Answers / Assmt 4 / MATH 318 / Fall 2007

[1]

(i) $\Phi_1(z) :=: \forall x (x \leq z)$

(ii) $\Phi_2(z) :=: \forall x (z \leq x)$

(iii) $\Phi_3(x, y, z) :=:$

$$z \leq x \;\wedge\; z \leq y \;\wedge\; \forall u((u \leq x \wedge u \leq y) \rightarrow u \leq z)$$

Also good:

$$\Phi_3(x,y,z) :=: \forall u (u \leq z \leftrightarrow (u \leq x \wedge u \leq y))$$

(iv) $\Phi_4(x,y,z) :=: \forall u (z \leq u \leftrightarrow (x \leq u \wedge y \leq u))$

or (also good):

$\Phi_4(x,y,z) :=:$

$$x \leq z \;\wedge\; y \leq z \;\wedge\; \forall u((x \leq u \wedge y \leq u) \rightarrow z \leq u))$$

(v) $\Phi_5 :=: \exists z\, \Phi_1(z) :=: \exists z \forall x (x \leq z)$

(vi) $\Phi_6 :=: \exists z\, \Phi_2(z)$

(vii) $\Phi_7(x,y) :=: \forall x \forall y\; \exists z\; \Phi_3(x,y,z)$

$\Phi_5(x,y)$ was in the problem

misprint!

(viii) $\Phi_8(x,y) :=: \forall x\, \forall y\, \exists z\, \Phi_4(x,y,z)$

(ix) $\Phi_9 :=: \forall x (x \leq x) \wedge \forall x \forall y \forall z((x \leq y \wedge y \leq z) \rightarrow x \leq z))$

$$\wedge\; \forall x \forall y((x \leq y \wedge y \leq x) \rightarrow x = y)) \wedge \Phi_5 \wedge \Phi_6 \wedge \Phi_7 \wedge \Phi_8$$

[2] Φ_{10}: to express the distributive law in a lattice: ($\Phi_3, \Phi_4, \dots$ are from [1] above)

$$\Phi_{10} :=: \forall x \forall y \forall z \forall u \forall v \forall s \forall t$$

$$\Big[\Phi_4(x,y,u) \wedge \Phi_3(u,z,v) \wedge \Phi_3(x,z,s) \wedge \Phi_3(y,z,t)\Big]$$

Comments: { '$u = x \vee y$' '$v = u \wedge z$' '$s = x \wedge z$' $t =$ '$y \wedge z$'
$(x \vee y) \wedge z$

$$\longrightarrow \Phi_4(s,t,v)\Big].$$

Comments: '$s \vee t = v$'
$(x \wedge z) \vee (y \wedge z) = v$
$= (x \vee y) \wedge z$

$$\Sigma_{dl} :=: \Phi_9 \wedge \Phi_{10}$$

$$\Sigma_{Ba} :=: \Sigma_{dl} \wedge \forall x \exists y \Big[\forall z \big(\Phi_3(x,y,z) \to \Phi_2(z)\big) \wedge \forall z \big(\Phi_4(x,y,z) \to \Phi_1(z)\big)\Big]$$

Alternative (equivalent) form for Φ_{10}:

$$\forall x \forall y \forall z \exists u \exists v \exists s \exists t$$

$$\Big[\Phi_4(x,y,u) \wedge \Phi_4(u,z,v) \wedge \Phi_3(x,z,s) \wedge \Phi_3(y,z,t) \wedge \Phi_4(s,t,v)\Big]$$

comments: $\underbrace{\underbrace{(x \vee y)}_{u} \wedge z}_{v} = \underbrace{\underbrace{(x \wedge z)}_{s} \vee \underbrace{(y \wedge z)}_{t}}_{s}$

[3] Sub formulas of Φ_1:

		Free variables
1	$z = y$	y, z
2	Rxz	x, z
3	$Rxz \to z = y$	x, y, z
4	$\forall z(Rxz \to z = y)$	x, y
5	Rxy	x, y
6	$Rxy \wedge \forall z(Rxz \to z = y)$	x, y
7	$\exists y(Rxy \wedge \forall z(Rxz \to z = y))$	x
8	$\forall x \exists y(Rxy \wedge \forall z(Rxz \to z = y))$	$\emptyset$

Truth-value tables:

1:

