Assignment 3/Math 338/Fall, 2009
Due: Monday, October 5

[1] In the diagram below (Figure 1) of (straight) lines and points on them, *we assume* that

the triangle $\triangle ABC$ is isosceles ($\overline{AC} = \overline{BC}$),

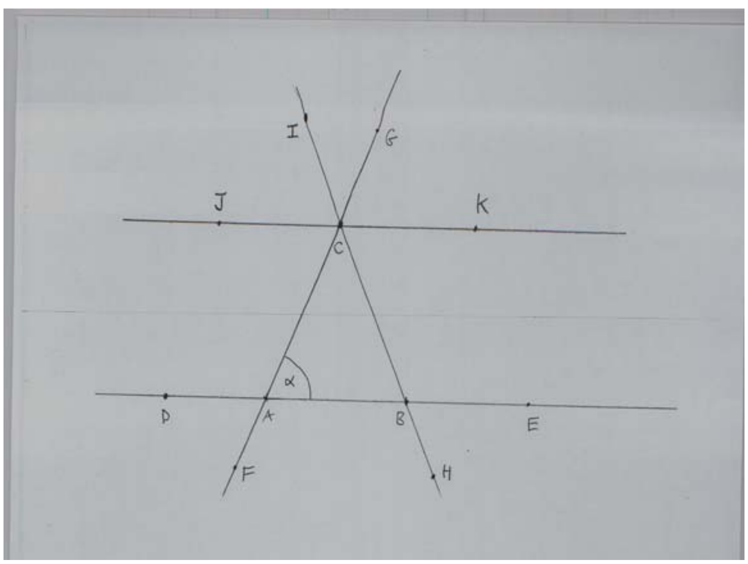
and that

the line JK is parallel to the line AB .

At each of the three points A, B, C there are several angles. For instance, from C , there are six rays emanating, to the six points A, B, G, I, J and K . Any two of these rays -- excepting the pairs on the same line such as CJ and CK , for instance -- will enclose an angle of size less than 180° . There are 12 such angles at the point C . There are several others at A and B .

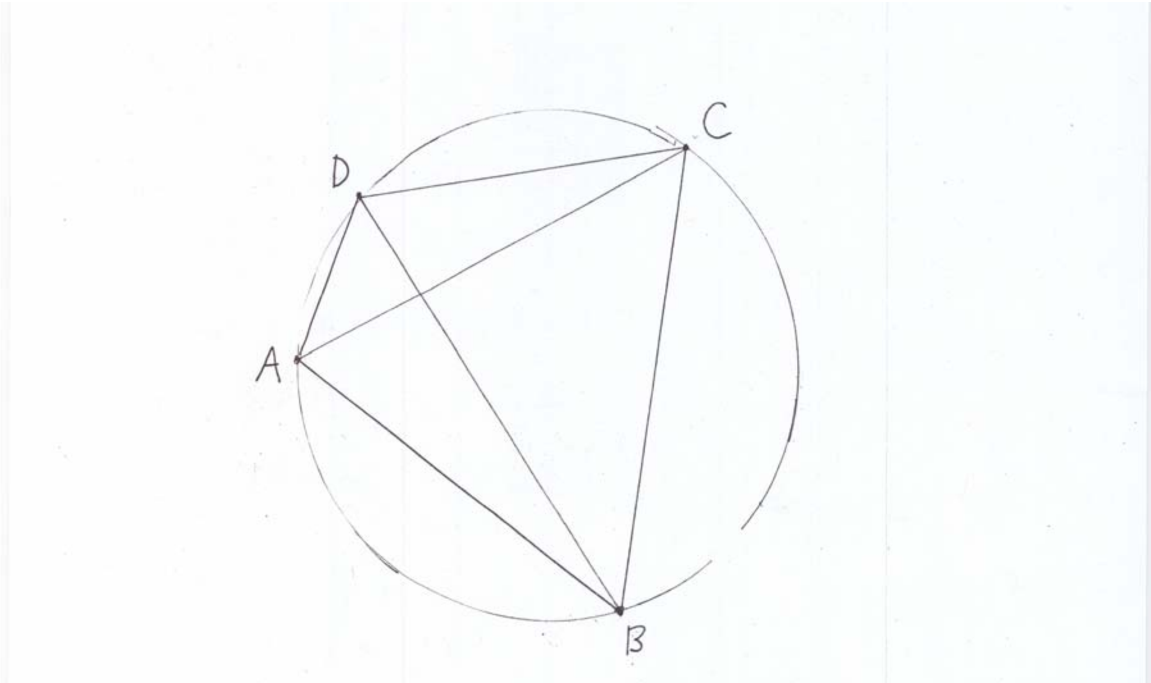
The task is to **determine** the value of each of the angles described above, at each of the three points A, B, C , as a formula in terms of the single value $\alpha = \sphericalangle CAB$. (Remember the assumptions.)

