

**Assignment 2/MATH 338/Fall 2009**  
**Due: Wednesday, September 23**

**[1] 1) Calculate**  $\sqrt{7}$  and  $\sqrt{11}$  with an accuracy of  $10^{-10}$ : find a simple fraction  $\frac{p}{q}$ , with  $p$  and  $q$  integers, such that  $\sqrt{7} < \frac{p}{q}$  and  $\frac{p}{q} - \sqrt{7} < 10^{-10}$ ; and similarly for  $\sqrt{11}$ . **Prove** that your answer is correct, *without* calculating the square roots by some other method, or by calculator. Follow the instructions.

**Instructions:** The method to be used is the so-called Newton's method; there is evidence that the Babylonian's were aware of the essentials of the method.

The general method, *as we learned in class*, is as follows. Start with a non-square integer  $N$ , and a rational number  $a$  such that  $\sqrt{N} < a$ . We define the following quantities:

$$a_0 = a, \quad b_0 = b = a^2 - N, \quad c_0 = \frac{b}{2a}$$

and, when  $n = 0, 1, 2, 3, \dots$  and  $a_n$ ,  $b_n$  and  $c_n$  are already given, then we define

$a_{n+1}$ ,  $b_{n+1}$  and  $c_{n+1}$  by

$$a_{n+1} = a_n - c_n, \quad b_{n+1} = a_{n+1}^2 - N = c_n^2, \quad c_{n+1} = \frac{b_{n+1}}{2a_{n+1}} = \frac{c_n^2}{2a_{n+1}}.$$

Then the rational numbers  $a_0, a_1, a_2, \dots, a_n, \dots$  are better and better approximations of

$\sqrt{N}$ :  $\sqrt{N} < \dots < a_n < \dots < a_2 < a_1 < a_0$ , and we have  $b_{n+1} < \frac{b_n^2}{2N}$ , and also

$$b_n < \frac{b^{2^n}}{(4N)^{2^n - 1}}. \quad (1)$$

For estimating the error  $e_n = a_n - \sqrt{N}$ , let  $d$  be largest integer such that  $d^2 < N$ , and use the inequality

$$e_n < \frac{b_n}{2d}. \quad (2)$$

For “best results” (see part 3) below), **choose**  $a = \frac{p}{q}$ , the starting approximation, in such a manner that  $p^2 - q^2N = 1$ . For the examples  $N = 7$  and  $11$ , this can be done by easy trial and error: one tries the numbers  $q = 1, 2, 3, \dots$  one by one until  $1 + q^2N$  becomes a complete square  $p^2$ . It is an important theorem, one that we will learn more about later, in connection with Indian mathematics, *that for any non-square integer  $N$ , there are, in fact infinitely many, pairs  $(p, q)$  of integers such that  $p^2 - q^2N = 1$ .*

**2) Determine** an integer  $n$  such that, for  $N = 7$ , and your choice of the starting approximation  $a$ , the error  $e_n$  of the approximation is less than  $10^{-40}$  (that is, the approximation  $a_n$  is correct for the first 40 decimals). *Do not* calculate  $a_n$  itself.

**3) (for bonus marks) Prove**, for a general non-square integer  $N$ , that if the starting approximation  $a = \frac{p}{q}$  is chosen as stated at the end of part 1), then for all  $n$ ,  $b_n$  and  $c_n$  are all unit fractions: of the form  $\frac{1}{s}$  for an integer  $s$ .

**[2] Find** a primitive Pythagorean triple  $(a, b, c)$  such that  $100 < a$  and  $\frac{3}{2}a < b < 2a$ , and one such that  $100 < a$  and  $2a < b < \frac{5}{2}a$ . Use the formulas  $\hat{a} = \frac{\hat{b}}{2}(t - \frac{1}{t})$ ,  $\hat{c} = \frac{\hat{b}}{2}(t + \frac{1}{t})$  learned in class, to get suitable rational triples  $(\hat{a}, \hat{b}, \hat{c})$ , and obtain the integers  $(a, b, c)$  by scaling  $(\hat{a}, \hat{b}, \hat{c})$  with a common denominator. Remember that “primitive” means that the integers  $a, b, c$  have no common non-unit factor. Some trial and error in choosing the values for  $\hat{b}$  and  $\hat{t}$  is appropriate in this case.

**[3]** We have a right triangle with legs  $a$  and  $b$ , and an *isosceles* right triangle with side  $e$ . We are given that the hypotenuses of the two triangles are equal. **Find** a method that will give you all triples  $(a, b, e)$  of integers that will fit this description, and give two primitive examples with each number being at least 50 (and it is cheating to give  $(a, b, e)$  and  $(b, a, e)$ ).

**Hints:** Note that it is not required that the hypotenuse be an integer.

We apply a method similar to what happened for the Pythagorean triples. Write down the equation that the three numbers  $a, b, e$  satisfy, and re-write it so that, on both sides, you have a difference of two squares. Be sure that you don't write  $\sqrt{2}$  anywhere; everything has to be rational. Write each difference of squares as a product. Re-write the equation as the equation of two ratios. Write the new parameter  $t$  for the equal ratios. Choose  $t$  and one of the three quantities  $a, b, e$ , and express rationally (no square roots!) the other two of the quantities  $a, b, e$ . You have obtained a parametrization of all rational solutions of the problem. The integer triples are obtained by scaling as in the case of the Pythagorean triples.

**[4]** Here are two methods for converting a simple fraction  $a = \frac{r}{s}$  into the sum of an integer and distinct unit fractions. Note that both methods provide a systematic way of doing division of integers in the Egyptian system of notation.

