 1-1 function, we have that f_β is a 1-1 function. The domain of f_β is $\bigcup_{\gamma<\beta}\mathrm{dom}(f_\gamma)=\bigcup_{\gamma<\beta}\lambda_\gamma=\lambda_\beta$ (see (5)). The range of f_β is $\bigcup_{\gamma<\beta}\mathrm{ran}(f_\gamma)=\bigcup_{\gamma<\beta}C_\gamma=C_\beta$ (see (4)). Therefore, f_β is a bijection,

$$f_\beta:\lambda_\beta \xrightarrow{\cong} C_\beta .$$

The construction of f_β also ensures that $f_\gamma\subset f_\beta$ for all $\gamma<\beta$.

This completes the recursive construction of the functions f_β for all $\beta\le\alpha$ with the stated properties. Since we have $f_\alpha:\lambda_\alpha \xrightarrow{\cong} C_\alpha$, we have proved (6) as desired.

(7) Corollary. $|\mathbf{V}_{\omega+\alpha}|=\mathrm{beth}_\alpha$ for all $\alpha\in\mathbf{Ord}$.

This follows from the lemma, by an easy transfinite induction (**exercise**).

(8) Proposition

(i) For any $\alpha\in\mathbf{Ord}$, $\mathbf{V}_\alpha$ is closed under P_{TR} , P_{AUS} and P_{AS} .
(ii) For any $\alpha>0$, $\mathbf{V}_\alpha$ is closed under P_{AES} .
(iii) For any $\alpha>\omega$, $\mathbf{V}_\alpha$ is closed under P_{AI} .
(iv) For any limit ordinal α , $\mathbf{V}_\alpha$ is closed under P_{APS} and P_{APoS} .
(v) Let θ be a strong limit regular cardinal (an inaccessible cardinal). Then $\mathbf{V}_\theta$ is closed under P_{AR} .

Proof. (i): We have that $\mathbf{V}_\alpha$ is transitive, $x\in\mathbf{V}_\alpha \Longrightarrow \bigcup x\in\mathbf{V}_\alpha$, and $y\subset x\in\mathbf{V}_\alpha \Longrightarrow y\in\mathbf{V}_\alpha$; the first of these facts was pointed out before; the other two are left as easy **exercises** (hint: use transfinite induction).

(ii) and (iii): obvious.

(iv): **exercise**.

(v): First, let us note that $\mathrm{beth}_\theta=\theta$. Since θ is a strong limit cardinal, $\theta=\mathrm{beth}_\alpha$ for some limit ordinal α; of course, $\alpha\le\theta$. If we had $\alpha<\theta$, then θ would be singular: $\theta=\mathrm{beth}_\alpha=\mathrm{lub}_{\beta<\alpha}\,\mathrm{beth}_\beta$, with each $\mathrm{beth}_\beta<\theta$. Therefore, we must have $\alpha=\theta$. Thus, $beth_\theta=\theta$.

Next, I claim that for all $x\in\mathbf{V}_\theta$, we have $|x|<\theta$. From $x\in\mathbf{V}_\theta$, it follows that $x\in\mathbf{V}_\alpha$ for some $\alpha<\theta$; therefore, $x\subset\mathbf{V}_\alpha$, and $|x|\le|\mathbf{V}_\alpha|\le\mathrm{beth}_\alpha$ by (7), and so $|x|<\mathrm{beth}_\theta=\theta$.

Now, we can show that $\mathbf{V}_\theta$ is closed under P_{AR}. Let F be a Function such that $D \underset{\mathrm{def}}{=} \mathrm{Dom}(F)\in\mathbf{V}_\theta$ and $\mathrm{Range}(F)\subset\mathbf{V}_\theta$. For every $x\in D$, we have $F(x)\in\mathbf{V}_\theta$, and therefore, there is $\alpha_x<\theta$ such that $F(x)\in\mathbf{V}_{\alpha_x}$. We have the system of ordinals $\langle\alpha_x\rangle_{x\in D}$, with each α_x less than θ, indexed by the set D whose cardinality is $|D|<\theta$ (since $D\in\mathbf{V}_\theta$). By the regularity of θ, we conclude that $\alpha \underset{\mathrm{def}}{=} \mathrm{lub}_{x\in D}\,\alpha_x<\theta$. Since $\alpha_x\le\alpha$ implies that $\mathbf{V}_{\alpha_x}\subset\mathbf{V}_\alpha$, we get that $F(x)\in\mathbf{V}_\alpha$ for every $x\in D$. This says that $\mathrm{Range}(F)\subset\mathbf{V}_\alpha$, and $\mathrm{Range}(F)\in\mathcal{P}(\mathbf{V}_\alpha)=\mathbf{V}_{\alpha+1}\subset\mathbf{V}_\theta$ as desired.

The last proposition implies the "if" part of the Theorem: if A is $\mathbf{V}$ or $\mathbf{V}_\theta$ for an inaccessible cardinal θ, then $\mathcal{SC}(A)$ is closed under the said eight closure conditions. But we also get, for instance, that

(9) Proposition Let $A=\mathbf{V}_\delta$ for a limit ordinal $\delta>\omega$. Then $\mathcal{SC}(A)$ satisfies all axioms of set theory except possibly the Axiom of Replacement.

(10) Lemma

(i) $\alpha\subset\mathbf{V}_\alpha$, but $\alpha\notin\mathbf{V}_\alpha$; $\mathbf{V}_\alpha\cap\mathbf{Ord}=\alpha$; $\beta\in\mathbf{V}_\alpha \Longleftrightarrow \beta<\alpha$.

(ii) $\mathbf{V}_\alpha=\bigcup_{\beta<\alpha}\mathcal{P}(\mathbf{V}_\beta)$.

(iii) Define the so-called *rank function* $\mathrm{r}:\mathbf{V}\to\mathbf{Ord}$ by recursion on the well-founded relation $\in$ as follows:

$$\mathrm{r}(x)=\mathrm{lsub}_{y\in x}\,\mathrm{r}(y).$$

We have that $\mathrm{r}(x)\in\mathbf{Ord}$, and $\mathrm{r}(x)$ is the least α such that $x\subset\mathbf{V}_\alpha$.

(iv) $x\in\mathbf{V}_\alpha \iff \mathrm{r}(x) < \alpha$.
(v) $x\in\mathbf{V}_\alpha \iff \mathrm{r}(x)\in\mathbf{V}_\alpha$.

Proof: (i), (ii): **exercises**.

