[1] $\boxed{\forall x(Pxy \wedge Rxy) \vdash \forall x Pxy \wedge \forall x Rxy}$:

Informal: "Assume premiss. We want to prove two things: 1. $\forall x Pxy$, 2. $\forall x Rxy$. For 1: let x be arbitrary element (of U), to show Pxy. Apply premiss to this x, to obtain $Pxy \wedge Rxy$. It follows that Pxy. Done. Proving 2. is similar"

Formal:

1	1	$\forall x(Pxy \wedge Rxy)$	P
1	2	$Pxy \wedge Rxy$	US:1
1	3	Pxy	T:2
1	4	$\forall x Pxy$	UG:3 (x not free in 1)
1	5	Rxy	T:2
1	6	$\forall x Rxy$	UG:5 (x not free in 1)
1	7	$\forall x Pxy \wedge \forall x Rxy$	T:4,6

$\boxed{\forall x Pxy \wedge \forall x Rxy \vdash \forall x(Pxy \wedge Rxy)}$:

Informal: "Assume premiss; let x be arbitrary, to show $Pxy \wedge Rxy$. By premiss, we have $\forall x Pxy$, hence, applied to x, Pxy. We get Rxy similarly. We have shown $Pxy \wedge Rxy$."

Formal:

1	1	$\forall x Pxy \wedge \forall x Rxy$	P
1	2	$\forall x Pxy$	T:1
1	3	Pxy	US:2
1	4	$\forall x Rxy$	T:1
1	5	Rxy	US:4
1	6	$Pxy \wedge Rxy$	T:3,5
1	7	$\forall x(Pxy \wedge Rxy)$	UG:6 (x not free in 1)

[2] $\boxed{\forall x(Py \vee Rxy) \vdash Py \vee \forall x Rxy}$:

Informal: see "Hints" on question sheet.

Formal:

1	1	$\forall x(Py \vee Rxy)$	P
2	2	$\overline{Py}$	P
1	3	$Py \vee Rxy$	US: 1
1, 2	4	Rxy	T: 2, 3 [$\bar{A}, A \vee B \vdash B$: ✓]
1, 2	5	$\forall x Rxy$	UG: 4 [x not free in 1, 2 !]
1	6	$\overline{P}y \to \forall x Rxy$	D: 5
1	7	$Py \vee \forall x Rxy$	T: 6 [$\bar{A} \to B \vdash A \vee B$: ✓]

$\boxed{Py \vee \forall x Rxy \vdash \forall x(Py \vee Rxy)}$:

Informal: omitted

Formal:

1	1	$Py \vee \forall x Rxy$	P
2	2	Py	P (case 1)
3	3	$\forall x Rxy$	P (case 2)
3	4	Rxy	US: 3
3	5	$Py \vee Rxy$	T: 4 [$B \vdash A \vee B$: ✓]
2	6	$Py \vee Rxy$	T: 2 [$A \vdash A \vee B$: ✓]
1	7	$Py \vee Rxy$	AC: 5, 6
1	8	$\forall x(Py \vee Rxy)$	UG: 7 (x not free in 1)

[3] $\boxed{\exists x(Pxy \vee Rxy) \vdash \exists x Pxy \vee \exists x Rxy}$:

Informal: "Assume premises. Hence, there is x such that either Pxy (Case 1) or Rxy (Case 2). In Case 1, we have $\exists x Pxy$ hence $\exists x Pxy \vee \exists x Rxy$. Similarly, in Case 2, we conclude the same"

Formal:

1	1	$\exists x(Pxy \vee Rxy)$	P
2	2	$Pxy \vee Rxy$	P
3	3	Pxy	P (Case 1)
4	4	Rxy	P (Case 2)
3	5	$\exists x Pxy$	EG: 3
4	6	$\exists x Rxy$	EG: 4
3	7	$\exists x Pxy \vee \exists x Rxy$	T: 5
4	8	$\exists x Pxy \vee \exists x Rxy$	T: 6
2	9	$\exists x Pxy \vee \exists x Rxy$	AC: 7,8
1	10	$\exists x Pxy \vee \exists x Rxy$	ES: 9

[x not free in 10]

$\boxed{\exists x Pxy \vee \exists x Rxy \vdash \exists x(Pxy \vee Rxy)}$:

Informal: omitted

Formal:

