Take-home final for Mathematical Logic 2, MATH 592, Winter 2010

Part 1. Model theory

1) Prove the Downward Lowenheim-Skolem Theorem (DLSThm) : Suppose L is a countable first-order language, M an infinite L-structure, X a subset of $|M|$, the underlying set of M, such that $\aleph_0 \le \#(X) \le \#|M|$ (I write # for cardinality). Then there is an elementary substructure N of M such that $X \subset |N|$ and $\#|N| = \# X$. ("Countable" allows "finite".).

For the proof, **prove** and use the following:

Lemma: Suppose that B is a subset of $|M|$ such that for every formula $\exists x A(x)$ in the language $L(B)$ (see below!), if $\mathfrak{A}(\exists x A(x)) = \mathsf{T}$, then there is $b \in B$ such that $\mathfrak{A}(A_x[\underline{b}]) = \mathsf{T}$. Then there is a unique elementary substructure of M whose underlying set equals B.

(The DLSThm is a purely semantical theorem. No connection to deductive logic should be invoked in its proof).

NB. $L(B)$ is the sublanguage of $L(\mathfrak{A})$ (see line -4. p.18 in Shoenfield) with constants only from $L \cup \{\underline{b} : b \in B\}$; I wrote $\underline{b}$ for the name of b. In other words, in the lemma, $\exists x A(x)$ is allowed to have parameters (names) from the set B but not from $|M| - B$.

Elementary substructures are defined and used in Shoenfield, but, apparently, the DLSThm is not mentioned.

2): Problem 5/Ch 4/pp. 65&66. [For terminology, consult Kelley: General Topology.]

3): Problems 1 and 2/Ch 5/p. 92.

Let L be a first-order language. Let Γ_L denote the set of all closed formulas of L. Let us write $\mathfrak{T}_L$ for the set $\mathfrak{T}\mathfrak{S}(\Gamma_L)$; see the Shoenfield references in 3) above. Put slightly differently: writing $e_{\mathfrak{A}}$ (evaluation at $\mathfrak{A}$) for the function $e_{\mathfrak{A}} : \Gamma_L \to \{\mathsf{T}, \mathsf{F}\}$ for which $e_{\mathfrak{A}}(A) = \mathfrak{A}(A)$ $(A \in \Gamma_L)$, $\mathfrak{T}_L$ is the set $\{ e_{\mathfrak{A}} : \mathfrak{A}$ is any L-structure$\}$. According to 2) above, $\mathfrak{T}_L$ is a compact set (closed subset of the (compact) product space $\prod_{A\in\Gamma_L}\{\{\mathsf{T},\mathsf{F}\}\}$).

Let $\mathfrak{I}$ be a class of L-structures. Let us write $\mathfrak{T}_L(\mathfrak{I})$ for the set $\{e_{\mathfrak{A}} : \mathfrak{A} \in \mathfrak{I}\}$. Call the class $\mathfrak{I}$ *compact* if $\mathfrak{T}_L(\mathfrak{I})$ is a closed (hence, compact) subset of $\mathfrak{T}_L$.

Now, let L, L' be first-order languages, $L \subset L'$. If $\mathfrak{A}$ is an L'-structure, $\mathfrak{A} \restriction L$ is the L*-reduct* of $\mathfrak{A}$: the L structure $\mathfrak{B}$ whose underlying set is that of $\mathfrak{A}$, and which interprets all L symbols (that are, as a consequence, also L'-symbols) the same way as $\mathfrak{A}$. It follows that for all L formulas A, $(\mathfrak{A}\restriction L)(A) = \mathfrak{A}(A)$. (Shoenfield calls "reduct" by the name "restriction" : line -6, p. 43)

For a class $\mathfrak{I}$ of L'-structures, $\mathfrak{I}\restriction L \overset{def}{=} \{\mathfrak{A}\restriction L : \mathfrak{A} \in \mathfrak{I}\}$. A *generalized pseudo-elementary*, or PC_Δ- class, is one of the form $\mathfrak{I}\restriction L$ for some L' and some class $\mathfrak{I}$ of L'-structures such that $L \subset L'$. ("$\subset$" allows "$=$").

4) **Prove** that every PC_Δ- class is compact.

5) A *non-standard model* of Peano arithmetic is a model (in the language $L = \{0, S, +, \times, <\}$) of Peano arithmetic that is *not* isomorphic to the standard model $(\mathbb{N}; 0, S, +, \times, <)$. **Prove** that the class of non-standard models of Peano arithmetic is a PC_Δ- class, which is *not* generalized elementary.