$z = y$	y	z
$\top$	0	0
$\perp$	0	1
$\perp$	1	0
$\top$	1	1

2:

Rxz	x	z
$\perp$	0	0
$\top$	0	1
$\top$	1	0
$\perp$	1	1

3:

$Rxz \to z = y$	x	y	z
$\top$	0	0	0
$\perp$	0	0	1
$\top$	0	1	0
$\top$	0	1	1
$\top$	1	0	0
$\top$	1	0	1
$\perp$	1	1	0
$\top$	1	1	1

4:

$\forall z(Rxz \to z = y)$	x	y
$\perp$	0	0
$\perp$	0	1
$\perp$	1	0
$\perp$	1	1

5:

Rxy	x	y
$\bot$	0	0
$\top$	0	1
$\top$	1	0
$\bot$	1	1

6:

$Rxy \land \forall z(Rxz \to z=y)$	x	y
$\bot$	0	0
$\top$	0	1
$\top$	1	0
$\bot$	1	1

7:

$\exists x(Rxy \land \forall z(Rxz \to z=y))$	x
$\top$	0
$\top$	1

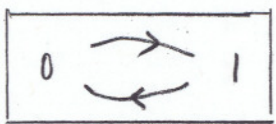
8:

$\forall x \exists y(Rxy \land \forall z(Rxz \to z=y))$
$\top$

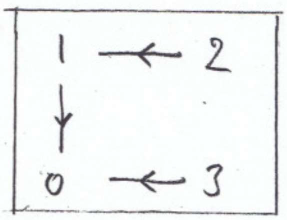
[4] (i)

	Φ_1	Φ_2	Φ_3	Φ_4
M_1	YES	YES	NO	YES
M_2	NO	NO	NO	YES
M_3	NO	NO	NO	NO
M_4	NO	YES	YES	YES

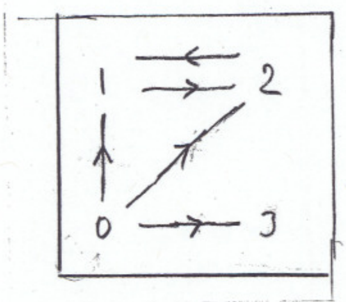
M_1:



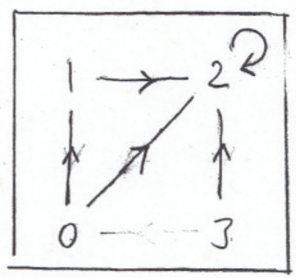
M_2:



M_3:



M_4:



(ii) (a) $\Phi_1 \equiv \forall x \exists y \, (Rxy \wedge \forall z \, (\bar{R}xz \vee z=y))$

$\neg\Phi_1 \equiv \exists x \forall y \, (\bar{R}xy \vee \exists z \, (Rxz \wedge z\neq y))$

$\neg\Phi_2 \equiv \exists x \forall y \, \bar{R}xy$

$\neg\Phi_3 \equiv \forall y \exists x \, \bar{R}xy$

$\neg\Phi_4 \equiv \exists x \forall y \, (\bar{R}xy \wedge \exists z \, (z\neq x \wedge \bar{R}zx \wedge \bar{R}zy))$

(iii) (b) Φ_1 : $y=f(x)$: $\forall x \, (R(x,f(x)) \wedge \forall z \, (\bar{R}xz \vee z=f(x)))$

Φ_2 : $y=g(x)$: $\forall x \, R(x,g(x))$

Φ_3 : $y=b$: $\forall x \, Rxb$

Φ_4 : $y=h(x)$: $\forall x \, (R(x,h(x)) \vee \forall z \, (z=x \vee R(z,x) \vee R(z,h(x))))$

$\neg\Phi_1$: $x=c,\ z=k(y)$: $\forall y \, (\bar{R}cy \vee (R(c,k(y)) \wedge k(y)\neq y))$

$\neg\Phi_2$: $x=d$: $\forall y \, \bar{R}dy$

$\neg\Phi_3$: $x=l(y)$: $\forall y \, \bar{R}(l(y),y)$

$\neg\Phi_4$: $x=e,\ z=m(y)$: $\forall y \, (\bar{R}ey \wedge (m(y)\neq e \wedge \bar{R}(m(y),e) \wedge \bar{R}(m(y),y)))$