1	1	$\exists x Pxy \vee \exists x Rxy$	P
2	2	$\exists x Pxy$	P (Case 1)
3	3	$\exists x Rxy$	P (Case 2)
4	4	Pxy	P
5	5	Rxy	P
4	6	$Pxy \vee Rxy$	T: 4
5	7	$Pxy \vee Rxy$	T: 5
4	8	$\exists x(Pxy \vee Rxy)$	EG: 6
5	9	$\exists x(Pxy \vee Rxy)$	EG: 7
2	10	$\exists x(Pxy \vee Rxy)$	ES: 8

[x not free in 8]

3	11	$\exists x(Pxy \vee Rxy)$	ES: 9 [x not free in 9]
1	12	$\exists x(Pxy \vee Rxy)$	AC: 10, 11

[14] $\boxed{\exists x(Py \wedge Rxy) \vdash Py \wedge \exists x Rxy}$:

Informal: omitted

Formal:

1	1	$\exists x(Py \wedge Rxy)$	P
2	2	$Py \wedge Rxy$	P
2	3	Py	T: 2
2	4	Rxy	T: 2
2	5	$\exists x Rxy$	EG: 4
2	6	$Py \wedge \exists x Rxy$	T: 3, 5
1	7	$Py \wedge \exists x Rxy$	ES: 6 [x not free in 1, 6]

$\boxed{Py \wedge \exists x Rxy \vdash \exists x(Py \wedge Rxy)}$:

Informal: omitted

Formal:

1	1	$Py \wedge \exists x Rxy$	P
1	2	$\exists x Rxy$	T: 1
3	3	Rxy	P
1, 3	4	$Py \wedge Rxy$	T: 1, 3
1, 3	5	$\exists x(Py \wedge Rxy)$	EG: 4
1	6	$\exists x(Py \wedge Rxy)$	ES: 5 (x not free in 1, 5)

[15] $\boxed{\forall x Pxy \vee \forall x Rxy \vdash \forall x(Pxy \vee Rxy)}$:

Informal: omitted

Formal:

1	1	$\forall x Pxy \vee \forall x Rxy$	1

2	2	$\forall x Pxy$	P (<u>Case 1</u>)
3	3	$\forall x Rxy$	P (<u>Case 2</u>)
2	4	Pxy	US: 2
3	5	Rxy	US: 3
2	6	$Pxy \vee Rxy$	T: 4
3	7	$Pxy \vee Rxy$	T: 5
1	8	$Pxy \vee Rxy$	AC: 6, 7
1	9	$\forall x(Pxy \vee Rxy)$	UG: 8 [x not free in 1]

[6] $\exists x(Pxy \wedge Rxy) \vdash \exists x Pxy \wedge \exists x Rxy$

<u>Informal</u>: omitted

<u>Formal</u>:

1	1	$\exists x(Pxy \wedge Rxy)$	P
2	2	$Pxy \wedge Rxy$	P
2	3	Pxy	T: 2
2	4	$\exists x Pxy$	EG: 3
2	5	Rxy	T: 2
2	6	$\exists x Rxy$	EG: 5
2	7	$\exists x Pxy \wedge \exists x Rxy$	T: 4, 6
1	8	$\exists x Pxy \wedge \exists x Rxy$	ES: 7 (x not free in 7)

[7] <u>Informal</u>: omitted

<u>Formal</u>:

1	1	$\forall x \exists y (x = fy)$	P
1	2	$\exists y (x = fy)$	US: 1
1	3]z	$\exists z (x = fz)$	CBV: 2
4	4	$x = fz$	P
1	5	$\exists y (z = fy)$	US: 1 [z for x]

6	6 ⌋ y	$z = fy$	P
4, 6	7	$x = ffy$	E : 4,6
4, 6	8	$\exists y(x = ffy)$	EG : 7
4, 1	9	$\exists y(x = ffy)$	ES : 8 (active : y; not free in 4 and (9) 6 is traded in for 5)
1	10	$\exists y(x = ffy)$	ES : 9 (active : z; not free in 1)
1	11	$\forall x \exists y(x = ffy)$	UG (active : x; not free in 1)

[8]