(iii): We prove that for any $\alpha\in\mathbf{Ord}$, $x\subset\mathbf{V}_\alpha \iff \mathrm{r}(x)\le\alpha$; this will suffice. We proceed by $\in$-induction on x. Thus, we take x, and assume that for any $y\in x$ and $\beta\in\mathbf{Ord}$, $y\subset\mathbf{V}_\beta \iff \mathrm{r}(y)\le\beta$. We have:

$$x\subset\mathbf{V}_\alpha \iff \forall y\in x.\, y\in\mathbf{V}_\alpha \underset{\text{(ii)}}{\overset{}{\iff}} \forall y\in x.\, y\in\bigcup_{\beta<\alpha}\mathcal{P}(\mathbf{V}_\beta)$$

$$\iff \forall y\in x.\exists\beta<\alpha.\, y\in\mathcal{P}(\mathbf{V}_\beta) \iff \forall y\in x.\exists\beta<\alpha.\, y\subset\mathbf{V}_\beta$$
$$\underset{\text{ind.hyp.}}{\iff} \forall y\in x.\exists\beta<\alpha.\,\mathrm{r}(y)\le\beta \underset{\text{def. of r}(x)}{\iff} \forall y\in x.\,\mathrm{r}(y)<\alpha$$

$$\iff \operatorname*{lub}_{y\in x} \mathrm{r}(x) \le \alpha .$$

(iv): $$x\in\mathbf{V}_\alpha \iff x\in\bigcup_{\beta<\alpha}\mathcal{P}(\mathbf{V}_\beta) \iff \exists\beta<\alpha.\, x\in\mathcal{P}(\mathbf{V}_\beta) \iff \exists\beta<\alpha.\, x\subset\mathbf{V}_\beta$$
$$\underset{\text{(iii)}}{\iff} \exists\beta<\alpha.\,\mathrm{r}(x)\le\beta \iff \mathrm{r}(x)<\alpha$$

(v): $$x\in\mathbf{V}_\alpha \underset{\text{(iv)}}{\iff} \mathrm{r}(x)<\alpha \underset{\text{(i)}}{\iff} \mathrm{r}(x)\in\mathbf{V}_\alpha .$$

Proof of the "only if" part of (1') Theorem. Assume that $\mathcal{SC}(A)$ satisfies the closure conditions in question. I assert that

(11) $x \in A$ implies $r(x)+1 \in A$.

We prove this by $\in$-induction on x . Suppose $x \in A$, and for all $y \in x$, we have $r(y)+1 \in A$. We have $r(x)=\mathrm{lsub}_{y\in x}\, r(y) = \mathrm{lub}_{y\in x}\, (r(y)+1)$. Define the Function F to have domain equal to x ($\in A$), and to satisfy $F(y)=r(y)+1$ for each $y \in x$; $F(y) \in A$ for all $y \in x$; that is, $\mathrm{Range}(F) \subset A$. Therefore, by A being P_{AR}-closed, $\mathrm{Range}(F) \in A$. Using that A is P_{AUS}-closed, we have $\bigcup \mathrm{Range}(F) \in A$. But

$$\bigcup \mathrm{Range}(F) = \bigcup_{y\in x} (r(y)+1) = \mathrm{lub}_{y\in x}\, (r(y)+1) = r(x) \;;$$

thus, $r(x) \in A$. Since A is closed under pair-sets (under P_{APS}), and union-sets (under P_{AUS}), $u \in A$ implies $S(u) \in A$. Therefore, $r(x)+1 \in A$ as asserted.

Next, I claim that

(12) $\alpha \in \mathbf{A} \cap \mathbf{Ord} \implies \mathbf{V}_\alpha \in A$.

This is done by induction on $\alpha \in \mathbf{Ord}$. For $\alpha = 0$, the assertion is true since A is P_{AES}=closed ($\emptyset \in A$). For α a successor ordinal, we use that $u \in A$ implies $\mathcal{P}(u) \in A$ (closure under P_{APoS}), and for α a limit ordinal, we use that A is P_{AR}- and P_{AUS}-closed; the details are easy.

I can now demonstrate that

$$A = \bigcup_{\alpha \in A \cap \mathbf{Ord}} \mathbf{V}_\alpha . \qquad (13)$$

Suppose first that $x \in A$. Then, by (11), $\alpha =_{\mathrm{def}} r(x)+1 \in A \cap \mathbf{Ord}$. Since $x \in \mathbf{V}_\alpha$ (see (10) Lemma (iv)), x belongs to the right-hand side.

Suppose x belongs to the right-hand side: there is $\alpha \in A \cap \mathbf{Ord}$ such that $x \in \mathbf{V}_\alpha$. By (10) Lemma (iv), $r(x)+1 \le \alpha$. By A being transitive, $r(x)+1 \in A$. By (12), $\mathbf{V}_{r(x)+1} \in A$, hence $\mathbf{V}_{r(x)+1} \subset A$ (A is transitive). Since $x \in \mathbf{V}_{r(x)+1}$, $x \in A$ follows.

Consider the class $\theta = A \cap \mathbf{Ord}$. Since A is transitive, θ is a transitive class of ordinals.

Either θ=**Ord**, or there is some $\alpha\in$**Ord**$-\theta$. But in the latter case, for all $\beta>\alpha$, $\beta\in$**Ord**$-\theta$, by the transitivity of θ; therefore, **Ord**$-\theta\supset$**Ord**$-\alpha$, and $\theta\subset\alpha$; thus, θ is a set, a transitive set of ordinals; hence, θ is an ordinal. We see that either θ=**Ord**, or $\theta\in$**Ord**. Let us also note that, in case $\theta\in$**Ord**, θ is a *limit* ordinal. This is because

$$\beta<\theta \Longrightarrow \beta\in A \Longrightarrow \beta+1\in A \Longrightarrow \beta+1<\theta .$$

By (13), $A = \bigcup_{\alpha\in\theta} \mathbf{V}_\alpha$. When θ=**Ord**, $A = \mathbf{V}$. Otherwise, since θ is a limit ordinal, $A=\mathbf{V}_\theta$. We show that, in this case, θ is a strong limit regular cardinal. Recall that κ is a strong limit cardinal means that $\kappa=\mathrm{beth}_\delta$ for a limit ordinal δ. This is the same as $\kappa>\aleph_0$ and for all λ, $\lambda<\kappa$, λ a cardinal implies $2^\lambda<\kappa$.

First, for the regularity of θ, in the "purely ordinal" formulation (') in §11. θ is a limit ordinal. Suppose $\beta<\theta$, and $\alpha_\iota<\theta$ for each $\iota<\beta$. By the definition of θ as $A\cap Ord$, we have $\beta\in A$, and $\alpha_\iota\in A$ for $\iota<\beta$. Consider the function F whose domain is $\beta\in A$, and for which $F(\iota)=\alpha_\iota\in A$. By A being P_{AR}-closed, $\mathrm{range}(F)=\{\alpha_\iota : \iota\in\beta\}\in A$; thus, $\mathrm{lub}_{\iota\in\beta}\,\alpha_\iota = \bigcup \mathrm{range}(F) \in A$ by A being P_{AUS}-closed. $\mathrm{lub}_{\iota\in\beta}\,\alpha_\iota \in A$ implies that $\mathrm{lub}_{\iota\in\beta}\,\alpha_\iota < \theta$ as desired.

