[1]

$$X = ((A \to B) \to C) \to \bar B \bar C D =$$

$$= \overline{\overline{\bar A \vee B} \vee C} \vee \bar B \bar C D = (\overline{A \bar B} \wedge \bar C) \vee \bar B \bar C D$$

$$= (\bar A \vee B)\bar C \vee \bar B \bar C D = \bar A \bar C \vee B \bar C \vee \bar B \bar C D \leftarrow \text{(i) for } X$$

$$= \underbrace{\bar A B \bar C D \vee \bar A B \bar C \bar D \vee \bar A \bar B \bar C D \vee \bar A \bar B \bar C \bar D}_{\bar A \bar C} \vee \underbrace{A B \bar C D \vee A B \bar C \bar D \vee \bcancel{\bar A B \bar C D} \vee \bcancel{\bar A B \bar C \bar D}}_{B \bar C}$$

$$\vee \underbrace{A \bar B \bar C D \vee \bcancel{\bar A \bar B \bar C D}}_{\bar B \bar C D} =$$

(ii) for X

$$= A B \bar C D \vee A B \bar C \bar D \vee A \bar B \bar C D \vee \bar A B \bar C D \vee \bar A B \bar C \bar D \vee \bar A \bar B \bar C D \vee \bar A \bar B \bar C \bar D$$

$$Y = \neg((A \to C) \to (B \to D)) = \overline{\overline{\bar A \vee C} \vee \bar B \vee D} = \overline{A \bar C \vee \bar B \vee D} =$$

$$= (\bar A \vee C) B \bar D = \bar A B \bar D \vee B C \bar D = \leftarrow \text{(i) for } Y$$

(ii) for Y

$$\bar A B C \bar D \vee \bar A B \bar C \bar D \vee A B C \bar D \vee \bcancel{\bar A B C \bar D} = A B C \bar D \vee \bar A B C \bar D \vee \bar A B \bar C \bar D$$

$$Z = \overline{\bar B \vee \bar{\bar C} \vee A B} = B \bar C (\bar A \vee \bar B) = \bar A B \bar C \vee \bcancel{B \bar B \bar C} = \bar A B \bar C = \leftarrow \text{(i) for } Z$$

$$= \bar A B \bar C D \vee \bar A B \bar C \bar D \leftarrow \text{(ii) for } Z$$

$$U = \overset{①}{(\bar A \vee C)}\overset{②}{(B \vee C)}\overset{③}{(\bar D \vee C)} \vee \overset{④}{\bar B (A \vee D)}; \quad ①\cdot② = \bar A B \vee \bar A C \vee B C \vee C = \bar A B \vee C$$

$$①\cdot②\cdot③ = (\bar A B \vee C)(\bar D \vee C) = \bar A B \bar D \vee \bar A B C \vee C \bar D \vee C = \bar A B \bar D \vee C;$$

$$U = ①\cdot②\cdot③ \vee A \bar B \vee \bar B D = \bar A B \bar D \vee C \vee A \bar B \vee \bar B D \leftarrow \text{(i) for } U$$

$$= \boxed{\bar A B C \bar D \vee \bar A B \bar C \bar D} \vee \underbrace{ABCD}_{1} \vee \underbrace{\bar A B C D}_{2} \vee \underbrace{A \bar B C D}_{3} \vee \underbrace{A \bar B C \bar D}_{4} \vee \underbrace{\bar A B C D}_{5} \vee \underbrace{\bcancel{\bar A B C \bar D}}_{6}$$

$\vee \bar{A}\bar{B}CD \vee \bar{A}\bar{B}C\bar{D}$ (7, 8) $\vee$ [~~$A\bar{B}CD$~~ $\vee$ ~~$A\bar{B}\bar{C}\bar{D}$~~ $\vee A\bar{B}\bar{C}\bar{D} \vee A\bar{B}\bar{C}\bar{D}$]

$\vee$ [~~$A\bar{B}CD$~~ $\vee$ ~~$A\bar{B}\bar{C}D$~~ $\vee$ ~~$\bar{A}\bar{B}CD$~~ $\vee \bar{A}\bar{B}\bar{C}D$] $=$