	1	1	$\forall x \exists y(x = fy \vee x = gy)$	P	
	2	2	$\forall x R(fx, x)$	P	
	3	3	$\forall x R(gx, x)$	P	
	1	4 ⌉ y	$\exists y(x = fy \vee x = gy)$	US : 1	
	5	5 ⌋ ⌉	$x = fy \vee x = gy$	P	
	6	6 (AC)	$x = fy$	P	
	7	7 ⌋	$x = gy$	P	
	2	8	$R(fy, y)$	US : 2 (y for x)	
	6,2	9	$R(x, y)$	E : 6,8	
	6,2	10	$\exists y R(x, y)$	EG : 9	
	3	11	$R(gy, y)$	US : 3 (y for x)	
	7,3	12	$R(x, y)$	E : 7,11	
	7,3	13	$\exists y R(x, y)$	EG : 12	
!	5,2,3	14	$\exists y R(x, y)$	AC : 10, 13	!
	1,2,3	15	$\exists y R(x, y)$	ES : 14 (active : y)	
	1,2,3	16	$\forall x \exists y R(x, y)$	UG : 15	

[9] (i) $\exists y Rxy \vdash \neg \forall y \neg Rxy$:

1	1	$\exists y Rxy$	P
2	2	$\forall y \neg Rxy$	P
2	3	$\neg Rxy$	US: 2
4	4	Rxy	P
4	5	$\neg \forall y \neg Rxy$	C: 3, 4

Explanation of last inference:

$$\underbrace{Rxy \vdash Rxy}_{\text{line 4}} \qquad\qquad \underbrace{\forall y \neg Rxy \vdash \neg Rxy}_{\text{line 3}}$$

by weakening (W): $\Downarrow (W)$ $\qquad\qquad \Downarrow (W)$

$$C\ \dfrac{Rxy,\ \forall y \neg Rxy \vdash Rxy \qquad Rxy,\ \forall y \neg Rxy \vdash \neg Rxy}{Rxy \vdash \neg \forall y \neg Rxy}$$

1	6	$\neg \forall y \neg Rxy$	ES: 5

(ii) $\neg \forall y \neg Rxy \vdash \exists y Rxy$:

1	1	$\neg \forall y \neg Rxy$	P
2	2	$\neg \exists y Rxy$	P
3	3	Rxy	P
3	4	$\exists y Rxy$	EG: 3
2	5	$\neg Rxy$	C: 2, 4
2	6	$\forall y \neg Rxy$	UG: 5
1	7	$\exists y Rxy$	C: 1, 6

[10] (i) $\forall y Rxy \vdash \neg \exists y \neg Rxy$

1	1	$\forall x\, Rxy$	P
2	2	$\exists y \neg Rxy$	P
3	3	$\neg Rxy$	P
1	4	Rxy	US : 1
1, 3	5	$\neg \exists y \neg Rxy$	C : 3, 4
1, 2	6	$\neg \exists y \neg Rxy$	ES : 5
1	7	$\neg \exists y \neg Rxy$	C : 2, 6

(ii) $\neg \exists y \neg Rxy \vdash \forall y Rxy$:

1	1	$\neg \exists y \neg Rxy$	P
2	2	$\neg Rxy$	P
2	3	$\exists y \neg Rxy$	EG : 2
1	4	$\neg\neg Rxy$	C : 1, 3
1	5	Rxy	T : 4
1	6	$\forall y Rxy$	UG : 5

!! [11], [12] : at END !!

[13] $\forall xyz\,(x+y=z \leftrightarrow ADDxyz) \vdash \forall xyz\,((x+y)+z = x+(y+z))$

$\leftrightarrow \forall xyz\, \exists uvw\, (ADDxyu \wedge ADDyzv \wedge ADDuzw \wedge ADDxvw)$

1	1	$\forall xyz\,(x+y=z \leftrightarrow ADDxyz)$	P	
2	2	$\forall xyz\,((x+y)+z = x+(y+z))$	P	
$\emptyset$	3	$x+y = x+y$	E	
$\emptyset$	4	$\exists u(x+y = u)$	EG : 3	
5	5	$x+y = u$	P	←
$\emptyset$	6	$y+z = y+z$	E	
$\emptyset$	7	$\exists v(y+z = v)$	EG : 7	
8	8	$y+z = v$	P	←

