Assignment 9/MATH 338/Fall 2009
Due: Thursday, December 3

[1] We imagine a Cartesian coordinate system fixed in a plane. As usual, we talk about the x-coordinate (*abscissa*), and the y-coordinate (*ordinate*) of a point.

Consider the equation

$$x^2 + y^2 + Ax + By + C = 0\,, \qquad (1)$$

where A, B, C are given constants.

1) Prove that the equation describes a circle, a single point, or the empty set, depending on how the constants A, B, C are related. In the proof, use

(i) the definition of the circle, which says that a circle is the set of points $P(x, y)$ whose distance to a fixed point $C(a,b)$ is a fixed value r,

and

(ii) the fact that the distance between points $P(x, y)$ and $C(a,b)$ is $\sqrt{(x-a)^2 + (y-b)^2}$.

2) Give formulas for the radius and the coordinates of the centre of the circle (1), valid when the constants A, B, C are appropriate (in fact, the formulas in question will be implicit in your work for 1)).

3) Give the equation, in the form (1), of the circumcircle of the triangle XYZ, where X, Y, Z are the points $X(0,0)$, $Y(1,3)$, $Z(3,1)$.

[2] **1) Prove** the following theorem: Given distinct points A and B in the plane, and a positive real number $a \neq 1$, the locus of the points P in the plane such that the ratio of the distances of P to A and B equals a ($\frac{AP}{BP} = a$) is a circle (what is the locus when $a = 1$?). **Hints:** Use a Cartesian coordinate system chosen appropriately: set up a coordinate system, including the choice of the unit length, so that the given points A and B have simple coordinates: its origin is A: $A(0,0)$; and its unit length is AB: the point B has coordinates $B(0,1)$. Refer to results mentioned in [1].

2) Take the special value $a = 2$, and by a purely geometric reasoning (using the fact stated in 1) but not its proof), **determine** the centre and the radius of the circle in question in terms of the fixed points A and B. As a second step, go back to question [1], part 2), and determine the same data using the formulas obtained there. Of course, you should be getting the same result.

Remark: I do not know of a reasonably simple purely geometric proof, not using coordinates, of the theorem stated in 1). If you have one, I will be interested in hearing about it.

3)* (for bonus marks only) This will be an exercise to show that coordinate geometry has the power to replace ordinary geometric arguments, but, sometimes, with a price to pay: the coordinate proof may become quite a bit longer than the original geometric proof.

Using coordinates, **prove** the following familiar (right?) theorem: Given distinct points A and B in the plane, and an angle α, $0<\alpha<180^\circ$, the points $P(x,y)$ in the plane such that $\sphericalangle APB=\alpha$ all lie on a single circle (they do not take up the whole circle, only an arc of it).

Hints: Start the work as in 1). Express the condition by the cosine law, applied to the triangle APB; write c for $\cos(\theta)$ to abbreviate. As it is, the expression will not look like an equation of the form (1); among others, it will contain square roots. Rearrange the equation with the square-roots on one side, the rest on the other side; take squares of both sides to clear the square roots, and simplify.

Recall the Euclidean proof of this theorem: "the central angle on a chord is double of the angle from the circumference on the same chord". Using this, it is not difficult to write down the equation of the circle. Compare that with what you got by the mechanical coordinate methods above.

Recall oblique coordinate systems, and their transformations.

(Thomas L. Heath, in his introduction of his edition of Apollonius's *Conics*, discusses at length Apollonius's geometric algebra and its uses to coordinate transformations between oblique coordinate systems; thus, the subject is well justified from the historical point of view.) The fundamental facts are as follows.

We are allowing different units on the two coordinate axes. Thus, the system is given by three points, say O, E_1 and E_2, not on a single line. O is the origin. The two coordinate axes are the lines OE_1 and OE_2; the positive rays of the coordinate axes are $\vec{OE_1}$ and $\vec{OE_2}$. The unit lengths on the two axes are the lengths $\overline{OE_1}$ and $\overline{OE_2}$.

We write e_1 for the vector $\overrightarrow{OE_1}$, e_2 for $\overrightarrow{OE_2}$.

Saying that the point P has coordinates x and y, in notation $P(x,y)$, is the same as saying that the vector equation $\overrightarrow{OP} = x\mathrm{e}_1 + y\mathrm{e}_2$ holds. The base-points E_1 and E_2 are given by coordinates as $E_1(1,0)$ and $E_2(0,1)$.

