

Assignment 8/MATH 338/Winter, 2009

Due: Friday, November 20

In what follows, until otherwise stated, a, b, x, y, u, v denote (arbitrary) real numbers, c, z, w denote complex numbers.

Recall the formal definition of “complex number”: a complex number z is given by two real numbers x and y ; in short, $z \equiv (x, y)$. Accordingly, complex numbers $z \equiv (x, y)$, $w \equiv (u, v)$ are *equal*, $z = w$, if and only if $x = u$ and $y = v$. Operations on complex numbers are defined by

$$(x, y) + (u, v) = (x + u, y + v) ,$$

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu) .$$

[1] Prove, directly from the definitions, the following laws (involving complex numbers):

- 1) $(z + w) + c = z + (w + c)$
- 2) $(z + w) \cdot c = z \cdot c + w \cdot c$.
- 3) For every $z \neq (0, 0)$, there is a unique w such that $z \cdot w = (1, 0)$.
- 4) There are exactly two distinct complex values of w such that $w^2 = (-1, 0)$.
- 5) $(x, y) = (x, 0) + (0, 1) \cdot (y, 0)$.

For dealing with complex numbers, the trigonometric functions have to be used for all values of the (angle) argument, not just for acute and obtuse angles as in Ptolemaic trigonometry. In what follows, we measure angles in radians.

The definitions of $\cos(\alpha)$ and $\sin(\alpha)$ for an arbitrary value of α is given as the coordinates x and y of the point $P(x, y)$ on the unit circle with centre the origin, obtained by rotating the point $P_0(1, 0)$ through the angle α (counterclockwise if $\alpha \geq 0$, clockwise if $\alpha \leq 0$) about the origin.

The addition formulas

$$\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$$

remain valid for arbitrary values of α and β . The half-angle formulas, however, have to be modified by appropriate signs:

$$\cos\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1+\cos(\alpha)}{2}}, \quad \sin\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1-\cos(\alpha)}{2}}$$

the signs depending on α in a certain way, which is the subject of one of the questions below.

Note that if $\beta \equiv \alpha \pmod{4\pi}$, that is, $\beta - \alpha$ is an integer multiple of 4π , then

$\cos\left(\frac{\beta}{2}\right) = \cos\left(\frac{\alpha}{2}\right)$ and $\sin\left(\frac{\beta}{2}\right) = \sin\left(\frac{\alpha}{2}\right)$ (**why?**). Therefore, when investigating the values of $\cos\left(\frac{\alpha}{2}\right)$ and $\sin\left(\frac{\alpha}{2}\right)$, we may restrict attention to the range (set) $\{\alpha : 0 \leq \alpha \leq 4\pi\}$.

[2] 1) Determine, in the form of quadratic irrationalities, both $\cos(\alpha)$ and $\sin(\alpha)$ for the following values of α :

$$\frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, \frac{5\pi}{4}, -\frac{5\pi}{4}, \frac{7\pi}{4}, -\frac{7\pi}{4}, \frac{9\pi}{4}, -\frac{9\pi}{4}, \frac{11\pi}{4}, -\frac{11\pi}{4}.$$

2) Discuss fully the signs of the values of $\cos\left(\frac{\alpha}{2}\right)$ and $\sin\left(\frac{\alpha}{2}\right)$ as α ranges over the values $0 \leq \alpha \leq 4\pi$. You should come up with certain particular subsets of the set $\{\alpha : 0 \leq \alpha \leq 4\pi\}$ on which the signs $\pm$ in the half-angle formulas both have a definite value.

3) Determine as a quadratic irrationality the values of $\cos\left(\frac{\alpha}{2}\right)$ and $\sin\left(\frac{\alpha}{2}\right)$ for all the values of α listed in part 1).

We return to complex numbers, but now in their standard notation: what was $z = (x, y)$ in [1], is $z = x + iy$ from now on.

Recall the general rule for the n th roots of complex numbers. Assume $z = r \cdot e^{i\theta}$ ($r, \theta \in \mathbb{R}$, $r \geq 0$), and n is an integer ≥ 2 . If $z = 0$ (equivalently, $r = 0$), then $\sqrt[n]{z}$ has

the single value 0. If $z \neq 0$, then $\sqrt[n]{z}$ has n distinct values w_k , $k = 0, 1, \dots, n-1$, determined as follows: $w_0 = r^{(1/n)} \cdot e^{i \frac{\theta}{n}}$, and $w_k = w_0 \cdot (a_n)^k$, where $a_n = e^{\frac{2\pi i}{n}}$.

[3] 1) **Calculate** the three values of

$$\sqrt[3]{\frac{1}{1-i\sqrt{3}} + \frac{1}{1+i\sqrt{3}}}$$

precisely: each one in the form of $u + iv$, with u and v real quadratic irrationalities.

2) **Complete** the determination of the five values of $\sqrt[5]{-32}$ started in class; all values are to be written in the form $u + iv$, with both u and v real quadratic irrationalities. **Hints**: start with proving that

$$\cos\left(\frac{2\pi}{5}\right) = \cos(72^\circ) = \sin(18^\circ) = \frac{\sqrt{5}-1}{4} = \frac{c_{10}}{2}$$

and

$$\cos(36^\circ) = \cos^2(18^\circ) - \sin^2(18^\circ) = 1 - 2\sin^2(18^\circ) = \frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{c_5}{2}$$

(remember c_{10} and c_5 ?).

3) Use the calculator and **determine** two different values of $\sqrt[6]{-1+2i}$, both in the form $u + iv$, with u and v reals, determined with an accuracy of four decimals after the decimal point.

[4] Here we give formulas for the two values of the square-root of $z = x + iy$ ($x, y \in \mathbb{R}$) in terms of x and y .

1) First, assume that $x \geq 0$, $y \geq 0$ and $z \neq 0$. **Write down** appropriate *real* quadratic irrationalities u and v , in terms of x and y , such that, for $w = u + iv$, we have $w^2 = z$. w is one value of $\sqrt{z}$; what is the other value of $\sqrt{z}$?

Hints to 1): Start with the general scheme of taking n th roots, and apply it to the special case $n = 2$. In order to obtain results in terms x and y , you will need to use the half-angle formulas. The assumption $x \geq 0$, $y \geq 0$ will be used for knowing that

$\arg(z)$ is between 0 and $\frac{\pi}{2}$, and therefore both the $\cos(\arg(z)/2)$ and $\sin(\arg(z)/2)$ are non-negative, that is, the square-roots that express these two values, by the half-angle formulas, are to be taken with the positive sign.

2) Discuss the general case of the square-roots of $z = x + iy$ as follows. You will obtain expressions U and V that are quadratic irrationalities in terms of x and y such that each of the square-roots of $z = x + iy$ and of $\bar{z} = x - iy$ is of the form $(\pm U) + (\pm V)i$. (In fact, U and V are the expressions you got in part 1)). Your task is to describe precisely which signs are to be given for the square-roots in question, depending on which quadrant $z = x + iy$ is in (if $x \geq 0$, $y \geq 0$, z is in the first quadrant; if $x \leq 0$, $y \geq 0$, then z is in the second quadrant, etc. Points on the coordinate axes are counted in more than one quadrant.)

3) As an application of the above, **determine** all eight values of $\sqrt{\pm 1 \pm i}$ (two values of each of four complex numbers), each of the eight values in the form of $w = u + iv$, with u and v *real* quadratic irrationalities.