Finally, for θ being a strong limit cardinal. $\theta>\aleph_0$ follows from $\mathbb{N}\in A$ (A is P_{AI}-closed). Suppose $\kappa<\theta$, κ a cardinal. Then $\kappa\in A$, and $\mathcal{P}(\kappa)\in A$ (A is P_{APS}-closed). Let $\lambda=|\mathcal{P}(\kappa)|$; we have some $f:\mathcal{P}(\kappa)\xrightarrow{\cong}\lambda$. We want to see that $\lambda\in A$, or equivalently, that $\lambda<\theta$. Suppose otherwise; $\theta\le\lambda$. But then, we can let

$$X = f^{-1}[\theta]=\{f^{-1}(\alpha):\alpha<\theta\}\subset\mathcal{P}(\kappa) ;$$

by A being P_{AS}-closed, $X\in A$. We have $f\restriction X:X\xrightarrow{\cong}\theta$, $\mathrm{dom}(f\restriction X)=X\in A$, and $\mathrm{range}(f\restriction X)=\theta\subset A$; from which by A being P_{AR}-closed, it follows that $\theta\in A\cap$**Ord**$=\theta$, contradiction. Therefore, we must have $\lambda<\theta$. The proof is complete.

Theorem (1') says that for a class being closed under set-existence is equivalent to being the same as $\mathbf{V}_\theta$, the θth stage in the cumulative hierarchy, for θ a strongly inaccessible cardinal, or $\theta = \mathbf{Ord}$, where, by convention, $\mathbf{V}_{\mathbf{Ord}} = \mathbf{V}$.

Given any closure predicate, it is natural consider the least class closed under it. Let $\mathbf{U}$ denote the least class closed under P_{SE}. What do we know about $\mathbf{U}$? *Either*, $\mathbf{U}$ equals $\mathbf{V}$ (in particular, $\mathbf{U}$ is a proper class), and there is no inaccessible cardinal (if there is such θ, $\mathbf{V}_\theta \subsetneqq \mathbf{V}$, $\mathbf{V}_\theta$ is P_{SE}-closed, contradicting $\mathbf{U}$ being least such); *or else*, $\mathbf{U}$ is a set, in fact, $\mathbf{U}$ is $\mathbf{V}_\theta$ for the *least* inaccessible cardinal θ (there cannot be any inaccessible $\theta' < \theta$, since then $\mathbf{V}_{\theta'} \subsetneqq \mathbf{V}_\theta$, $\mathbf{V}_{\theta'}$ is P_{SE}-closed (by (1')), and $\mathbf{V}_\theta$ is not least among such classes). Let us write θ_0 for the least inaccessible cardinal *if any*. We have

(14) *Either* there is no inaccessible cardinal, and $\mathbf{U}$ is $\mathbf{V}$ (*first alternative*)
or there is at least one inaccessible cardinal, and $\mathbf{U}$ is $\mathbf{V}_{\theta_0}$, for θ_0 the least inaccessible (*second alternative*).

Which one of the two alternatives is the case? Whatever the answer, we can prove that

(15) Proposition $\mathcal{SC}(\mathbf{U}) \models$ "there is no inaccessible cardinal".

To establish this, we need

(16) Lemma Let A be a class closed under set-existence. For any $a \in A$, a is an inaccessible cardinal in the sense of the model $\mathcal{SC}(A)$ if an only if a is an inaccessible cardinal (in the sense of the universe).

Proof of (15) from (16). Suppose $\mathcal{SC}(\mathbf{U}) \models$ "there is at least one inaccessible cardinal". This means that there is $a \in \mathbf{U}$ which is an inaccessible cardinal in the sense of the model $\mathcal{SC}(\mathbf{U})$. By (16), a is an inaccessible cardinal. This excludes the first alternative in (14). But under the second alternative, we have $\mathbf{U} = \mathbf{V}_{\theta_0}$, and then $a \in \mathbf{V}_{\theta_0}$ implies that $a < \theta_0$ (see (10)(i)), which contradicts θ_0 being the least inaccessible. We have reached a contradiction, which completes the proof.

Outline of the proof of (16). We have that

a is an inaccessible cardinal $\Longleftrightarrow$

$\mathtt{Ord}(a)$ & a is limit ordinal & $a>\omega$ (17)
§ $\forall F((\mathtt{Function}(F)$ & $\mathtt{Dom}(F)\in a$ & $\mathtt{Range}(F)\subset a) \Longrightarrow \bigcup \mathtt{Range}(F)\in a)$ (18)
§ $\forall y\forall z\forall G((_1 y\in a$ & $\mathtt{Ord}(z)$ & $\mathtt{Function}(G)$ & $\mathtt{Dom}(g)=\mathcal{P}(y)$
& $\mathtt{Range}(G)=z$ & G is a bijection$)_1 \Longrightarrow z\in a)$ (19)

Line (18) expresses condition (1') in §11. Line (19) expresses that for all cardinals $\kappa<a$, we have $2^{\kappa}<a$. To be sure, what it says directly is that for all ordinals y, z, if $\mathcal{P}(y) \sim z$, then $z<a$ -- but this is an equivalent statement of the above, given that a is a cardinal by the previous conditions.

Remember that

$\mathtt{Ord}(a)$ $\Longleftrightarrow$ a is transitive and trichotome
$\Longleftrightarrow$ $\forall b\in a.\forall c\in b.c\in a$ & $\forall b\in a.\forall c\in a(b\in c$ or $b=c$ or $c\in b)$.

a is limit $\Longleftrightarrow$ $\forall b\in a.\exists c\in a(b\in a)$

As we saw in the discussion of AR in §12, the fact that these expressions contain only bounded quantifiers, ensure, on the basis that A is transitive, that the predicates in question are *absolute*: for $a\in A$,

$$\mathtt{Ord}^{(A)}(a) \Longleftrightarrow \mathtt{Ord}(a) ;$$

$$(a \text{ is limit})^{(A)} \Longleftrightarrow a \text{ is limit}$$

Further, we saw in *loc. cit.* that, for reasons identical to the ones just cited, the predicate $\mathtt{Function}(F)$ is absolute, and $\mathtt{Dom}^{(A)}(F) = \mathtt{Dom}(F)$,
$\mathtt{Range}^{(A)}(F) = \mathtt{Dom}^{(A)}(F)$, $\bigcup^{(A)} F = \bigcup F$ for $F\subset A$. Being a bijection is also

absolute.

Assume that a is an inaccessible in the sense of $\mathcal{SC}(A)$. Thus, $a\in A$, and the relativization to A of lines (17), (18) and (19) hold true. By what we said above, the relativization does not change (17) at all (note that $\omega\in A$); in (18), the quantifier $\forall F$ at the front is replaced by $\forall F\subset A$; and in (19), the quantifiers $\forall y$, $\forall z$, $\forall G$ are replaced, respectively, by $\forall y\in A$, $\forall z\in A$ and $\forall G\subset A$; otherwise, there is no change in the expressions (we also use that A is P_{APoS}-closed). We have to prove that from the assumed relativized statement, the unrelativized version follows. As for line (18), now we have to show something for something for all classes F, and not just for subclasses F of A. But the assumption for F is that

$$\mathrm{Function}(F)\ \&\ \mathrm{Dom}(F)\in a\ \&\ \mathrm{Range}(F)\subset a$$

holds; and this implies that $F\subset A$ since $F\subset\mathrm{Dom}(F)\times\mathrm{Range}(F)$, and $X\subset A$, $Y\subset A$ imply, by A being P_{APS}-closed, that $X\times Y\subset A$. This takes care of line (18).

For a similar treatment of line (19), we now have y, z and G satisfying the $(\ldots)_1^1$. It immediately follows that $y\in A$. But it also follows that $z=\mathrm{Range}(G)\in A$, by A being P_{AR}-closed. $G\subset A$ follows as before.