(iv): For $M_1 \models \Phi_1$: $f=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. for $M_1 \models \Phi_2$: $g=f$. For $M_1 \models \neg\Phi_3$: $l=\begin{pmatrix}0&1\\0&1\end{pmatrix}$

For $M_1 \models \Phi_4$: $h=g=f$. For $M_2 \models \neg\Phi_1$: $c=0$, k = any function $U \to U$.
For $M_2 \models \neg\Phi_2$: $d=0$. For $M_2 \models \neg\Phi_3$: $l=\begin{pmatrix}0&1&2&3\\0&0&0&0\end{pmatrix}$. For $M_2 \models \Phi_4$:
$h \begin{pmatrix}0&1&2&3\\1&0&1&0\end{pmatrix}$. For $M_3 \models \neg\Phi_1$: $c=3$, k any function $U \to U$.

!

For $M_3 \models \neg\Phi_2$: $d = 3$. For $M_3 \models \neg\Phi_3$: $l = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}$

For $M_3 \models \neg\Phi_4$: $e = 3$, $m = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

$\uparrow\uparrow$

these two values cannot be changed;
the values at 0 & 3 could be either 1 or 2.

For $M_4 \models \neg\Phi_1$: $c = 0$, $k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix}$

$\uparrow\uparrow$ (essential)

For $M_4 \models \Phi_2$: $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 \end{pmatrix}$.

For $M_4 \models \Phi_3$: $b = 2$

For $M_4 \models \Phi_4$: $h = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 \end{pmatrix}$

$\boxed{\text{Explanations re [4](iv):}}$

consider the cases

$$M_2 \models \neg \Phi_1 \,,\quad M_3 \models \neg \Phi_1 \,,\quad M_4 \models \neg \Phi_1$$

(these facts are taken from the table in [4](i))

Consider the scheme for the Tarski tables for NNF($\neg\Phi_1$),

or, rather, the part that is relevant for the existential quantifiers in NNF($\neg\Phi_1$):

① (z,y,x)	$Rxz \wedge z \neq y$	$z \; y \; x$
② (y,x)	$\boxed{\exists z}(Rxz \wedge z \neq y)$	$y \; x$
③	$\bar{R}xy \vee \exists z(Rxz \wedge z \neq y)$	$y \; x$
④	$\forall y\,(\bar{R}xy \vee \exists z(Rxz \wedge z \neq y))$	x
⑤	$\boxed{\exists x}$ ④(x)	$\emptyset$

We need, in all three cases (M_2, M_3, M_4), to find c, an element in the underlying set of M, so that

④(c) is true $(\underline{x = c})$ (1)

and then, a function $k : A \to A$, such that

①$(k(y), y, c)$ is true (2)

$\uparrow$ for z, $z = k(y)$; $\uparrow$ for x

(continued next page)

$$\left.\begin{array}{l}\text{every time when } ②(y,c) \text{ is true} \\ \quad [\text{when } ②(y,c) \text{ is false, we} \\ \quad \text{do not worry about } k(y)\text{: it is} \\ \quad \underline{\text{anything}}\,] \end{array}\right\} (2)$$

Look at M_2: see p. ② above.

Without having calculated the table for ④, we see that ④(c) is OK for $\boxed{c=0}$: simply because now $\forall y\,(\bar{R}(0,y))$ is true (no arrow out of 0).

For M_3, this is similar: $\boxed{c=3}$ can be taken.

But for M_4, it is more complicated. Now $\boxed{c=0}$ works, since for all y, either there is no $0 \to y$, or, if there is one, there is another one $0 \to z$ with $z \neq y$.

This conclusion, namely, that $c=0$ works now can, of course, be reached by the mechanical calculation of the table for ④ – but that requires working 'from the beginning'.

This takes care of (1) in all 3 cases.

Next, to (2).:

we have to look at

$\exists z(R(c,z) \wedge z \neq y)$	y	$(x=c!)$

and see for which y this is true

<u>In M_2:</u>

$c=0$

$\exists z(R(c,z) \wedge z \neq y)$	y
$\perp$	0
$\perp$	1
$\perp$	2
$\perp$	3

since : no arrow out 0 at all!
This tells us that the function k is anything: there is no requirement on it. $k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ works.