Suppose we have two oblique systems with common origin O; the first one with base-points E_1 and E_2, the second with base-points U_1 and U_2. Writing $P(x,y)$ and $P[u,v]$ indicates that the point P has coordinates x and y in the first system (the "$x,y-$system"), and coordinates u and v in the second system (the "$u,v-$system").

Fact: There are constants a,b,c,d depending on the two systems such that we have the formulas

$$x = au + bv \quad \text{and} \quad y = cu + dv; \qquad (2)$$

connecting the coordinates of the *same* variable point $P(x,y) \equiv P[u,v]$. Those of you who have seen matrices will recognize that this is the matrix equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix},$$

but it is not necessary at all to think in terms of matrices (until quite late in the history of math, people did not think in terms of matrices).

Why is this true? The points U_1 and U_2 have certain x,y coordinates: let us denote them as follows: $U_1(a,c)$ and $U_2(b,d)$. This means that the vectors u_1 and u_2 are expressed as $\mathrm{u}_1 = a\mathrm{e}_1 + c\mathrm{e}_2$ and $\mathrm{u}_2 = b\mathrm{e}_1 + d\mathrm{e}_2$. To say that we have $P[u,v]$ with u,v coordinates u,v means that $\overrightarrow{OP} = u\mathrm{u}_1 + v\mathrm{u}_2$. Therefore,

$$\overrightarrow{OP} = u\mathrm{u}_1 + v\mathrm{u}_2 = u(a\mathrm{e}_1 + c\mathrm{e}_2) + v(b\mathrm{e}_1 + d\mathrm{e}_2) = (au + bv)\mathrm{e}_1 + (cu + dv)\mathrm{e}_2$$

which *means* that $x = au + bv, \quad y = cu + dv$.

Solving the equations (2) for u and v, we pbtain

$$u = \frac{1}{ad - bc}(dx - by) \quad \text{and} \quad v = \frac{1}{ad - bc}(-cx + ay)\,. \qquad (3)$$

(Those of you who know about matrices will recognize that we have written

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

involving the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. But it is not at all necessary to use matrices to get the formulas in (3): just solve the system (2) for the unknowns u and v in the way you learnt it in high school).

If we start, conversely, with

$$u = Ax + By \quad \text{and} \quad v = Cx + Dy\,,$$

then, of course, we will have

$$x = \frac{1}{AD - BC}(Du - Bv) \quad \text{and} \quad y = -\frac{1}{AD - BC}(-Cx + Ay)\,.$$

In other words,

$$A = \frac{1}{ad - bc}d, \quad B = -\frac{1}{ad - bc}b, \quad C = -\frac{1}{ad - bc}c, \quad D = \frac{1}{ad - bc}a$$

and, of course,

$$a = \frac{1}{AD - BC}D, \quad b = -\frac{1}{AD - BC}B, \quad c = -\frac{1}{AD - BC}C, \quad d = \frac{1}{AD - BC}A\,.$$

The meaning of the constants can be also explained by saying that the y-axis is the line $au + bv = 0$ in u, v-coordinates (since this just says that $x = 0$), the u-axis is the line $Cx + Dy = 0$ (meaning $v = 0$), etc.

A *Cartesian coordinate system* is an oblique system, given by origin O and base-points E_1, E_2, such that the lines OE_1 and OE_2 are perpendicular, and the segments OE_1, OE_2 are of unit length.

As we said before, in a Cartesian system, we have a simple formula for the distance between two points $C(a,b)$ and $P(x,y)$: the distance is dist$(C,P)=\sqrt{(x-a)^2+(y-b)^2}$. If we are working in an oblique system, the distance formula becomes more complicated, and will be dependent also on the lengths of the basis vectors $\mathrm{e}_1 = \overrightarrow{OE_1}$, $\mathrm{e}_2 = \overrightarrow{OE_2}$ and the angle between them. This is the main reason why Cartesian systems are preferred.

For instance, if we have the equation $x^2+y^2=1$ in an oblique system, we cannot conclude that the equation describes a circle.

Apollonius proves that every ellipse has the equation

$$Ax^2+By^2=1 \qquad (4)$$

with suitable positive constants A and B, in a suitable (not at all uniquely determined) oblique coordinate system; and conversely, in any oblique coordinate system, and positive constants A and B, (4) describes an ellipse (remember that, for us, circles are particular ellipses).