This completes our somewhat sketchy verification of (16).

Using the axioms, we *prove* theorems of set theory. The methods of proof are clear to us in practice, but they have not yet been clarified explicitly: this will be the task of *logic*. In any case, proving a theorem from the axioms takes the form of some kind of *deduction*; a statement Φ is *provable* from certain axioms $\mathcal{A}$ if a deduction "obeying the rules of logic" of the statement from the axioms exists. Now, a fundamental intuition that we have about provability is that it is *sound* with respect to states of affairs concerning the truth of the statements involved. In particular, we *should*, and in fact, we *will* have that

(**Soundness of Provability**) *If* Φ *is provable from* $\mathcal{A}$, *then every model of* $\mathcal{A}$ *will also satisfy* Φ *.*

Granting Soundness, we now draw some conclusions from what we saw above.

Let us return to (9) Proposition. Let us consider the set $A=\mathbf{V}_{\omega+\omega}=\mathbf{V}_{\omega\cdot 2}$. According to (9), $\mathcal{SC}(A)$ satisfies all axioms of set theory, except perhaps AR , the Axiom of Replacement. We now point out that in fact, $\mathcal{SC}(A)$ does *not* satisfy AR. Although this follows from (1') Theorem , since $\omega\cdot 2$ is not regular (**exercise**), we point out a direct argument. Consider the Function F whose domain is $\mathrm{Dom}(F)=\omega$, and for which $F(n)=\mathbf{V}_{\omega+n}$ ($n\in\omega$). Since $\omega\subset\mathbf{V}_{\omega}$, $\omega\in\mathcal{P}(\mathbf{V}_{\omega})=\mathbf{V}_{\omega+1}\subset\mathbf{V}_{\omega\cdot 2}=A$, and $\mathbf{V}_{\omega+n}\in A$ for all $n\in\omega$, we have that $\mathrm{Dom}(F)\in A$ and $\mathrm{Range}(F)\subset A$. If $\mathcal{SC}(A)$ satisfied AR, we would have $\mathrm{Range}(F)=\{\mathbf{V}_{\omega+n}:n\in\omega\}\in A$; but $\mathcal{SC}(A)\models$AUS , A is $\mathrm{P}_{\mathrm{AUS}}$-closed, thus, $A=\mathbf{V}_{\omega+\omega}=\bigcup\{\mathbf{V}_{\omega+n}:n\in\omega\}\in A$ would follow; contradiction. We have shown

(20) $\mathcal{SC}(\mathbf{V}_{\omega\cdot 2})$ satisfies all axioms of MK class-set theory, except the Axiom of Replacement, which it does not satisfy.

As a consequence, by Soundness of Provability, we conclude

(21) Metatheorem The Axiom of Replacement is not provable from the remaining axioms.

Remember ((1) Theorem) that every P_{SE}-closed class, in particular $\mathbf{U}$, will give rise to a model $\mathcal{SC}(\mathbf{U})$ of all of MK. Therefore, by (15), we similarly conclude

(22) Metatheorem The assertion "there is an inaccessible cardinal" is not provable from the axioms of MK.

Now, recall the alternatives of (14). We have that

(23) The statements

(i) "There is an inaccessible cardinal"

and

(ii) "The least class $\mathbf{U}$ closed under set existence is a set"

are equivalent within MK.

It is generally accepted that either (23)(i) or (23)(ii) is a *reasonable additional* axiom of set theory.

In fact, we have

(24) The statements

(i) "For every ordinal α, there is an inaccessible cardinal greater than α"

and

(ii) "For every set x, there is a set U closed under set existence for which $x \in U$."

are equivalent within MK.

Exercise. Convince yourself of the truth of (24).

Alexander Grothendieck made the equivalent statements (24)(i), (24(ii) axioms in his set-theory, because he needed them in the Theory of Categories. A set closed under set-existence is also called a Grothendieck universe.

§14 The Boole/Stone algebra of sets

14.1. Lattices and Boolean algebras.

Given a set A, the subsets of A admit the following simple and familiar operations on them: $\cap$ (*intersection*), $\cup$ (*union*) and $-$ (*complementation*). If $X, Y \subset A$, then $X \cap Y$, $X \cup Y$ are also subsets of A. With A fixed (and suppressed in the notation), we write $-X = A - X$ for any $X \subset A$; of course, $-X \subset A$ again. Intersection and union are binary operations on $\mathcal{P}(A)$, $-$ is a unary operation on $\mathcal{P}(A)$:

$$\cap : \mathcal{P}(A) \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A) ,$$

$$\cup : \mathcal{P}(A) \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A) ,$$

$$- : \mathcal{P}(A) \longrightarrow \mathcal{P}(A) .$$

Of course, intersection and union are defined for any number of arguments; using the binary versions repeatedly, we can reproduce finite intersections and union, *except* the empty intersection and the empty union. For the empty intersection, we take the set A itself; for the empty union, the empty set.

What is the justification? For any family $\mathcal{F} \subset \mathcal{P}(A)$ of subsets of A, we have

$$\bigcap \mathcal{F} = \{x \in A : \text{for all } X \in \mathcal{F} , x \in X\}$$

and

$$\bigcup \mathcal{F} = \{x \in A : \text{for some } X \in \mathcal{F} , x \in X\} .$$

Note that the expression for $\bigcap \mathcal{F}$ is the same as that in Section 3, page 35 except for the clause " $\in A$ " ; the expression for $\bigcup \mathcal{F}$ has a similar difference to the earlier expression on p. 30.

For the union, there is no actual difference in meaning; the old and the new expressions give the same set. For the intersection, the same is true except for the empty family $\mathcal{F}$; the old expression gives $\mathbf{V}$, a non-set; the new expression gives A itself. Of course, the union of the empty family, according to the general formula, is the empty set. It goes without saying that $X\cap Y = \bigcap \{X, Y\}$, $X\cup Y = \bigcup \{X, Y\}$.

The composite object

$$(\mathcal{P}(A); \cap, \cup, -, A, 0) \tag{1}$$

is an example of what we call an *algebra*: a set (in this case $\mathcal{P}(A)$), called the *underlying set of* the algebra, with certain particular *operations* on it (in this case, the binary operations $\cap$, $\cup$, the unary operation $-$, and the 0-*ary operations* A , 0 : 0-ary operations are distinguished elements of the underlying set). *Any* object of the form

$$(B ; \wedge , \vee , \neg , 1 , 0)$$

with B a set, $\wedge$, $\vee$ both $B\times B \longrightarrow B$, $\neg : B \longrightarrow B$, and $1, 0 \in B$, is an algebra *similar* to (1). Speaking in very general terms, we will seek, and at least partly find, properties of algebras of the form (1) that distinguish them among all the algebras similar to them; the result will be the notion of *Boolean algebra*.

For future reference, let's say that when we denote an algebra by a single letter, say B , $|B|$ denotes the underlying set of B . This, of course, conflicts with the notation for "cardinality"; it is advisable to use $\#A$ for the cardinality of the set A when the underlying set of an algebra is also to be used.