$= \bar{A}BCD \vee ABC\bar{D} \vee A\bar{B}CD \vee A\bar{B}C\bar{D} \vee A\bar{B}\bar{C}D \vee A\bar{B}\bar{C}\bar{D} \vee$

$\vee \bar{A}BCD \vee \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}\bar{D} \vee \bar{A}\bar{B}CD \vee \bar{A}\bar{B}C\bar{D} \vee \bar{A}\bar{B}\bar{C}D$

(ii) for U

$V = \bar{A}B\bar{D} \vee C \vee (A \vee C)(\bar{D} \vee A)(\bar{B} \vee C)(\bar{B} \vee \bar{D}) =$

$= \bar{A}B\bar{D} \vee C \vee A(\bar{D} \vee A)(\bar{B} \vee C)(\bar{B} \vee \bar{D})$

$= \bar{A}B\bar{D} \vee C \vee A(\bar{D} \vee A)\bar{B}(\bar{B} \vee \bar{D})$

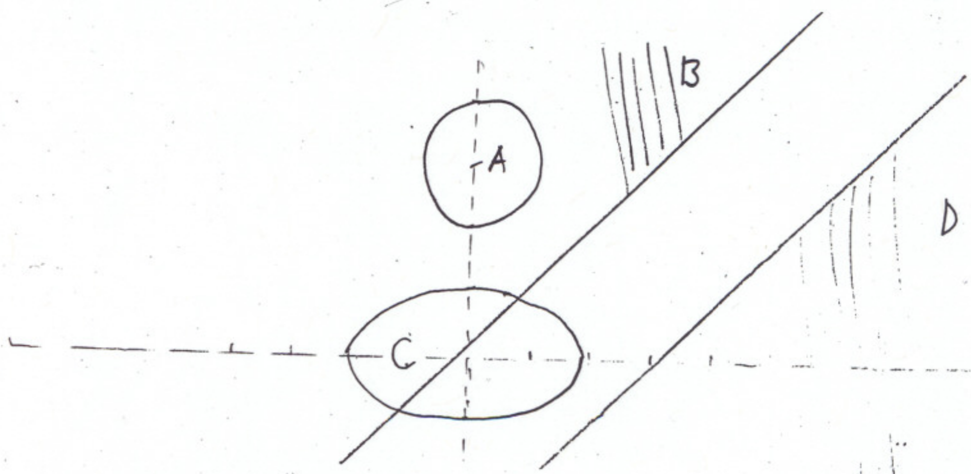
$= \bar{A}B\bar{D} \vee C \vee A\bar{B}$ ← (i) for V

$= ABCD \vee ABC\bar{D} \vee A\bar{B}CD \vee A\bar{B}C\bar{D} \vee A\bar{B}\bar{C}D \vee A\bar{B}\bar{C}\bar{D} \vee$

$\vee \bar{A}BCD \vee \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}\bar{D} \vee \bar{A}\bar{B}CD \vee \bar{A}\bar{B}C\bar{D}$

(ii) for V

[2]



(i) ATOMS of $\langle A, B, C, D \rangle$

- A
  - AB
    - $ABC = \bot$
    - $AB\bar{C}$
      - $AB\bar{C}D = \bot$
      - $AB\bar{C}\bar{D} = A \neq \bot$
  - $A\bar{B} = \bot$
- $\bar{A}$
  - $\bar{A}B$
    - $\bar{A}BC$
      - $\bar{A}BCD = \bot$
      - $\bar{A}BC\bar{D} = \bar{A}BC \neq \bot$
    - $\bar{A}B\bar{C}$
      - $\bar{A}B\bar{C}D = \bot$
      - $\bar{A}B\bar{C}\bar{D} = \bar{A}B\bar{C} \neq \bot$
  - $\bar{A}\bar{B}$
    - $\bar{A}\bar{B}C$
      - $\bar{A}\bar{B}CD = \bot$
      - $\bar{A}\bar{B}C\bar{D} = \bar{A}\bar{B}C \neq \bot$
    - $\bar{A}\bar{B}\bar{C}$
      - $\bar{A}\bar{B}\bar{C}D \neq \bot$
      - $\bar{A}\bar{B}\bar{C}\bar{D} \neq \bot$