<u>In M_3:</u>

$c=3$

$\exists z(R(c,z) \wedge z \neq y)$	y
$\perp$	0
$\perp$	1
$\perp$	2
$\perp$	3

$\boxed{\text{Same!}}$

<u>In M_4:</u>

$c=0$

$\exists z(R(c,z) \wedge z \neq y)$	y	(!!)
T	0	[z = 1, z=2 both work]
T	1	[z = 2 works]
T	2	[z = 1 works]
T	3	[z = 1 & z = 2 both work]

We have to make k so that

$$R(c, k(y)) \wedge k(y) \neq y \quad \text{for all } y;$$

but the column (!!) gives the answer:

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 \text{ or } 2 & 2 & 1 & 1 \text{ or } 2 \end{pmatrix}$$

[5]

Ψ_1 is not logically valid. Example: $U=\{0,1\}$, $R=\{(0,0),(1,1)\}$

Ψ_2 is logically valid. If $(U;R)\models \exists y\forall x Rxy$, we have a $y\in U$ such that Rxy holds for all $x\in U$; but then, for any x, we can choose this y to make Rxy hold, that is, $(U;R)\models \forall x\exists y Rxy$.

Ψ_3 is not logically valid. Example: $U=\{0,1,2,3\}$, $R=$

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 0 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

$(U;R)$ is strictly antisymmetric; from every node there is an arrow; to every node there is an arrow; but for $x=1$ & $y=3$? $Rxy \vee x=y \vee Ryx \sim \bot$

Ψ_4 is logically valid. Suppose $(U;R)\models \forall x\exists y Rxy$; let $x\in U$, to find y and z such that $Rxy\wedge Ryz$. First, apply premiss to get y with Rxy; then apply premiss for this y as x to find z such that Ryz. Done

Ψ_5 is not logically valid. $U=\{0,1,2\}$, $R=\{(0,0),(1,1),(2,2)\}$.

Ψ_6 is logically valid. Suppose $(U;R)\models$ antecedent of $\rightarrow$. $(U;R)$ is symmetric and antisymmetric. Therefore, for any $x,y\in U$, if Rxy, then also Ryx, and so: $x=y$. To prove succedent of $\rightarrow$: let $x,y,z\in U$ and assume $Rxy\wedge Ryz$. It follows that: $y=z$. Since Rxy and $y=z$, we have Rxz. Done

[6] $A = \{1, 2, 3, 4\}$

R :

$$\begin{tikzcd} 2 \arrow[r] & 3 \arrow[d] \\ 1 \arrow[u] & 4 \arrow[l] \end{tikzcd}$$

Skolem form:

$$\forall x\, R(x, \widetilde{f(x)}) \wedge \forall x \forall y\, (\bar{R}xy \vee Ryx)$$
$$\wedge\ \forall xyuv\, (\bar{R}xy \vee \bar{R}uv \vee ((x \neq u \vee y = v) \wedge (y \neq v \vee x = u)))$$
$$\wedge\ \neg R(c,d) \wedge \neg R(d,c) \wedge \neg c = d.$$

Skolem functions: $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$c = 2,\ d = 4$ (or: $c = 1,\ d = 3$) ...)

[7]

	u_1	u_2	u_3	u_4	u_5	u_6
Φ_1	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
Φ_2	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
Φ_3	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\top$
Φ_4	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$
Φ_5	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$

$$\Phi_1 \equiv \forall x \forall y (\bar{R}xz \vee \exists y (Rxy \wedge Ryz)); \qquad \boxed{y = f(x,z)}$$

$$\Phi_1^{Sk} :=: \forall x \forall y (\bar{R}xz \vee (R(x, f(x,z)) \wedge R(f(x,z), z)))$$

$$\neg \Phi_1 \equiv \exists x \exists z (Rxz \wedge \forall y (\bar{R}xy \vee \bar{R}yz)); \qquad \boxed{x = c,\ z = d}$$

$$(\neg \Phi_1)^{SK} :=: R(c,d) \wedge \forall y (\bar{R}(c,y) \vee \bar{R}(y,d))$$

$$\Phi_2 \equiv \forall x \exists y [Rxy \wedge \forall z (\bar{R}xz \vee y = z \vee Ryz)] \qquad \boxed{y = f(x)}$$

$$(\Phi_2)^{SK} :=: \forall x [R(x, f(x)) \wedge \forall z (\bar{R}xz \vee f(x) = z \vee R(f(x), z))]$$

$$\neg \Phi_2 \equiv \exists x \forall y [\bar{R}xy \vee \exists z (Rxz \wedge y \neq z \wedge \bar{R}yz] \qquad \boxed{x = c,\ z = f(y)}$$

$$(\neg \Phi_2)^{SK} :=: \forall y [\bar{R}(c,y) \vee (R(c, f(y)) \wedge y \neq f(y) \wedge \bar{R}(y, f(y))]$$