Let us first look at the basic operations from another point of view, namely the context of the *poset* $(\mathcal{P}(A), \subset)$. We have, for any $\mathcal{F} \subset \mathcal{P}(A)$, that

$\bigcap \mathcal{F}$ is the *largest* subset Y of A for which $Y \subset X$ for all $X \in \mathcal{F}$:

$\bigcap \mathcal{F} \subset X$ for all $X \in \mathcal{F}$, and

if $Y \subset X$ for all $X \in \mathcal{F}$, then $Y \subset \bigcap \mathcal{F}$;

and similarly,

$\bigcup \mathcal{F}$ is the *least* subset Y of A for which $X \subset Y$ for all $X \in \mathcal{F}$:

$X \subset \bigcup \mathcal{F}$ for all $X \in \mathcal{F}$, and
if $X \subset Y$ for all $X \in \mathcal{F}$, then $\bigcup \mathcal{F} \subset Y$.

(**verify** this statement).

In general, in any poset $(B, \leq)$, and for any family $\mathcal{F} \subset B$ of elements of B , a *lower bound of* $\mathcal{F}$ is any $y \in B$ such that $y \leq x$ for all $x \in \mathcal{F}$; the *greatest lower bound* (g.l.b), or *infimum* (inf) of $\mathcal{F}$ (if it exists!) is the *maximum* element of the set L of all lower bounds of $\mathcal{F}$: $y_0 \in L$ such that $y \leq y_0$ for all $y \in L$. (Note that the requirement is more than to say that y_0 be a *maximal* element of L !). The g.l.b. of $\mathcal{F}$ is denoted by $\bigwedge \mathcal{F}$; $\bigwedge \mathcal{F}$ does not necessarily exist (in an arbitrary poset $(B, \leq)$), but if it does, it is uniquely determined by the definition. The notions of *upper bound, least upper bound (l.u.b., supremum, sup*), with the notation $\bigvee \mathcal{F}$, are defined similarly ("dually"). [In the context of ordinals and well-orderings, we have already used lub's extensively.]

Now, notice that what we said above about intersections and unions amounts to this that in the poset $(\mathcal{P}(A), \subset)$, $\bigwedge \mathcal{F}$, $\bigvee \mathcal{F}$ exist for all $\mathcal{F} \subset \mathcal{P}(A)$, and in fact $\bigvee \mathcal{F} = \bigcap \mathcal{F}$, $\bigwedge \mathcal{F} = \bigcup \mathcal{F}$.

It is worth remarking that the definitions of inf (sup) can be put in the following form:

$$y \leq \bigwedge \mathcal{F} \iff y \leq x \text{ for all } x \in \mathcal{F} ;$$

$$\bigvee \mathcal{F} \leq y \iff x \leq y \text{ for all } x \in \mathcal{F} ;$$

y ranges over all the elements of the poset.

Note also that $\bigwedge \emptyset$ is the maximum element of the poset (if such exists); $\bigvee \emptyset$ is the minimum element (if exists). We write 1 for the maximum element, 0 for the minimum element (if they exist).

A poset $(B, \leq)$ is called a *lattice* if $\bigwedge \mathcal{F}$, $\bigvee \mathcal{F}$ exist for all *finite* sets $\mathcal{F} \subset B$. Thus, in a lattice $(B, \leq)$, there always are a maximum element 1 , a minimum element 0 ; moreover, for any $x, y \in B$, $x \wedge y \underset{\text{def}}{=} \bigwedge \{x, y\}$, $x \vee y \underset{\text{def}}{=} \bigvee \{x, y\}$ always exist. The poset $(\mathcal{P}(A), \subset)$ is a lattice; in fact, it is what is called a *complete lattice*, meaning that $\bigwedge \mathcal{F}$, $\bigvee \mathcal{F}$ exist for all $\mathcal{F} \subset B = \mathcal{P}(A)$.

Note the following laws that always hold in any lattice:

$x \wedge y = y \wedge x$, $x \vee y = y \vee x$ (commutative laws)

$(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$
(associative laws)

$x \wedge x = x$, $x \vee x = x$ (idempotent laws)

$x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$ (absorption laws)

$x \wedge 1 = x$, $x \vee 1 = 1$, $x \wedge 0 = 0$, $x \vee 0 = x$.

Exercises. (i) Verify that the above hold in any lattice.

(ii) Assume an algebra $(B, \wedge, \vee, 1, 0)$ satisfying the above laws. Show that there is a unique partial ordering $\leq$ on B that makes $(B, \leq)$ a lattice in such a way that the given $\wedge$, $\vee$, 1 , 0 become the lattice operations.

(iii) Suppose that in a poset, $\bigwedge \mathcal{F}$ exists for all sets of elements of the poset. Show that then also $\bigvee \mathcal{F}$ always exists. Show that if, in this assertion, we restrict $\mathcal{F}$ to be a finite set in both occurrences, then the resulting statement is not always true any more.

Exercises (i) and (ii) say that the concept of lattice can be given a purely "operational" ("algebraic") formulation.

The set-theoretic complement $-X = A - X$ also can be given a "lattice" description. The set $Y = -X$ is distinguished among all the subsets of A by the following two properties:

$Y \cup X = A$ and $Y \cap X = 0$,

(**verify!**). In a lattice, y is *a complement* of x if $y \vee x = 1$ and $y \wedge x = 0$. In a general lattice, the complement of an element may not exist, and it is also possible that there are two different complements of the same element.

A particular property of $(\mathcal{P}(A), \subset)$ as a lattice is that it is *distributive*. A lattice $(B, \leq)$ is distributive if

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

for all $x, y, z \in B$.

Indeed, the distributive law is familiar for $(\mathcal{P}(A), \subset)$ (see Assignment 1).

Exercises. (iv) Show that in a distributive lattice, the dual of the distributive law, that is

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$

holds too.

(v) Show that in a distributive lattice, every element has at most one complement.

(vi) Show that any linear ordering with a minimal and a maximal element is a distributive lattice.

A *Boolean algebra* is a distributive lattice in which every element has a complement. Of course, $(\mathcal{P}(A), \subset)$ is a Boolean algebra.

One particular Boolean algebra, $(\mathcal{P}(1), \subset)$, plays a central role in our theory. This one has two elements: 0 and 1 (right?) ; note that $\mathcal{P}(1) = 2$. The binary Boolean operations are tabulated as follows:

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

In addition, we have $\neg(1) = 0$, $\neg(0) = 1$. We call this algebra the *two-element Boolean algebra*, and denote it by $\mathit{2}$.

Let us point out that $\mathit{2}$ is also considered to be the *algebra of truth values* **t**=**true** and

$\mathbf{f}=\mathbf{false}$; $\mathbf{t}$ is identified with 1 , $\mathbf{f}$ with 0 . Under this identification, the above operations $\wedge$, $\vee$ and $\neg$ become the logical operations of *conjunction* ("and"), *disjunction* ("or"), and *negation* ("not").