Atoms:
- $AB\bar{C}\bar{D}$
- $\bar{A}BC\bar{D}$
- $\bar{A}B\bar{C}\bar{D}$
- $\bar{A}\bar{B}C\bar{D}$
- $\bar{A}\bar{B}\bar{C}D$
- $\bar{A}\bar{B}\bar{C}\bar{D}$

(ii)

$$X = AB\bar{C}\bar{D} \vee \bar{A}B\bar{C}\bar{D} \vee \bar{A}\bar{B}\bar{C}D \vee \bar{A}\bar{B}\bar{C}\bar{D}$$

$$Y = \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}\bar{D}$$

$$Z = \bar{A}B\bar{C}\bar{D}$$

(iii) $Z < X$ because all atoms below Z (one atom) are also below X (therefore $Z \leq X$), and not vice versa (so $Z < Y$ also.

But $Y \nless X$ since the atom $\bar{A}BC\bar{D}$ which is below Y is not below X. Thus, for $U, V \in \{X, Y, Z\}$, $U < V$ holds just in case $U = Z$ and $V \in \{X, Y\}$.

[3]* (i) [see, for def'n of atom, in the general case: p. 99 in text]

"only if": Assuming $x \in Q$, we want to conclude that: for all atoms a of $(Q, \leq)$,

either $a \wedge x = a$

or $a \wedge x = \bot$

Let a be an atom of $(Q, \leq)$. Since $x \in Q$, $b = a \wedge x$ also belongs to Q. But $a \wedge x \leq a$. By definition of 'atom', for $b \in Q$, $b \leq a$ implies $b = a$ or $b = \bot$; for $b = a \wedge x$, this is the required assertion.

"if": Assume: $x \in P$ and for all atoms 'a' of $(Q, \leq)$, either $a \wedge x = a$ (Case 1), or $a \wedge x = \bot$ (Case 2).

Let $a_1, \dots, a_m$ be all the atoms of $(Q, \leq)$ such that 'Case 1' happens, $b_1, \dots, b_n$ all those for which "Case 2" happens. Then

$$a_1, \dots, a_m, b_1, \dots, b_n$$

is a list of all atoms of $(Q, \leq)$; therefore

$$a_1 \vee \cdots \vee a_m \vee b_1 \vee \cdots \vee b_n = \top$$ (top element in both $(P, \leq)$ and $(Q, \leq)$!)

Therefore, $x = x \wedge \top = x \wedge (a_1 \vee \cdots \vee a_m \vee b_1 \vee \cdots \vee b_n) =$
$= (x \wedge a_1) \vee \cdots \vee (x \wedge a_m) \vee (x \wedge b_1) \vee \cdots \vee (x \vee b_n) =$
$= a_1 \vee \cdots \vee a_m \vee \bot \vee \cdots \vee \bot = a_1 \vee \cdots \vee a_m$

Since $a_1, \dots, a_m$ are all elements of $(Q, \leq)$, a subalgebra of $(P, \leq)$, and

$$x = a_1 \vee \cdots \vee a_m,$$

we have that $x \in Q$ as desired.

(ii):

For S:

$$(x,y) \in S \iff (y-x)(x-y-3) < 0$$

$$\iff \text{either } \boxed{y-x<0} \text{ and } \underbrace{x-y-3>0}_{(x,y)\in D}$$

$$\text{or } \underbrace{y-x>0}_{(x,y)\in B} \text{ and } x-y-3<0$$

But $x-y-3>0 \Rightarrow y-x+3<0 \Rightarrow \boxed{y-x<0}$

and $y-x>0 \Rightarrow x-y<0 \Rightarrow x-y-3<0$

Thus: $(x,y) \in S \iff (x,y) \in D$ or $(x,y) \in B$

$$S = D \vee B \in \langle A, B, C, D \rangle.$$

$$\therefore S \in \langle A, B, C, D \rangle$$

For T:

To show that $T \notin \langle A, B, C, D \rangle$, we find an atom a of $\langle A, B, C, D \rangle$ for which $T \wedge a \neq a$, $T \wedge a \neq \bot$.