Figure 1:



[2] **1)** We learned in class that a convex quadrangle $ABCD$ is *cyclic*, that is, can be inscribed in a circle (there is a circle passing through each of its four vertices *if and only if* $\sphericalangle ACB = \sphericalangle ADB$ (see Figure 2 below). (This is, essentially, Euclid's Proposition III-21.). It follows that *if in a convex quadrangle* $ABCD$ *we have the angle equality* $\sphericalangle ACB = \sphericalangle ADB$ *of the two angles based on the side* AB*, then, as a consequence, we also have the three other angle equalities, corresponding to the three other sides* BC, CD, DA *of the quadrangle*.

Figure 2:



Write down these three angle equalities, and **briefly justify** why they indeed follow from the single one $\sphericalangle ACB = \sphericalangle ADB$. (I think this result is quite surprising, something that is far from being intuitively obvious!).

2) Use part 1) to **prove** that the *orthocenter of any given triangle is the incenter of the orthic triangle of the given triangle*. **Draw** a careful diagram, with appropriate notations, displaying the full situation.

Reminders: The *orthocenter* of a triangle is the common point of the three *altitudes* of the triangle: the three lines perpendicular to a side through a vertex. The *orthic triangle* of a triangle is the triangle whose vertices are the three feet of the altitudes, the *foot* of an altitude being its point of intersection with the side to which it is orthogonal. The *incenter* of a triangle is the common point of its three angle bisectors. The incenter of a triangle is the center of the circle that touches all three sides of the triangle from the inside (we say that a circle *touches* a line when the line is a tangent to the circle).

[3] This exercise is to calculate the side s_5 of the regular pentagon and the side s_{10} of the regular decagon in terms of the radius of circle they are inscribed in. For simplicity, we take the radius of the circle to be equal to 1.

Euclid gives the constructions of the regular pentagon and the regular decagon in Book 3; these are discussed in detail in our text.

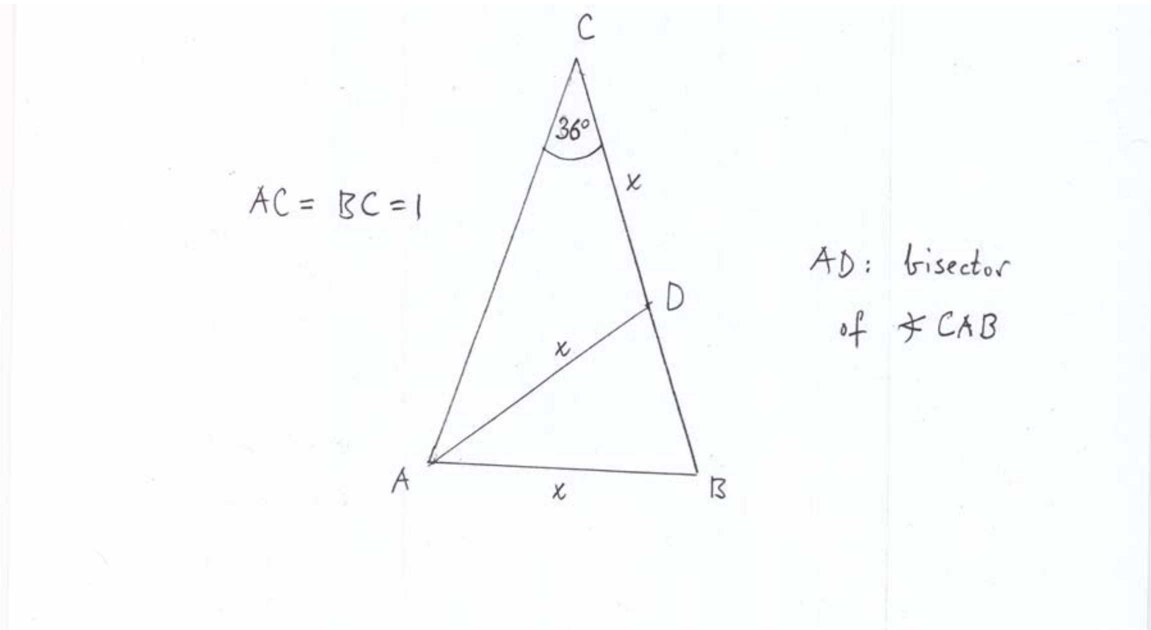
The Euclidean constructions can be turned into calculations, resulting in algebraic expressions, with possibly repeated square-roots used in addition to addition, subtraction, multiplication and division. By "calculation" here, we mean the providing of such formulas, without further "calculating" them approximately.