$\emptyset$	9	$u+z = u+z$	E
$\emptyset$	10	$\exists w(u+z = w)$	EG: 9
11	11 $]w$	$u+z = w$	P ←
2	12	$(x+y)+z = x+(y+z)$	US: 2
2, 5	13	$u+z = x+(y+z)$	E: 5, 12
2, 5, 11	14	$w = x+(y+z)$	E: 11, 13
2, 5, 8, 11	15	$w = x+v$	E: 8, 14
2, 5, 8, 11	15'	$x+v = w$	E (symmetry)
1	16	$x+y = u \Leftrightarrow ADDxyu$	US: 1
1	17	$y+z = v \Leftrightarrow ADDyzv$	US: 1
1	18	$u+z = w \Leftrightarrow ADDuzw$	US: 1
1	19	$x+v = w \Leftrightarrow ADDxvw$	US: 1
1, 5	20	$ADDxyu$	T: 16, 5
1, 8	21	$ADDyzv$	T: 17, 8
1, 11	22	$ADDuzw$	T: 18, 11
1, 2, 5, 8, 11	23	$ADDxvw$	T: 19, 15
1, 2, 5, 8, 11	24	$ADDxyu \wedge ADDyzv \wedge ADDuzw \wedge ADDxvw$	T: 20, 21, 22, 23
1, 2, 5, 8, 11	25	$\exists uvw\,(ADDxyu \wedge ADDxvw \wedge ADDuzw \wedge ADDxvw)$	EG: 24 (3×)
1, 2, 5, 8	26	25	ES: 25 ($\exists w$)
1, 2, 5	27	25	ES: 26 ($\exists v$)
1, 2	28	25	ES: 27 ($\exists u$)
1, 2	29	$\forall xyz$ 25	UG: 28 (3×)
1	30	2 $\rightarrow \forall xyz$ 25	D: 29
31	31	$\forall xyz$ 25	P

31	32	25	US: 31 (3x)
33	33 (32–33: u, v, w)	24	P
1, 33	34	$x+y=u$	T: 33, 16
1, 33	35	$y+z=v$	T: 33, 17
1, 33	36	$u+z=w$	T: 33, 18
1, 33	37	$x+v=w$	T: 33, 19
1, 33	38	$w=x+v$	E: 37
1, 33	39	$w=x+(y+z)$	E: 35, 38
1, 33	40	$u+z=x+(y+z)$	E: 39, 36
1, 33	41	$(x+y)+z=x+(y+z)$	E: 40, 37
1, 31	42	$(x+y)+z=x+(y+z)$	ES: 41 (triple: $\exists w$, $\exists v$, $\exists u$)
1, 31	43	$\forall xyz$ 42 (:=: 2)	UG: 42 (3x)
1	44	$\forall xyz$ 25 $\rightarrow$ 2	D: 43
1	45	2 $\leftrightarrow$ $\forall xyz$ 25	T: 30, 44

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[14] (i) I ~~omit~~ the proof of reflexivity and symmetry for R, and do transitivity: for all $x, y, z \in A$, Rxy & Ryz imply Rxy.

Take arbitrary $x, y, z \in A$, and assume Rxy & Ryz.

By the definition of $R(-,-)$, we have: u and v in $A = \mathbb{N} - \{0\}$ such that $x^u = y^v$; and also, w and t in $\cancel{A} = \mathbb{N} - \{0\}$ such that $y^w = z^t$. From $x^u = y^v$, we obtain $(x^u)^w = x^{uw} = (y^v)^w = y^{vw}$, and from $y^w = z^t$, we obtain $(y^w)^v = y^{wv} = (z^t)^v = z^{tv}$. Thus:

$$x^{uw} = y^{vw} = y^{wv} = z^{tv}.$$

This means that uw and tv work as witnesses for $\exists u \exists v (x^u = z^v)$, that is, Rxz.

(ii) $\forall x \forall y (Rxy \leftrightarrow \exists u \exists v (e(x,u) = e(y,v)))$
$\forall x \forall u \forall w (e(x, m(u,w)) = e(e(x,u),w))$, $\forall x \forall y (m(x,y) = m(y,x))$

$$\vdash$$

$\forall x \forall y ((Rxy \wedge Ryz) \rightarrow Rxz)$

1	1	$\forall x \forall y (Rxy \leftrightarrow \exists u \exists v (e(x,u) = e(y,v)))$	P
2	2	$\forall x \forall u \forall w (e(x, m(u,w)) = e(e(x,u),w)$	P
1	3	$Rxy \leftrightarrow \exists u \exists v (e(x,u) = e(y,v))$	US(2x):1
4	4	$Rxy \wedge Ryz$	P
4	5	$\exists u \exists v (e(x,u) = e(y,v))$	T:4
6	6	$\exists v (e(x,u) = e(y,v))$	P
7	7	$e(x,u) = e(y,v)$	P
1	8	$Ryz \leftrightarrow \exists u \exists v (e(y,u) = e(z,v))$	US(2x):1
1	9	$Ryz \leftrightarrow \exists w \exists t (e(y,w) = e(z,t))$	CBV:8

[no line 10]