Take $a = \bar{A}\bar{B}\bar{C}\bar{D}$. By similar calculation as before:

$$(x,y) \in T \iff x-y-3 \geq 0 \ \underset{\sim}{\text{or}} \ y-x \geq 0$$

$$\iff (x,y) \in D \text{ or } x-y-3=0 \text{ or } (x,y) \in B \text{ or } y-x=0.$$

Therefore, $T \wedge a = T \wedge ([x-y-3=0] \vee [y-x=0])$

$T \wedge a \neq a$, since $(3,1) \in a = \bar{A}\bar{B}\bar{C}\bar{D}$

but $(3,1) \notin T \wedge a$

$T \wedge a \neq \bot$ since $(3,3) \in T \wedge a$.

[4] (i) $\Phi(A,B,C,D) \equiv ABCD \vee ABC\bar{D} \vee AB\bar{C}D \vee AB\bar{C}\bar{D}$

$\vee A\bar{B}CD \vee A\bar{B}C\bar{D} \vee A\bar{B}\bar{C}D \vee \bar{A}BCD \vee \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}D \vee \bar{A}\bar{B}CD$

(ii) at least two of the values of A, B, C, D are T

$\Longleftrightarrow$

it is not the case that at most one of the values of A, B, C, D is T

Therefore:

$$\Phi(A,B,C,D) \equiv \neg(A\bar{B}\bar{C}\bar{D} \vee \bar{A}B\bar{C}\bar{D} \vee \bar{A}\bar{B}C\bar{D} \vee \bar{A}\bar{B}\bar{C}D \vee \bar{A}\bar{B}\bar{C}\bar{D})$$

This can be further 'minimized':

$$\equiv \neg((A\bar{B} \vee \bar{A}B)\bar{C}\bar{D} \vee \bar{A}\bar{B}C\bar{D} \vee \bar{A}\bar{B}\bar{C})$$

$$\equiv \neg((A\bar{B} \vee \bar{A}B)\bar{C}\bar{D} \vee \bar{A}\bar{B}(C\bar{D} \vee \bar{C}))$$

↖ ANSWER

Note: these are not the only possible, (correct) answers!

[5] (i) If $x+\frac{1}{2}$ were an integer, then we would have $x > 20$ (by X). But then, by V, we would have that x is an integer. $x+\frac{1}{2}$ and x cannot both be integers. We conclude that $x+\frac{1}{2}$ is not an integer.

By W, therefore, x is an integer. By V, $x > 20$. By U, $x < 22$. Knowing that $x > 20$, $x < 22$ and that x is an integer, we conclude that $x = 21$.

(ii)

$$A \equiv \text{"}x+\tfrac{1}{2} \text{ is an integer"}$$

$$B \equiv \text{"}x<22\text{"}$$

$$C \equiv \text{"}x \text{ is an integer"}$$

$$D \equiv \text{"}x>20\text{"}$$

$$E \equiv \text{"}x=21\text{"}$$

Then

$$U \equiv A \vee B$$

$$V \equiv C \Leftrightarrow D$$

$$W \equiv C \vee A$$

$$X \equiv A \rightarrow D$$

Let: $Y \overset{}{\underset{\text{def}}{:\equiv}} C \rightarrow \neg A \equiv$ "if x is an integer, then $x+\frac{1}{2}$ is not an integer"

$$Z \overset{\text{def}}{\equiv} (B \wedge D \wedge C) \rightarrow E \equiv$$

$\equiv$ "if $x<22$, $x>20$ and x is an integer, then $x=21$".