$(U_3, <) \models \Phi_1$: with Skolem function:

$f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$: $f(x, z) = \frac{x+z}{2}$

$(U_1, <) \models \neg \Phi_1$ with Skolem functions

$c, d \in \mathbb{N}$: $c = 0,\ d = 1$

$(U_2, <) \models \Phi_2$ with Skolem function

$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$: $f(x) = x + 1$

$(U_4, <) \models \neg \Phi_2$ with Skolem functions

$c \in \mathbb{R},\ f : \mathbb{R} \to \mathbb{R}$: $c = 0,\ f(y) = \frac{y}{2}$

[8] (i) $\Phi :=: \forall x \forall \varepsilon (P(\varepsilon) \to \exists \delta (P(\delta) \wedge \forall u (D(x,u,\delta) \to$

$\to D(fx, fu, \varepsilon))))$

(ii) $NN\Phi(\Phi) :=:$

$$\forall x \forall \varepsilon (\bar{P}(\varepsilon) \vee \exists \delta (P(\delta) \wedge \forall u (\bar{D}(x,u,\delta) \vee D(fx, fu, \varepsilon))))$$

$NN\Phi(\neg\Phi) :=:$

$$\exists x \exists \varepsilon (P(\varepsilon) \wedge \forall \delta (\bar{P}(\delta) \vee \exists u (D(x,u,\delta) \wedge \bar{D}(fx, fu, \varepsilon)))).$$

(iii) Skolem forms:

$Sk(\Phi) :=: \forall x \forall \varepsilon (\bar{P}(\varepsilon) \vee (P(\delta(x,\varepsilon)) \wedge \forall u (\bar{D}(x,u,\delta(x,\varepsilon)) \vee D(fx, fu, \varepsilon))))$

$\boxed{\delta = \delta(x,\varepsilon)}$

$Sk(\neg\Phi) :=: P(\varepsilon_0) \wedge \forall \delta (\bar{P}(\delta) \vee (D(x_0, u(\delta), \delta) \wedge \bar{D}(fx_0, f(u(\delta)), \varepsilon_0)).$

$\boxed{x = x_0,\ \varepsilon = \varepsilon_0,\ u = u(\delta)}$

$\boxed{(\mathbb{R}; P, D, \widetilde{exp}) \models \Phi ::}$ we let $\boxed{\delta(x,\varepsilon) = \min(1, \varepsilon/3e^x)}$

This is OK, since: if $P(\varepsilon)$, i.e., $\varepsilon > 0$, then $\delta(x,\varepsilon) > 0$ and

(claim:) if $\underbrace{|x-u| < \delta(x,\varepsilon)}_{(1)}$, then $|fx - fu| < \varepsilon$ $[fx = e^x, fu = e^u]$

(proof:) Assume (1). $|e^x - e^u| = e^x |1 - e^{u-x}|$; write $v = u - x$;

We have: $|v| < 1$ and $|v| < \varepsilon/3e^x$ by assumption.

Therefore: $|1 - e^v| = |v + \frac{v^2}{2} + \frac{v^3}{3!} + \dots| = |v|\,|1 + \frac{v}{2} + \frac{v^2}{3!} + \dots| <$

$< |v|\,|1 + \frac{1}{2} + \frac{1}{3!} + \dots| = |v| \cdot e < |v| \cdot 3 \leq \frac{\varepsilon}{e^x}$; $|e^x - e^u| = e^x |1 - e^v| < e^x \frac{\varepsilon}{e^x} = \varepsilon$

because $|v| \leq 1$.

$\boxed{\text{For } (\mathbb{R}; P, D, s) \models \neg\Phi ::}$

$x_0 = 0,\ \varepsilon_0 = \frac{1}{2},\ u(\delta) = -\frac{\delta}{2}$ work