With any poset $B = (|B|, \leq)$, we have its *opposite*, B^{o} . The underlying set of B^{o} is the same, $|B|$, as of B ; the ordering in B^{o} is the *opposite* of that in B : $x \leq_{B^{o}} y \underset{\text{def}}{\Longleftrightarrow} y \leq x$. It is clear that B^{o} so defined is a poset too. Moreover, it is also clear that the inf of a set $\mathcal{F}$ in the sense of B^{o} is the same as the sup of $\mathcal{F}$ in the sense of B , and *vice versa*. Thus, of B is a lattice, so is B^{o} . Moreover, as exercise (iv) above shows, if B is a distributive lattice, then B^{o} is distributive too. Also, the definition of complement shows that the notions of complement in B and B^{o} are the same. Briefly put, the notion of "lattice", "distributive lattice", and "Boolean algebra" are each self-dual concepts: if a poset falls in any of these categories, so does its opposite.

14.2. Some algebraic ideas.

Note that the notion of Boolean algebra is defined in terms of the operations $\wedge$, $\vee$, $\neg$, 1 and 0 by *identities* : the laws describing lattices, the distributive law, and the laws defining the complement. In general, an identity, for any kind of algebra, is an equality of two *terms* built up of the basic operations of the algebra, required to hold for all values of the variables involved. In the definition of Boolean algebra, we have found some *particular* identities that hold in the set-algebra $(\mathcal{P}(A), \wedge, \vee, \neg, 1, 0)$; *have we found them all*?

As it is, this question is not very intelligent since, e.g., $1 \wedge x = x$ is an identity not listed above that obviously holds in the set-algebra, and in fact, in all lattices, as a consequence of two of the axioms (why?). However, we may ask:

(*) is it the case that all identities that hold in the set-algebras are *consequences* of the Boolean axioms, that is, are true in *all* Boolean algebras?

Put this way, the question amounts to asking whether we have found, in the Boolean axioms, a sufficient basis to deduce all identities formulated in terms $\cap$, $\cup$, $-$, 1 , 0 that are true for sets; if the answer is "no", then there is another, still undiscovered, *essentially new* identity concerning these set-operations.

We will give an affirmative answer to the question just asked, by deducing it from a more abstract theorem to be stated soon.

Example. The so-called De-Morgan law: $\neg(x\wedge y) = (\neg x)\vee(\neg y)$ holds in set-algebras; in fact, it holds, in all Boolean algebras (**exercise** (vii)).

A *homomorphism* of lattices L and M , in notation $f:L\longrightarrow M$, is a mapping $f: |L| \longrightarrow |M|$ between the underlying sets that *preserves* the lattice operations:

$$f(x\wedge y) = f(x)\wedge f(y) ,$$

$$f(x\vee y) = f(x)\vee f(y) ,$$

$$f(1) = 1 ,$$

$$f(0) = 0 .$$

These equalities are required to hold for all $x, y \in |L|$; of course, on the left sides, $\wedge$, $\vee$, 1 , 0 refer to the lattice operations of L , on the right to those of M .

An *embedding* of lattices is a 1-1 homomorphism; an *isomorphism* is a bijective homomorphism.

Exercises. (viii) A lattice homomorphism f between Boolean algebras is a *Boolean homomorphism* in the sense that it also preserves complements: $f(\neg x) = \neg f(x)$.

(ix) Find a Boolean *embedding* of $(\mathcal{P}(2), \leq)$ into $(\mathcal{P}(3), \leq)$.

(x) Any lattice homomorphism preserves the partial ordering relation: $x \le y \implies fx \le fy$. If L, M are lattices, and f is a poset isomorphism $f:(|L|, \le) \longrightarrow (|M|, \le)$ (i.e., f is a bijection $f: |L| \xrightarrow{\cong} |M|$, and $x \le y \iff fx \le fy$ $(x, y \in |L|)$), then f is a lattice isomorphism as well. However, a poset-homomorphism between lattices (map preserving the order) is not necessarily a lattice homomorphism.

There are the following points to be made about homomorphisms and embeddings:

(1) given a (Boolean) homomorphism $f:B \longrightarrow C$, and a Boolean term $t(\vec{x})$ built up of variables and the symbols for the Boolean operations, then for any values $\vec{b}$ from B for the variables $\vec{x}$ we have

$$f(t^B(\vec{b})) = t^C(f\vec{b}) \ ;$$

that is, if we first evaluate t at $\vec{b}$ in B, then apply f, we obtain the same value as when we first apply f to each of the values in $\vec{b}$, and then evaluate t in C at those arguments;

and

(2) if an *identity* $s(\vec{x}) = t(\vec{x})$ holds in C (for all values in $|C|$), and $f:B \longrightarrow C$ is an *embedding*, then the same identity also holds in B.

(1) is a consequence of the definition of "homomorphism"; note that the "homomorphism" is defined in such a way that the assertion hold in case t is a simple term (has just one operation mentioned in it); the general statement is proved by "induction". (2) is a consequence of (1) as follows. Suppose $s(\vec{x}) = t(\vec{x})$ holds in C, and $f:B \longrightarrow C$ is an embedding. To show that the same identity holds in B, let $\vec{b}$ be arbitrary elements to evaluate the variables $\vec{x}$. Then

$$f(s^B(\vec{b})) = s^C(f\vec{b})$$

and

$$f(t^B(\vec{b})) = t^C(f\vec{b}) .$$

Since we have $s^C(f\vec{b}) = t^C(f\vec{b})$ by the assumption that the identity holds in C, we get that $f(s^B(\vec{b})) = f(t^B(\vec{b}))$. Since f is 1-1, it follows that $s^B(\vec{b}) = t^B(\vec{b})$ as desired.

Put briefly, (2) says that *any identity that holds in an algebra holds in any other that can be embedded into the given one*.

Exercise. (xi) Suppose the lattice L can be embedded into a distributive lattice. Then L itself is distributive.

Given a family $\langle L_i \rangle_{i\in I}$ of posets, their *Cartesian product*, $\prod_{i\in I} L_i$, is the poset L whose underlying set is $|L| = \times_{i\in I} |L_i|$, and for which

$$f \le g \iff f(i) \le g(i) \text{ for all } i \in I .$$

Here, f and g are arbitrary elements of $\times_{i\in I} |L_i|$ (remember that the latter is the set of certain functions with domain I); on the left side, $\le$ is the ordering of $\prod_{i\in I} L_i$ to be defined; on the right, $\le$ refers to the ordering given in (each) L_i.

Exercise. (xii) Verify that $\prod_{i\in I} L_i$ is indeed a poset; if each L_i is a lattice, then so is $\prod_{i\in I} L_i$; if each L_i is a distributive lattice, or a Boolean algebra, then so is $\prod_{i\in I} L_i$. In fact, the lattice (Boolean) operations on $\prod_{i\in I} L_i$ are defined *pointwise*: e.g., $(f \wedge g)(i) = f(i) \wedge g(i)$.

(xiii) The *projection mapping*

$$\pi_j \;:\; \prod_{i\in I} L_i \longrightarrow L_j$$
$$f \longmapsto f(j)$$

one for each $j \in I$, is a lattice homomorphism.

(xiv) Turning to Cartesian products of sets, let us note the following "mapping property" of Cartesian products: for any sets A_i for $i \in I$, and any further set B :

the maps $f:B \longrightarrow \times_{i\in I} A_i$ are in a one-to-one correspondence with families of the form $\langle f_i:B \longrightarrow A_i\rangle_{i\in I}$. Indeed, the correspondence, in one direction, associates with f the family where $f_i = \pi_i \circ f$ (with π_i defined as in (xiii)).