We accept Y & Z as true, on the basis of what we know from mathematics.

$$UVWXYZ \leq E:$$

$$(A \vee B)(C \Leftrightarrow D)(C \vee A)(A \rightarrow D)(C \rightarrow \neg A)((B \wedge D \wedge C) \rightarrow E) \leq E.$$

$$\underset{1}{(A \vee B)}\underset{2}{(\bar{C} \vee D)}\underset{3}{(\bar{D} \vee C)}\underset{4}{(C \vee A)}\underset{5}{(\bar{A} \vee D)}\underset{6}{(\bar{C} \vee \bar{A})}\underset{7}{(\bar{B} \vee \bar{D} \vee \bar{C} \vee E)}$$

1·2: $A\bar{C} \vee AD \vee B\bar{C} \vee BD$

1·2·3: $A\bar{C}\bar{D} \vee B\bar{C}\bar{D} \vee ACD \vee BCD$

1·2·3·4: $ACD \vee BCD \vee A\bar{C}\bar{D} \vee AB\bar{C}\bar{D} \vee ACD \vee ABCD =$

(equal; make $A\bar{C}\bar{D}$; $(ACD \vee ABCD = ACD)$)

$$= ACD \vee BCD \vee A\bar{C}\bar{D}$$

$1\cdot2\cdot3\cdot4\cdot5$ :: $\bar{A}\bar{B}CD \vee ACD \vee BCD = ACD \vee BCD$ (BCD)

$1\cdot2\cdot3\cdot4\cdot5\cdot6$ :: $\bar{A}BCD$

$1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7$; $\bar{A}BCD(\bar{B}\vee\bar{D}\vee\bar{C}\vee E) = \bar{A}BCDE$

and this is $\leq E$ identically

$\bar{A}BCDE \vdash E$ ✓

Next:

$U, V, W, X, Y \overset{?}{\vdash} E$

is to say that $1\cdot2\cdot3\cdot4\cdot5\cdot6 \vdash E$

$\equiv \bar{A}BCD \vdash E$

which is not the case: $\bar{A}BCD \leq E$ is not identically true: it is not true when

$A \sim \bot$, $B \sim C \sim D \sim T$ and $E \sim \bot$.

$U, V, W, X, Z \overset{?}{\vdash} E$

means

$(1\cdot2\cdot3\cdot4\cdot5)\cdot7 \vdash E$, $(ACD \vee BCD)(\bar{B}\vee\bar{D}\vee\bar{C}\vee E) \vdash E$

$\equiv A\bar{B}CD \vee ACDE \vee BCDE \vdash E$

not true, since $A\bar{B}CD \nleq E$ when $A \sim C \sim D \sim T$, $B \sim \bot$, $E \sim \bot$.

(iv) "Assume (the following all true:) (1): $A \vee B$, (2): $C \leftrightarrow D$, (3): $C \vee A$ (4): $A \to D$, (5): $C \to \neg A$ and (6): $(B \wedge D \wedge C) \to E$."

Show: $E = T$. If we had $A = T$, then by (4), $D = T$; by (2), $C = T$, by (5), $A = \bot$. Therefore, $A = T$ cannot be the case. Therefore, $A = \bot$. Then, by (1) and (3), $B = T$, $C = T$ and by (4), $D = T$. Thus, by (6), $E = T$."

[Instead of writing '$E = T$', we could have just said 'E'; or 'E is true'; etc.]

[6] (i) For $n=1$: the atoms of $\langle H_1, K_1 \rangle$:

$$H_1 \;\begin{cases} H_1K_1 = \emptyset \\ H_1\overline{K_1} = H_1 \end{cases}$$

$$\overline{H_1} \;\begin{cases} \overline{H_1}K_1 = K_1 \\ \overline{H_1}\,\overline{K_1} = \ell_1 \end{cases}$$

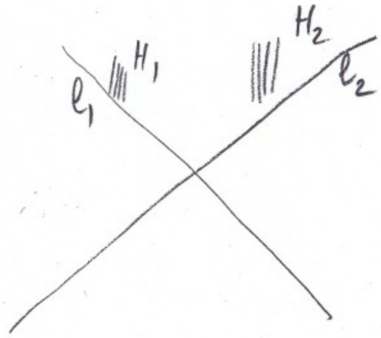
Atoms: H_1, K_1, ℓ_1;