1,4	11]w	$\exists w \exists t (e(y,w) = e(z,t))$	T: 4,9
12	12]z	$\exists t (e(y,w) = e(z,t))$	P
13	13	$e(y,w) = e(z,t)$	P
2	14	$e(x, m(u,w)) = e(e(x,u), w)$	US: 2
2	15	$e(y, m(v,w)) = e(e(y,v), w)$	US: 2
2,7	16	$e(x, m(u,w)) = e(y, m(v,w))$	E: 7, 14, 15
3	17	$m(v,w) = m(w,v)$	US: 3
2,3,7	18	$e(x, m(u,w)) = e(y, m(w,v))$	E: 16, 17
2	19	$e(y, m(w,v)) = e(e(y,w), v)$	US: 2
2	20	$e(z, m(t,v)) = e(e(z,t), v)$	US: 2
13,2	21	$e(y, m(w,v)) = e(z, m(t,v))$	E: 13, 19, 20
2,3,7,13	22	$e(x, m(u,w)) = e(z, m(t,v))$	E: 18, 21
2,3,7,13	23	$\exists v (e(x, m(u,w)) = e(z,v))$	EG: 22
2,3,7,13	24	$\exists u \exists v (e(x,u) = e(z,v))$	EG: 23
1	25	$Rxz \leftrightarrow \exists u \exists v (e(x,u) = e(z,v))$	US: 1
1,2,3,7,13	26	Rxz	T: 24, 25
1,2,3,7,12	27	Rxz	ES: 26 (12 vs 13)
1,2,3,7,4	28	Rxz	ES: 27 (11 vs 12)
1,2,3,6,4	29	Rxz	ES: 28 (7 vs 6)
1,2,3,4	30	Rxz	ES: 29 (6 vs 5)
1,2,3	31	$(Rxy \wedge Ryz) \rightarrow Rxz$	D: 30
1,2,3	32	$\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$	UG(3x): 31

[11] $\vdash: \forall x \exists y Rxy \to \forall x \exists y \exists z (Rxy \wedge Ryz)$

1	1	$\forall x \exists y Rxy$	P
1	2	$\exists y Rxy$	US:1
3	3	Rxy	P
1	4	$\forall y \exists z Ryz$	CBV:1
1	5	$\exists z Ryz$	US:4
6	6	Ryz	P
3,6	7	$Rxy \wedge Ryz$	T:3,6
3,6	8	$\exists y \exists z (Rxy \wedge Ryz)$	EG:7 (2x)
3,1	9	$\exists y \exists z (Rxy \wedge Ryz)$	ES:8 (active: z)
2,1	10	$\exists y \exists z (Rxy \wedge Ryz)$	ES:9 (active: y)
1	11	$\forall x \exists y \exists z (Rxy \wedge Ryz)$	UG:10
$\emptyset$	12	$\boxed{\forall x \exists y Rxy \to \forall x \exists y \exists z (Rxy \wedge Ryz)}$	D:11

[12] $\vdash \forall x \forall y (Rxy \to Ryx) \wedge \forall x \forall y ((Rxy \wedge Ryz) \to x=y)) \to \forall x \forall y \forall z ((Rxy \wedge Ryz) \to Rxz))$

SAME Remark as with [11]

1	1	$\forall x \forall y (Rxy \to Ryx) \wedge \forall x \forall y ((Rxy \wedge Ryx) \to x=y)$	P
2	2	Rxy	P

([12] continued)

1	3	$\forall x \forall y\, (Rxy \to Ryx)$	T : 1
1	4	$\forall x \forall y\, ((Rxy \wedge Ryx) \to x = y)$	T : 1
1	5	$Rxy \to Ryx$	US : 3 (2x)
1	6	$Rxy \wedge Ryx \to x = y$	US : 4 (2x)
1,2	7	$x = y$	T : 2, 5, 6
1	8	$Rxy \to x = y$	D : 7
1	9	$\forall x \forall y\, (Rxy \to x = y)$	UG : 7
10	10	$Rxy \wedge Ryz$	P
1	11	$Ryz \to y = z$	CBV & US : 9
1, 10	12	$y = z$	T : 10, 11
1, 10	13	Rxy	T : 10, 11
1, 10	14	Rxz	E : 12, 13
1	15	$(Rxy \wedge Ryz) \to Rxz$	D : 14
1	16	$\forall x \forall y \forall z\, ((Rxy \wedge Ryz) \to Rxz)$	UG : 15 (3x)
$\varnothing$	17	$\underline{\Psi_6}$	D : 16