6) Suppose M is a model of ZFC set-theory (in the language whose only non-logical symbol $\in$, a binary relation symbol). Let $L = \{0, S, +, \times, <\}$, and define the L-structure $\mathfrak{A}(M)$ as follows. $\mathfrak{A}(M)$ is the model of arithmetic defined by M.
The universe (underlying set) of $\mathfrak{A} = \mathfrak{A}(M)$ is $\omega^M = \{x \in |M| : M \vDash$ "x is a natural number"$\}$. $0^{\mathfrak{A}} = 0^M$ = the ordinal 0 in M, $+^{\mathfrak{A}} =$ addition of finite ordinals as given in M, etc.

Prove that $\mathrm{Ar(ZFC)} = \{\mathfrak{A}(M) : M \vDash ZFC\}$ is a PC_Δ-class.

Hints: 1. It makes things easier if you (prove and) use the fact that the class $\mathfrak{I}$=Ar(ZFC) is clsed under isomorphisms: if $\mathfrak{A} \in \mathfrak{I}$ & $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{B} \in \mathfrak{I}$.

2. Use the Downward Lowenheim-Skolem Theorem to show that if there is $M \vDash ZFC$ such that $\mathfrak{A} = \mathfrak{A}(M)$, then there is $M \vDash ZFC$ such that $\mathfrak{A} \cong \mathfrak{A}(M)$ (here it is easier to prove *isomorphism* directly than *equality*, I think) and $\#|M| = \#|\mathfrak{A}|$.

Remark: I think, Ar(ZFC) is not generalized elementary, *but I cannot prove it*. Can you?

7) Problems 28, 29 and 30, Chapter 5, pp.104 and 105.

8) Generalize 30 b) as follows: Suppose $\mathfrak{I}$ is a class of L-structures, $\mathfrak{B}$ an L-structure such that $e_{\mathfrak{B}}$ is in the *closure* of the set $\{e_{\mathfrak{A}} : \mathfrak{A} \in \mathfrak{I}\}$ (subset of $\prod_{A \in \Gamma_L} \{\{\mathsf{T},\mathsf{F}\}\}$). Then $\mathfrak{B}$ is isomorphic to an elementary substructure of an ultraproduct $\prod_{i \in I} \mathfrak{A}_i / U$ of structures $\mathfrak{A}_i \in \mathfrak{I}$.

9) **Conclude** that any compact class $\mathfrak{I}$ (in particular, any PC_{Δ}- class $\mathfrak{I}$) that is closed under elementary equivalence ($\mathfrak{A} \in \mathfrak{I}$ and $\mathfrak{A} \equiv \mathfrak{B}$ imply $\mathfrak{B} \in \mathfrak{I}$) is generalized elementary.

Part 2 Recursion theory

To be solved: Problems 1. to 6. inclusive, Chapter 7, pp. 190 to 192.

Comments:

1. You will have to (re-)read Chapter 7, up to 7.4 inclusive – but no further, to do these.

2. Re Problem 1. b): At first, Sh's instruction: "[Assign an index g to each primitive recursive function G. Set $F(a,g) = \langle G \rangle (a)$ if g is an index of G, and $F(a,g) = 0$ if g is not an index. …]" sounded cryptic to me. I now understand it – but I am reluctant to tell you about it: you may say that it is obvious what he means, and therefore telling you is an insult. However, upon request I will explain. The problem is a very good one!

3. The real thing in this group of problems is the cluster of problems 4, 5 and 6. They (and also problem 7) present the "classical" theory of r.e sets, given by Emile Post and John Myhill. The main *abstract* point of the theory is to show the fact that the *non-recursive* r.e. subsets do not form a single recursive isomorphism class: there are non-recursive r.e. sets $A, B \subset \mathbb{N}$ such that there is no recursive bijection $f : \mathbb{N} \overset{\cong}{\to} \mathbb{N}$ such that $f[A] = B$. This turned out to be the tip of a huge iceberg: there are all sorts of recursive-isomorphism-invariant properties of r.e. sets distinguishing among a huge variety of r.e. – and other – sets. The properties: "simple" and "creative", treated in these problems, are historically the first such.

The problems 2. and 3. are preparations for those mentioned above. They are more general than necessary for 4. 5. and 6: you may ignore the function variables (the Greek letters α, β …) in 2. and 3. if you only want tools for the later problems.

4. General piece of advice: always think of Church's thesis first to see, at least the plausibility if not the truth, of an assertion made. For instance, problem 3. about selectors is plausible under Church's thesis. Can you see that?

5. A general statement: you may replace problems in Part 2 by other ones from Chapters 6 and 7 that you find more interesting.