It turns out that s_{10} is the easier one; the calculation of s_5 uses s_{10}.

The results are as follows: $s_{10} = \frac{1}{2}(\sqrt{5}-1)$, $s_5 = \frac{1}{2}\sqrt{10-2\sqrt{5}}$. If the radius of the circle is r, these values have to be multiplied by r. (These results are also mentioned in the text, but only at a later stage).

The regular decagon is obtained by dividing the full central angle of 360° into 10 equal parts, and taking the vertices that are cut out of the circle by the 10 dividing radii. Therefore, s_{10} is the base of the isosceles triangle whose angle across the base equals 36°, and whose equal sides are equal to 1. In the following figure, we wrote x for s_{10}.

Figure 3:

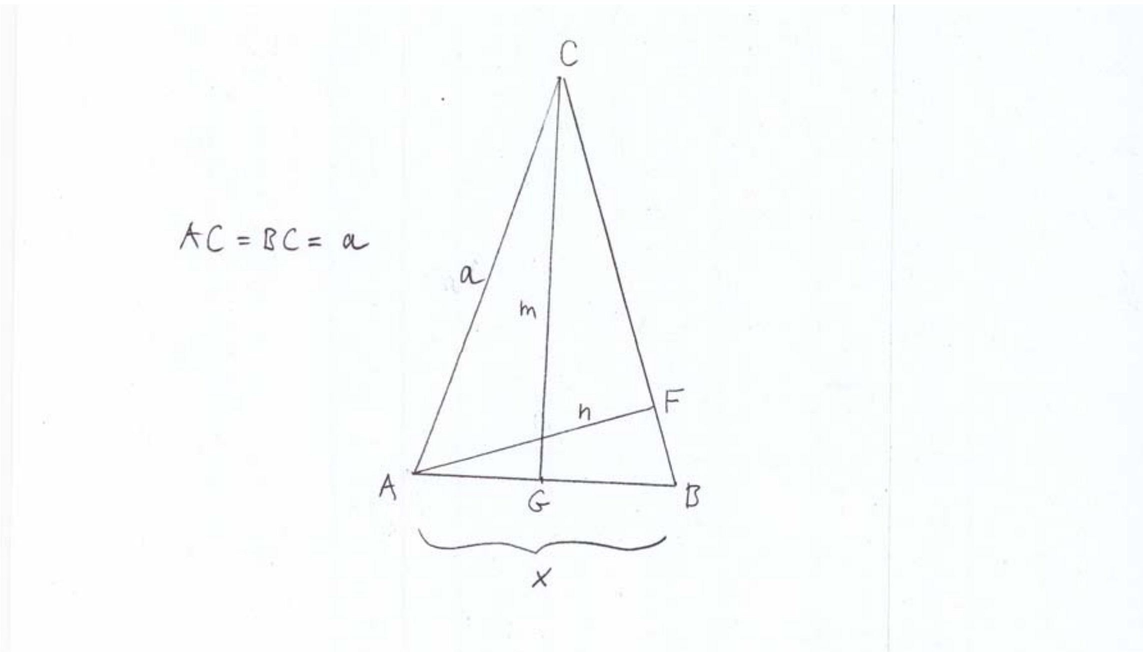


ABC is an isosceles triangle whose sides $AC = BC$ are equal to 1, and whose angle at C is equal to 36°. AD is the bisector of the angle at A.