(Similar facts are true for *hyperbolas*: then, one of A and B has to be positive, the other negative. *Parabolas* are quite different, since they are not centrally symmetric.)
Recall from class that, given any constants r, s, t, the equation

$$rx^2+sxy+ty^2=1 \qquad (5)$$

in the oblique system (x,y) describes either an ellipse, or a hyperbola, or, in degenerate cases, a pair of lines, or the empty set. The way we proved this was that we transformed, by suitably "completing a square", the equation into one of the form

$$Ru^2+Tv^2=1 \qquad (6)$$

in a new oblique system with coordinates u, v. Another "completion of squares" shows that any equation of the form

$$Ru^2+Tv^2+Mu+Nv=1$$

can be brought to the form

$$\hat{R}\hat{u}^2+\hat{T}\hat{v}^2=1$$

by a transformation of the form

$$u = \hat{u} + c_1, \quad v = \hat{v} + c_2 \quad \text{(equivalently: } \hat{u} = u - c_1, \quad \hat{v} = v - c_2 \text{)},$$

meaning that the new $(\hat{u}, \hat{v})-$system has origin $\hat{O}(c_1, c_2) \equiv \hat{O}[0,0]$, and $\hat{u}$ - and $\hat{v}$ - axes that are parallel to the u - and v -axes.

Parabolas may be defined as given by an equation

$$v = pu$$

in an oblique system with coordinates u, v.

We conclude that any equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

understood in any oblique coordinate system, represents either an ellipse, or a hyperbola, or a parabola, or, "in exceptional cases", one of several types of degenerate conics.

[3] In this problem, we have two coordinate systems, the Cartesian (x, y) -system, and the oblique (u, v) -system. The transformation from one system to the other is given by the formulas

$$x = 3u + 2v + 1, \quad y = 7u + 5v - 1. \qquad (7)$$

Notice that this is different from the way coordinates are related in (2) and (3) since here we have the constants 1 and -1 in addition to the linear combinations of the variables. This is due to the fact that the origins of the two systems are not the same. For instance, the origin $O'[0,0]$ with u, v coordinates $u_{O'} = 0, \; v_{O'} = 0$ has x, y coordinates

$$x_{O'} = 3 \cdot 0 + 2 \cdot 0 + 1 = 1, \quad \text{and} \quad y_{O'} = 7 \cdot 0 + 5 \cdot 0 - 1 = -1;$$

we have $O'(1, -1)$, with x, y coordinates indicated. There is no need, in fact, to use any general formula to answer the questions in this set. There will be a need to solve (7) for the unknowns u and v; but that should be done directly, without using any general formula.

1) Determine the equations of the circle $x^2 + y^2 + x + y = 1$ and of the line $3x + 4y = 2$ in u, v-coordinates.

2) Determine the equations of the ellipse $u^2 + v^2 = 1$ and of the line $3u + 4v = 2$ in x, y-coordinates.

3) Determine the center of the ellipse $u^2 + v^2 = 1$, both in u, v- and in x, y-coordinates.

4) Determine the equations of principal (symmetry) axes for the ellipse $u^2 + v^2 = 1$, both in u, v- and in x, y-coordinates. (**Hint:** recall Apollonius's solution to this problem: intersect the ellipse with a circle having the same center as the ellipse, to obtain a rectangle, and find the perpendicular bisectors of the sides of the rectangle.)

5) Let P be the point $P[u = \frac{3}{5}, v = \frac{4}{5}]$. **Check** that P is on the ellipse $u^2 + v^2 = 1$. **Determine** the further points Q, P', Q' on the ellipse, both in u, v- and in x, y-coordinates, such that PQ and $P'Q'$ are conjugate diameters of the ellipse.

6) Consider the point $A[u = 4, v = 5]$, and **find** the equations of the two tangents to the ellipse $u^2 + v^2 = 1$ that pass through the point A; find those equations both in u, v- and in x, y-coordinates.

7) Find the semi-axes a and b of the ellipse $u^2 + v^2 = 1$ (the lengths of the principal axes are $2a$ and $2b$).

8) Find the coordinates, both in the u, v- and the x, y- systems, of the two focus points (foci) of the ellipse $u^2 + v^2 = 1$: $S(x_S, y_S) \equiv S[u_S, v_S]$ and $S'(x_S, y_S) \equiv S'[u_S, v_S]$.

[4] We are given the equation $10x^2 + 12xy + 5y^2 = 50$ in the Cartesian coordinate system (x, y). Verify, by using a suitable oblique system, that the equation describes an ellipse, and, using Apollonius's method, find the equations (in x, y coordinates) of the principal (symmetry) axes of the ellipse, and the x, y- coordinates of its foci.