*First method:* called the greedy algorithm. If  $a > 1$ , then let  $b$  be the largest integer less than (or equal)  $a$ , and, instead of  $a$ , consider the number  $a - b$ , which is now less than 1. Assuming that  $a < 1$ , find the *largest* unit fraction  $\frac{1}{p}$  such that  $\frac{1}{p} < a$ . This means finding the *least* integer  $p$  such that  $p \geq \frac{1}{a}$ ; this is called the *ceiling* of  $\frac{1}{a}$ . (The ceiling of  $x$  is denoted  $\lceil x \rceil$ , except that the lower hooks are missing (sorry; I don't have the symbol!)). Pass to the number  $\hat{a} = a - \frac{1}{p}$ , and repeat the above for  $\hat{a}$ , etc, until you get zero.

**Example 1:**  $a = \frac{75}{64}$ . We have  $a = 1 + \frac{11}{64}$ ; we work with  $\frac{11}{64}$ . Find the ceiling of  $\frac{64}{11}$ :

$\frac{64}{11} = 5 + \frac{9}{11}$ , therefore  $\left\lceil \frac{64}{11} \right\rceil = 6 = p$ . Next, we take

$$\frac{11}{64} - \frac{1}{6} = \frac{66 - 64}{6 \cdot 64} = \frac{2}{6 \cdot 64} = \frac{1}{3 \cdot 64} = \frac{1}{192}.$$

Actually, this is now the end, since  $\frac{75}{64} = 1 + \frac{1}{6} + \frac{1}{192}$ .

**Example 2:**  $a = \frac{75}{62}$ . We have  $a = 1 + \frac{13}{62}$ ; we work with

$$a_0 = \frac{13}{62}.$$

Find the ceiling of  $\frac{1}{a_0} = \frac{62}{13}$ :  $\frac{62}{13} = 4 + \frac{10}{13}$ , therefore  $\left\lceil \frac{62}{13} \right\rceil = 5 = p_1$ . Next, we take

$$a_1 = a_0 - \frac{1}{p_1} = \frac{13}{62} - \frac{1}{5} = \frac{65 - 62}{5 \cdot 62} = \frac{3}{5 \cdot 62}.$$

We have to continue:  $\frac{1}{a_1} = \frac{5 \cdot 62}{3} = \frac{310}{3} = 103 + \frac{1}{3}$ ;  $\left\lceil \frac{5 \cdot 62}{3} \right\rceil = 104 = p_2$ , and

$$a_2 = a_1 - \frac{1}{p_2} = \frac{3}{5 \cdot 62} - \frac{1}{104} = \frac{3 \cdot 52 - 5 \cdot 31}{5 \cdot 62 \cdot 52} = \frac{156 - 155}{16,120} = \frac{1}{16,120}.$$

This finishes, and we have  $\frac{75}{62} = 1 + \frac{1}{5} + \frac{1}{104} + \frac{1}{16,120} = 1 \overline{5} \overline{104} \overline{16120}$ .

*Second method:* this is closer to what the Egyptians were actually doing. With the given  $a = \frac{r}{s}$ , and already assuming that  $a < 1$ , we write  $r$  as a sum of distinct powers of 2; in other words, we write in base-2. (This is certainly the basic piece of knowledge that the Egyptians had and used.) Then  $a$  becomes the sum of distinct fractions of the form  $\frac{2^k}{s}$ . In each term, we can then reduce until the denominator is odd (possibly =1). To deal

with a fraction of the form  $\frac{2^k}{s}$  where  $s$  is odd, we use, possibly repeatedly, the following formula:

$$\frac{1}{2n-1} = \frac{1}{n} + \frac{1}{n(2n-1)}$$

(check!). It may be necessary to do some adjustment when, in a resulting sum, equal terms appear; but eventually, we get a sum of distinct unit fractions.

**Example 1:**  $a = \frac{75}{64} = 1 + \frac{11}{64} = 1 + \frac{8+2+1}{64} = 1 + \frac{8}{64} + \frac{2}{64} + \frac{1}{64} = 1 + \frac{1}{8} + \frac{1}{32} + \frac{1}{64} = 1 \overline{8} \overline{32} \overline{64}$ .

**Example 2:**  $a = \frac{75}{62} = 1 + \frac{13}{62} = 1 + \frac{8+4+1}{62} = 1 + \frac{8}{62} + \frac{4}{62} + \frac{1}{62} = 1 + \frac{4}{31} + \frac{2}{31} + \frac{1}{62}.$

We now take  $\frac{2}{31}$ , and apply the above formula:  $\frac{2}{31} = \frac{2}{2 \cdot 16 - 1} = \frac{1}{16} + \frac{1}{16 \cdot 31}.$

Therefore,  $\frac{4}{31} = 2 \cdot \left(\frac{1}{16} + \frac{1}{16 \cdot 31}\right) = \frac{1}{8} + \frac{1}{8 \cdot 31}.$

Finally,  $a = \frac{75}{62} = 1 + \frac{1}{8} + \frac{1}{8 \cdot 31} + \frac{1}{16} + \frac{1}{16 \cdot 31} + \frac{1}{62} = 1 \frac{1}{8} \frac{1}{16} \frac{1}{62} \frac{1}{248} \frac{1}{496}.$

**Questions: 1):** Apply both methods to the following examples:

1.1)  $a = \frac{311}{120},$       1.2)  $a = \frac{311}{116}.$

**2) (for bonus marks)** Prove that the greedy algorithm applied to  $a = \frac{p}{q}$  will give  $a$  as the sum of at most  $p$  unit fractions.

**[5] Read** Chapter 2 of the text (this is a short one), and **write** an essay of about 800 words on the contents of the chapter.