1) Prove that **a)** $AB = AD = CD (= x)$, **b)** the triangles ABC and BDA are similar, and **c)** deduce that $s_{10} = \frac{1}{2}(\sqrt{5} - 1)$.

As a preliminary step, consider the following figure

Figure 4:



in which ABC is (any) isosceles triangle, whose equal sides are $AC = BC = a,$ the base is $AB = x$, and $m = CG$ and $n = AF$ are the two altitudes.

2) Give a formula for n in terms of a and x. (**Hint:** first obtain m, and use it to get n.)

Next:

3) Identify the triangles ABC in Figures 3 and 4, and, using parts 1) and 2), **give** the value for n (expressed algebraically, with square-roots, etc; not decimal approximation!).

4) Prove that the line-segment $AF = n$ in Figure 4, when ABC is as in Figure 3, is equal to $\frac{1}{2}s_5$.

5) Use a little algebra to show that the value obtained for n in part 3) gives, via part 4), the result $s_5 = \frac{1}{2}\sqrt{10-2\sqrt{5}}$.

[4] A *parallelogram* is a rectangle $ABCD$ whose opposite sides are parallel: $AB \parallel CD, \quad BC \parallel DA$. A *rectangle* is a parallelogram whose adjacent sides are orthogonal: $AB \perp BC$, etc. A *rhombus* is a parallelogram whose adjacent sides are of equal length: $AB = BC$. A square is a parallelogram which is both a rectangle and a rhombus. See Figure 5.

Prove the following assertions:

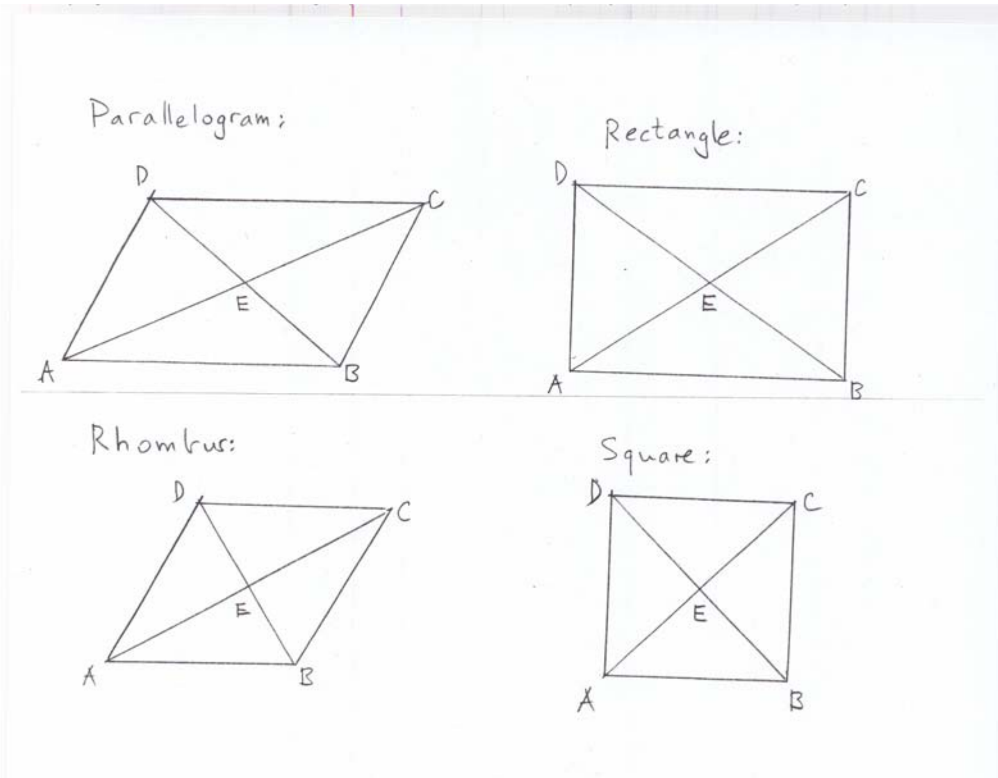
1) In a parallelogram $ABCD$, the opposite sides are equal: $AB = CD, \quad BC = DA$, and the diagonals bisect each other: $AE = EC, \quad BE = ED$.

