

Assignment 7/MATH 338/Fall 2009

Due: Wednesday, November 11

[1] 1) Let a and b be positive coprime integers. **Prove** that the fraction $\frac{1}{ab}$ can be written in the form $\frac{1}{ab} = \frac{x}{a} + \frac{y}{b}$ with (not necessarily positive) integers x and y .

2) Suppose that the positive integers a, b, c are pairwise coprime:
 $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$. **Prove** that $\frac{1}{abc} = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$ for suitable integers x, y, z . (**Hint:** you may use 1) to prove 2).)

Quadratic irrationalities are real numbers that are obtained by starting with integers and applying repeatedly the four basic algebraic operations (addition, subtraction, multiplication and division) and also taking square-roots. Examples are the sides of the regular decagon and the regular pentagon inscribed in the circle of radius 1: look up assignment 3. For short, we'll write QI for "quadratic irrationality".

The tenth book of Euclid's Elements, the largest one of the 13 books, is about the classification of a subclass of QI's.

Ptolemy and his helpers knew how to calculate QI's; that is, they were able to find a rational number (in fact, a *regular* fraction in the Babylonian sense, one that had a finite base-60 expansion) that was an approximation of a given QI with a desired accuracy.

By definition, each rational number is a QI; the sum, the difference, the product, the quotient (if the divisor is $\neq 0$) of QI's is a QI; and the square root of a non-negative QI is a QI.

3) **Prove** that if $\sin(\alpha)$ and $\sin(\beta)$ are QI's, then so are:

$$\cos(\alpha), \sin(\alpha + \beta), \cos(\alpha + \beta), \sin(\alpha - \beta), \cos(\alpha - \beta), \sin\left(\frac{1}{2}\alpha\right), \cos\left(\frac{1}{2}\alpha\right).$$

4) Let us write Φ for 360° . Note that $90^\circ = \frac{1}{2^2}\Phi$, $72^\circ = \frac{1}{5}\Phi$, $60^\circ = \frac{1}{2 \cdot 3}\Phi$.

Prove that for any angle α of measure $\alpha = \frac{a}{2^k \cdot 3 \cdot 5}\Phi$, where a and k are non-negative integers, $\sin(\alpha)$ and $\cos(\alpha)$ are quadratic irrationalities. (**Hint:** apply parts 2) and 3).)

5) **Write** down $\sin(3^\circ)$ and $\cos(3^\circ)$ as QI's.

6) Recall that a *minute*, as a measure of an angle, is one-sixtieth of a degree; a *second* is one-sixtieth of a minute. We write x' for x minutes, y'' for y seconds. **Prove** that if α is an integer multiple of the angle $22'30''$, then $\sin(\alpha)$ is a QI.

In what follows, angles are measured in radians. If x is a real number between 0 and 2π , then it means both an angle (x radians) and the arc of the unit circle supporting the central angle (of radian measure) x .

7) **Give** a geometric proof of the fact that, for angles $0 \leq \alpha \leq \beta \leq \frac{\pi}{2}$, then $0 \leq \sin(\beta) - \sin(\alpha) \leq \beta - \alpha$. (This is easy! Just uses the fact that the shortest path between two points is the straight line. You need to draw a figure that involves all the quantities that figure in the inequality.)

8) **Prove** that, for any angle α between 0 and $\frac{\pi}{2}$, and for any positive integer n , there is a QI a such that $|\sin(a) - a| < 10^{-n}$.

[2] We have four points in 3-space: A, B, X, Y . We use the following notations for various angles:

$$\alpha = \sphericalangle XAB, \hat{\alpha} = \sphericalangle YAB, \beta = \sphericalangle XBA, \hat{\beta} = \sphericalangle YBA, \tilde{\alpha} = \sphericalangle XAY, \tilde{\beta} = \sphericalangle XBY.$$

These are all the angles with vertices A or B , the points that are considered *accessible*. The points X and Y are visible from the accessible points, but one cannot make measurements of angles whose vertices are X or Y ; and one cannot measure distances between points that are not both accessible.

1) **Draw** a sketch of the 3D situation involved in the above description.

2) **Give** a formula for the length \overline{XY} (distance between X and Y) in terms $a = \overline{AB}$, and trigonometric functions of $\alpha, \hat{\alpha}, \beta, \hat{\beta}$ and $\tilde{\alpha}$. Use the abbreviation $c(\gamma)$ for $\cos(\gamma)$ and $s(\gamma)$ for $\sin(\gamma)$ (for γ any angle).

3) **Write** down the analogous formula for \overline{XY} in terms $\alpha, \hat{\alpha}, \beta, \hat{\beta}$ and $\tilde{\beta}$ (we changed $\tilde{\alpha}$ to $\tilde{\beta}$).

Consider the angles $\alpha, \hat{\alpha}, \beta, \hat{\beta}$ constant, the angles $\tilde{\alpha}, \tilde{\beta}$ variable. This corresponds to viewing the triangles ABX and ABY rigid, but freely and independently rotating about the AB axis. The angles $\tilde{\alpha}, \tilde{\beta}$ are not independent: in fact, one determines the other.

4) Prove that $c(\tilde{\alpha})$ and $c(\tilde{\beta})$ have a linear dependence on each other: there are constants C_1, C_2, C_3 (depending, of course, on the constant quantities $\alpha, \hat{\alpha}, \beta, \hat{\beta}$) such that $C_1 \cdot c(\tilde{\alpha}) + C_2 \cdot c(\tilde{\beta}) + C_3 = 0$.

5) Suppose we have *measured* two pairs $(\tilde{\alpha}_1, \tilde{\beta}_1), (\tilde{\alpha}_2, \tilde{\beta}_2)$ of values of the angles $\tilde{\alpha}, \tilde{\beta}$, with two different relative positions of the rigid triangles ABX and ABY . **Show** that, for an arbitrary further relative position of the same triangles, involving angles $\tilde{\alpha}, \tilde{\beta}$, we can *calculate* $c(\tilde{\beta})$ from $c(\tilde{\alpha})$, by a formula that uses, in addition to $c(\tilde{\alpha})$, only the trig functions $c(\tilde{\alpha}_1), c(\tilde{\beta}_1), c(\tilde{\alpha}_2), c(\tilde{\beta}_2)$ of the measured angles $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\alpha}_2, \tilde{\beta}_2$ and nothing else (in particular, without any reference to the angles $\alpha, \hat{\alpha}, \beta, \hat{\beta}$ given by the shapes of the triangles).