(xv) Now, if the A_i and B are lattices (say), then the correspondence of (iii) gives a one-to-one correspondence between homomorphisms $f:B \longrightarrow \prod_{i\in I} A_i$ and families of homomorphisms of the form $\langle f_i:B \longrightarrow A_i\rangle_{i\in I}$. Put in another way, to give a homomorphism $f:B \longrightarrow \prod_{i\in I} A_i$ is the same as to give a family of homomorphisms $\langle f_i:B \longrightarrow A_i\rangle_{i\in I}$.

When in the product $\prod_{i\in I} A_i$ all the algebras A_i are the same, say A , we write A^I for the product $\prod_{i\in I} A$; A^I is a *power* (the I^{th} *power*) of A . Note that the underlying set of the algebra A^I , $|A^I|$, is the same as $^I|A|$, where in the latter the notation of Section 3, p.36 is used.

The reason why we talk about products of algebras is because the power-set algebras $(\mathcal{P}(A), \subset)$ are all, essentially, powers of *2* , the two-element algebra, and this turns out to be a useful way of looking at power-set algebras. Recall the bijection

$$\begin{array}{rcl} \mathcal{P}(A) & \xrightarrow{\cong} & {}^{A}2 \\ X & \longmapsto & \mathrm{char}\, X\,. \end{array} \qquad (1)$$

Now, $\mathcal{P}(A)$ and ${}^{A}2$ are the respective underlying sets of the algebras $(\mathcal{P}(A), \subset)$ and 2^{A}. We have that

the mapping in (1) is an isomorphism $(\mathcal{P}(A), \subset) \xrightarrow{\cong} 2^{A}$.

Exercise (xvi): verify this important fact.

Combining the last fact with what we learned above about mappings into a product-algebra, we obtain

for any lattice L, and any set A, the lattice homomorphisms $f: L \longrightarrow (\mathcal{P}(A), \subset)$ are in a one-to-one correspondence with families of homomorphisms of the form $\langle f_a : L \longrightarrow 2 \rangle_{a \in A}$.

Moreover, in this correspondence,

f is an embedding (1-1) if and only if, for every pair (x, y) of distinct elements $x \neq y$ of L, there is $a \in A$ such that $f_a(x) \neq f_a(y)$.

Exercise (xvii). Verify the last two displayed assertions.

Stone representation theorem for distributive lattices (and Boolean algebras).

Any distributive lattice (hence, any Boolean algebra) has an embedding into a power-set algebra.

Equivalently, if L a distributive lattice, and $x \neq y$ are arbitrary elements of L, then there is a 2-valued homomorphism $f: L \longrightarrow 2$ such that $f(x) \neq f(y)$.

The proof of the Stone representation theorem is the subject of the next subsection.

Exercise (xviii)**.** Verify that the two version of the theorem are indeed equivalent. Note that the distributivity condition on the lattice is *necessary*. Note that the question asked under (*) (at the beginning of the present subsection 14.2) has, as a consequence of the Stone representation theorem, an affirmative answer.

14.3. Prime filters and ultrafilters

We now set out to prove the Stone representation theorem. First, we investigate the notion of a *2*-valued lattice homomorphism $f:L\longrightarrow 2$. Any such f is given by the set $F=\{x\in L : f(x)=1\}$; namely, f is then the characteristic function of X , $f=\mathrm{char}\,F : |L|\longrightarrow 2$. The question is what properties F must have in order for $\mathrm{char}\,F$ to be a lattice homomorphism. We introduce some standard terminology.

Let L be a lattice. $F\subset |L|$ is a *filter on* L if $(\mathrm{i})_F$ $1_L\in F$, $(\mathrm{ii})_F$ F is closed upward: $x\in F,\ x\le y \implies y\in F\ (x, y\in |L|)$ [as a consequence, in $(\mathrm{i})_F$, it would have been enough to require that F be non-empty], and $(\mathrm{iii})_F$ if x and y both belong to F , then so does $x\wedge y\ (x, y\in |L|)$.

Exercise (xix)**.** Verify that $F\subset |L|$ is a filter iff $\mathrm{char}\,F$ is an order-preserving map $L\longrightarrow 2$, and it also preserves $\wedge$ and 1 [for this, we say that f is a *meet-semilattice homomorphism*].

A filter F on L is *prime* if $(\mathrm{iv})_{PF}$ $0_L\notin F$ [equivalently, $F\ne |L|$; we say that F is a *proper* filter] and $(\mathrm{v})_{PF}$ whenever $x\vee y\in F$, then either $x\in F$, or $y\in F\ (x, y\in |L|)$.

Exercises. (xx) The prime filters on a lattice L are in a one-to-one correspondence with the homomorphisms $L\longrightarrow 2$.

(xxi) Let F be a filter on the Boolean algebra B . Then F is a prime filter on B iff for any $x\in |B|$, exactly one of x , $\neg x$ belongs to F .

In the case of a Boolean algebra, we may say "*ultrafilter*" to mean "prime filter".

In view of the reformulation of the notion of *2*-valued homomorphism as prime filter, and in view of second form of the Stone representation theorem (at the end of the second section), we now see that the Stone representation theorem is equivalent to the following statement:

For any distributive lattice L *, and any pair of distinct elements* $x\neq y$ *of* $|L|$ *, there is a prime filter* P *of* L *for which one of* x, y *belongs to* P *, and the other of* x, y *does not belong to* P *.*

We are going to show a stronger statement, which is also more specific concerning which of the two given elements can be made to belong, and which not to belong, to the prime filter. The stronger version can then be used to obtain other interesting consequences. The main feature of the stronger version is a certain symmetry with respect to "dualizing", that is, taking the opposite of the lattice in question.

Consider a lattice L . An *ideal* of L is, by definition, the same thing as a filter in L^{o} . Unraveling this, we obtain that an ideal is a subset I of $|L|$ such that $(i)_{I}$ $0_L\in I$, $(ii)_{I}$ I is closed downward: $x\in I$, $y\leq x \implies y\in I$ $(x, y\in |L|)$, and $(iii)_{I}$ if both x and y belong to I , then so does $x\vee y$ $(x, y\in |L|)$. A *prime ideal* of L is a prime filter of L^{o} , that is, an ideal I for which $(iv)_{PI}$ $1_L\notin I$, and $(v)_{PI}$ whenever $x\wedge y\in I$, then either $x\in I$ or $y\in I$.

Prime Filter Existence Theorem (PFET)**.** Given any filter F_0 and any ideal I_0 on the distributive lattice L such that F_0 and I_0 are disjoint: $F_0\cap I_0=\emptyset$, there is at least one prime filter P on L which contains F_0 as a subset and which is disjoint from I_0 :

$$F_0\subset P\ ,\ \ I_0\cap P=\emptyset\ .$$

Before we turn to the proof of the PFET, let us see how the latest formulation of the Stone representation theorem follows from it. Suppose $x, y\in |L|$, and $x\neq y$. Then either $x\not\leq y$, or $y\not\leq x$ (or both). Say, we have $x\not\leq y$. Now, consider the sets $F_0=\uparrow x \overset{=}{\text{def}} \{u\in |L| : u\geq x\}$, and $I_0=\downarrow y \overset{=}{\text{def}} \{v\in |L| : v\leq y\}$. We immediately see that $\uparrow x$ is a filter, and $\downarrow y$ is an ideal (**exercise**). Also, they are disjoint: if we had $u \in \uparrow x \cap \downarrow y$, then $x \le u$ and $u \le y$, and thus $x \le y$ would be the case. The PFET gives a prime filter P with $\uparrow x \subset P$ and $\downarrow y \cap P = \emptyset$. Then, since $x \in \uparrow x$ and $y \in \downarrow y$, we have that $x \in P$ and $y \notin P$ as desired.