2) A parallelogram $ABCD$ is a rectangle *if and only if* its diagonals are equal: $AC = BD$; and also, *if and only if* it can be inscribed in a circle (see problem [2] above).

3) A parallelogram $ABCD$ is a rhombus *if and only* if its diagonals are orthogonal: $AC \perp BD$.

4) A parallelogram $ABCD$ is a square *if and only* if its diagonals are orthogonal and equal: $AC \perp BD$ and $AC = BD$.

Figure 5:



Remarks: In the rest of the problems, Euclidean constructions, using straightedge and compass, are asked for. In specifying the constructions, certain elementary constructions may be used repeatedly. The latter include

drawing a line through a given point and perpendicular to, or parallel with, a given line;

copying a given line segment, or a given angle into a given (new) position;

bisecting a given angle or a given line segment;

etc.,

The "elementary constructions" should be used when needed without further explanation.

[5] Euclid gives many constructions of figures that have prescribed area and have other prescribed properties. For instance, in Proposition I-42 (not stated in the text), a parallelogram is constructed whose area equals the area of a given triangle, and in addition, (one of) the angle(s) of the parallelogram is a given angle. For simplicity, take the given angle to be a right angle.

1) Describe a construction that does what is required in Proposition I-42. Your construction may or may not be the same as Euclid's. **Give** sufficient justification for your construction.

In Proposition I-44, stated in the text, the construction of a rectangle is given whose area is equal to that of a given triangle, and one of whose sides is a given length. The construction proceeds by first replacing the given triangle by a rectangle of equal area, by I-22 discussed above. Therefore, the problem reduces to the following more reasonable problem:

Given a rectangle of sides a and b -- whose area is, therefore, ab --, construct a rectangle with a given side c and unknown side x whose area cx equals ab.

In other words, the problem is the geometric solution of the algebraic equation $cx = ab$; or what is the same, constructing length x that fits the proportion $x : a = b : c$. This construction is called the "construction of the fourth proportional".

2) Describe a construction that does what is required in the last paragraph. Your construction may or may not be the same as Euclid's (which is not described in the text). **Give** sufficient justification for your construction.

3) Here is what is happening in Euclid's Proposition I-45, an obviously very significant proposition.

Suppose an arbitrary polygon is given. **Construct** *a rectangle whose area equals the area of the given polygon.*

In you drawings for the question, start with an (irregular) pentagon as the given polygon.

(**Hints:** Subdivide the pentagon into triangles. Choose an arbitrary line segment a as one of the sides of rectangles, each of which is equal in area to one of the triangles. Since the rectangles agree in one side, they can be fitted together to form a single rectangle.)

Remark: Notice that thereby you determined by construction the area of an arbitrary polygon as being equal to $a \cdot (b_1 + b_2 + ...) = a \cdot b$, without using any formulas or measurements!

4) The construction of the square root of a given quantity is described in Proposition II-14; this is discussed on page 64 of the text. Justify the construction as given by using similarity, rather than areas as is done in Book II of Euclid. Explain that the construction is that of the "mean proportional" x proportion $c:x = x:d$, with given c and d.

5) Explain how to construct a square whose area equals that of a given, arbitrary, polygon.

[6](for bonus marks) Give Euclidean constructions, with full justifications, for the following problems. Hints are available upon demand.

1) Construct a triangle from the lengths of its three altitudes.

2) Given four points A, B, C, D in position, and two angles α and β by size, construct a point P such that $\sphericalangle APB = \alpha$ and $\sphericalangle CPD = \beta$.

3) Given a line l in position, and two lengths a and b, construct a point P on the line l such that the lengths AP and BP are in the given ratio: $AP:BP = a:b$.

Each of the above problems will or will not have a solution depending on the data provided. The construction should be successful when a solution exists; but no discussion of the conditions necessary for the existence of the solution is asked for.