McGill University/Department of Mathematics and Statistics
History and Philosophy of Mathematics/Midterm examination
November 2, 2009

1. All questions are worth the same marks.
2. Choose 5 (five) out of the ten questions to answer, with the restriction that at least 2 (two) should be chosen from each of Group 1 (questions [1] to [5]) and Group 2 (questions [6] to [10]).
3. Calculators are allowed. However, only integers are to be calculated directly by the calculator.
4. Results are accepted as satisfactory only if they are justified. Correct answers may receive nil marks if unjustified.

Group 1: choose at least two, and at most three from questions [1] to [5] to answer.

[1] **Describe** the Euclidean construction of a square whose area equals the area of a prescribed, but arbitrary (not necessarily regular) pentagon.

Draw figures with appropriate notation. Do not describe elementary constructions such as the bisecting of a line segment, or drawing of a perpendicular to a given line, etc. Do not give any proof. Use words sparingly.

[2] **1) Give** a proof of the Pythagorean theorem using laws of similarity and algebra. Do not refer to areas.

2) Give a proof of the Pythagorean theorem using areas, with as little algebra and as few words as possible.

[3] **Give** a formula forthe length of the side of the regular pentagon inscribed in the circle of radius 1. **Prove** that you formula is correct.

The formula should use integers and algebraic operations, including square roots, but it need not be as short as possible. Do not spend time with algebra in this case.

[4] **Formulate** Euclid's definition (Elements, Book V, Definition 5) of equality of ratios of like quantities (ratio of two lengths, ratio of two areas). Use Euclid's definition to **prove** that the areas of two rectangles sharing one side are to each other as their sides perpendicular to the shared side.

In the proof, use words sparingly. Use figures with appropriate notation, involving examples of the integers involved in Euclid's definition. Steps in the proof that are similar to previously done steps should not be repeated in detail.

[5] **1) Formulate** Euclid's parallel postulate (Elements, Book I, Postulate 5), in terms close Euclid's original wording.

2) Define "parallel lines", and formulate a theorem of the equality of certain angles formed by two parallel lines and a line intersecting them.

3) Give a proof the theorem in 2) based on the parallel postulate.

4) State and prove the theorem concerning the sum of the internal angles of a triangle, and the analogous result on quadrangles.

Use words sparingly. Use figures with notation explained.

Group 2: choose at least two, and at most three from questions [6] to [10] to answer.

[6] **1) Give** a system of three formulas, using two integer parameters, that always provides a Pythagorean triple of integers. **Verify** that indeed, the triple generated is always a Pythagorean triple.

2) State and prove a theorem to the effect that, by certain choices of the parameters, the formulas in 1) always yield primitive Pythagorean triples.

3) Determine three primitive Pythagorean triples (a,b,c) such that $a > \frac{1}{2}c$, $b > \frac{1}{2}c$ and $c > 50$ (it is cheating to give (a,b,c) together with (b,a,c)).

[7] **1) Calculate** the first three terms of the sum of unit fractions that the greedy algorithm gives when applied to the fraction $\frac{n}{2n+1}$, using the assumption that $n \geq 451$.

2) Give the full greedy-algorithm unit-fraction expansion of $\frac{n}{2n+1}$ for $n = 451$ and for $n = 452$. (You may find easier to do 2) before 1); however, if 1) is done, 2) requires almost no additional calculation.)

[8] **1) Perform** the multiplication $[47,32;09,16]_{60} \times [51;51]_{60}$ of numbers given in base-60 notation. The operations should be performed within the base-60 system as far as feasible. Converting to, and doing the operations in, base 10 (possibly by a calculator), and then reconverting, is not a legitimate method here.

2) Determine the base-60 expansion of the reciprocal fraction $\frac{1}{5^4}$ (the denominator is an exponential expression). Unlike 1), here any method is allowed – but it has to be given in full detail and justification.

[9] Let $a = 11790$, $b = 5715$, $c = 700$.

1) Determine the general solution of the Diophantine equation

$$ax + by = 180 \qquad (x, y \in \mathbb{Z}).$$

2) Determine a particular solution of the Diophantine equation

$$ax + by + cz = 15 \qquad (x, y, z \in \mathbb{Z}).$$

[10] **1) Describe** the Euclidean algorithm for calculating the gcd (greatest common divisor) of two positive integers in precise and general terms. The explanation should not be given on an example. Use a general notation; for instance, $a_0, a_1, ..., a_n$ for the sequence generated by the algorithm. For this part, do not give any proof.

2) Prove, using 1), the formula $\gcd(ca, cb) = c \cdot \gcd(a,b)$.

3) Deduce, as a consequence of 2), Euclid's theorem: if a prime p divides the product $a \cdot b$ of two positive integers, then p divides at least one of a and b.