The proof of the PFET is an application of Zorn's lemma. To emphasize the character of this proof, we isolate a part of it as a separate statement.

Criterion for a prime filter. Let F_0 be a filter, I_0 an ideal on the distributive lattice L. Then any filter on L which is *maximal* among those filters that contain F_0 and disjoint from I_0 is prime.

Proof of the PFET from the Criterion. Assuming the truth of the Criterion, we proceed as expected. Consider the set $\mathcal{F}$ of all filters on L that contain F_0 as a subset and are disjoint from I_0, partially ordered by inclusion, $\subset$. We apply Zorn's lemma to the poset $(\mathcal{F}, \subset)$. We **claim** that if $\mathcal{C}$ is any *non-empty* chain in $\mathcal{F}$, then $\bigcup \mathcal{C} \in \mathcal{F}$. Indeed, it is clear that condition $(\text{i})_{\text{F}}$ for filters holds, because $\mathcal{C}$ is non-empty; $(\text{ii})_{\text{F}}$ is also clear. To see $(\text{iii})_{\text{F}}$, if $x, y \in \bigcup \mathcal{C}$, then there are $F, F' \in \mathcal{C}$ with $x \in F$, $y \in F'$; since $\mathcal{C}$ is a chain, either $F \subset F'$, or $F' \subset F$; we conclude that both x and y belong either to F or to F', hence, so does $x \wedge y$ (since F, F' are filters!); but F, F' are both subsets of $\bigcup \mathcal{C}$, thus $x \wedge y$ belongs to $\bigcup \mathcal{C}$ as was to be shown. As to $I_0 \cap \bigcup \mathcal{C}$, if $a \in I_0$ belonged to $\bigcup \mathcal{C}$, then it would belong to an $F \in \mathcal{C}$, contradicting $F \in \mathcal{F}$ and the definition of $\mathcal{F}$. The **claim** is verified.

The condition of Zorn's lemma, namely that each chain have an upper bound is *almost* verified: for each non-empty chain $\mathcal{F}$, $\bigcup \mathcal{F}$ is such an upper bound. For the empty chain, take $\text{F}_0 \in \mathcal{F}$ as an upper bound.

By Zorn's lemma, there is a maximal element P of $(\mathcal{F}, \subset)$. By the Criterion, any such maximal element, that is, any filter maximal among those filters that contain F_0 and disjoint from I_0 is prime. This completes the proof.

Proof of the Criterion. Let P be any filter maximal among those filters that contain F_0 and disjoint from I_0. We verify the conditions $(\text{iv})_{\text{PF}}$ and $(\text{v})_{\text{PF}}$ for P. Since I_0 is an ideal, $0_L \in I_0$. Since $I_0 \cap P = \emptyset$, it follows that $0_L \notin P$; this is $(\text{iv})_{\text{PF}}$.

To see $(\text{v})_{\text{PF}}$, assume $x \vee y \in P$, and assume, contrary to what we want, that $x \notin P$ and $y \notin P$. We now construct a filter $P[x]$ containing $P \cup \{x\}$ as a subset; we put

$$P[x] = \{u \in |L| : u \geq s \wedge x \text{ for some } s \in P\} .$$

Indeed, $P[x]$ is a filter: conditions $(\text{i})_{\text{F}}$ and $(\text{ii})_{\text{F}}$ are clear; and if u , v both belong to $P[x]$, then there are s , $t \in P$ with

$$u \geq s \wedge x \quad \text{and} \quad v \geq t \wedge x ;$$

it follows that, for $r = s \wedge t$, we have

$$u \geq r \wedge x \quad \text{and} \quad v \geq r \wedge x ,$$

and hence, $u \wedge v \geq r \wedge x$ (why?); this shows that $u \wedge v \in P[x]$.

Since $x \notin P$, we have $P \underset{\neq}{\subset} P[x]$. By the maximality of P among those filters that contain F_0 and are disjoint from I_0 , and since clearly $F_0 \subset P[x]$ (because $F_0 \subset P$), it must be that $P[x]$ is not disjoint from I_0 ; there is $a \in I_0 \cap P[x]$. The definition of $P[x]$ gives that there is $s \in P$ such that

$$s \wedge x \leq a .$$

Doing the same with y as with x , we get $b \in I_0$ and $t \in P$ such that

$$t \wedge y \leq b .$$

Let $c = a \vee b$ and $r = s \wedge t$. Then, of course,

$$r \wedge x \leq c \quad \text{and} \quad r \wedge y \leq c ,$$

(why?); also, $c \in I_0$ and $r \in P$, since I_0 is an ideal and P is a filter. Now [and this is the one point where we use that L is distributive!],

$$r \wedge (x \vee y) = (r \wedge x) \vee (r \wedge y) \leq c$$

$\uparrow$ distributive law $\qquad$ $\uparrow$ $\vee$ means "sup"

Since we assumed that $x \vee y \in P$, and P is a filter, it follows that $c \in P$, in contradiction to the fact that $P \cap I_0 = \emptyset$. This contradiction shows that, indeed, $x \vee y \in P$ implies that either $x \in P$, $y \in P$, showing that P is a prime filter.

This completes the proof of the PFET.

Exercise. (xxii) A *principal* filter is one of the form $\uparrow x$ (for the latter notation, see above). Show that the principal ultrafilters of $(\mathbf{P}(A), \subset)$ are in a one-to-one correspondence with the elements of the set A.

(xxiii) The set $I_{\mathrm{rat}} = \{q : 0 \leq q \leq 1$ and q is rational$\}$, with the standard ordering of the rational (real) numbers is a distributive lattice. Give a description in familiar terms of the non-principal filters of $(I_{\mathrm{rat}}, \leq)$.

(xxiv) What can we say about prime filters of a total ordering with 0 and 1 as a distributive lattice?

(xxv) If U and V are prime filters (ultrafilters) in a Boolean algebra, then $U \subset V$ implies $U = V$.

(xxvi)* Conversely, if in a distributive lattice L, we have that $P \subset Q$ implies that $P = Q$ whenever P, Q are prime filters, then L is a Boolean algebra.

(xxvii) Apply the PFET to show the following. Let A be any non-empty set, and assume that $\mathcal{F}$ is a family of subsets of A with the property that the intersection of any finitely many sets in $\mathcal{F}$ is non-empty. Show that there is an ultrafilter of $(\mathcal{P}(A), \subset)$ which contains $\mathcal{F}$.

(xxviii) The set A is finite if and only if all ultrafilters of $(\mathcal{P}(A), \subset)$ are principal.