# of atoms : $3 = 1 + 2\cdot 1^2$

For $n=2$: the atoms of $\langle H_1, H_2, K_1, K_2 \rangle = \langle H_1, K_1, H_2, K_2 \rangle$

we start with the atoms of $\langle H_1, K_1 \rangle$, and intersect them

with $\overset{(-)}{H_2}$ and $\overset{(-)}{K_2}$:



$$H_1 \;\begin{cases} H_1H_2 \begin{cases} H_1H_2K_2 = \emptyset \\ H_1H_2\overline{K_2} = H_1H_2 \end{cases} \\ H_1\overline{H_2} \begin{cases} H_1\overline{H_2}K_2 = H_1K_2 \\ H_1\overline{H_2}\,\overline{K_2} = H_1\ell_2 \end{cases} \end{cases}$$

$$K_1 \;\begin{cases} K_1H_2 \begin{cases} K_1H_2K_2 = \emptyset \\ K_1H_2\overline{K_2} = K_1H_2 \end{cases} \\ K_1\overline{H_2} \begin{cases} K_1\overline{H_2}K_2 = K_1K_2 \\ K_1\overline{H_2}\,\overline{K_2} = K_1\ell_2 \end{cases} \end{cases}$$

$$\ell_1 \;\begin{cases} \ell_1H_2 \begin{cases} \ell_1H_2K_2 = \emptyset \\ \ell_1H_2\overline{K_2} = \ell_1H_2 \end{cases} \\ \ell_1\overline{H_2} \begin{cases} \ell_1\overline{H_2}K_2 = \ell_1K_2 \\ \ell_1\overline{H_2}\,\overline{K_2} = \ell_1\ell_2 \end{cases} \end{cases}$$

Atoms: # of atoms $= 9$

$= 1 + 2\cdot 2^2$ ✓

<u>four</u> 2D atoms:

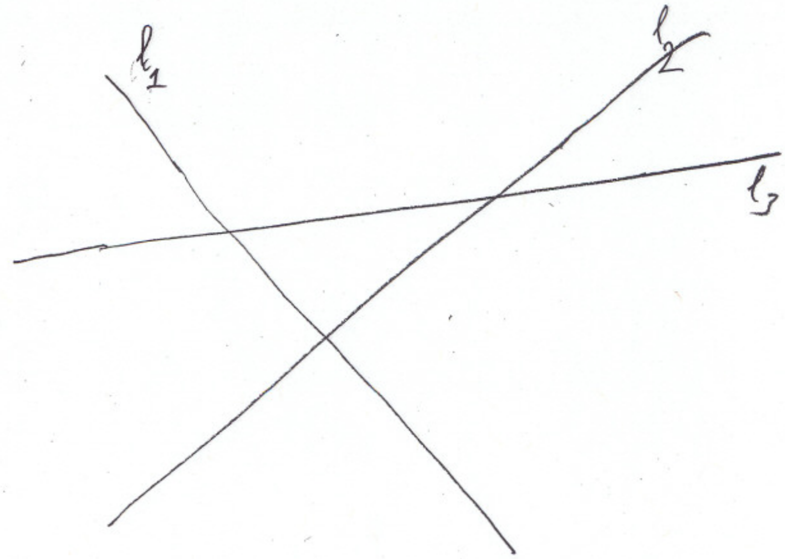
$H_1H_2,\ H_1K_2,\ K_1H_2,\ K_1K_2$

<u>four</u> 1D atoms:

$H_1\ell_2,\ K_1\ell_2,\ \ell_1H_2,\ \ell_1K_2$

<u>one</u> zero-D atom: $\ell_1\ell_2$

For $n=3$:



ℓ_3 intersects two of the 1D atoms of $\langle H_1, K_1, H_2, K_2 \rangle$:

$\ell_1 H_2$ and $H_1 \ell_2$

and three of the 2D atoms of $\langle H_1, K_1, H_2, K_2 \rangle$:

$K_1 H_2$, $H_1 H_2$ and $H_1 K_2$.

Each of these five 'old atoms' are split into three pieces: AH_3, $A\ell_3$ and AK_3.

(E.g., for $A = \ell_1 H_2$, we obtain three new atoms: $\ell_1 H_2 H_3$, $\ell_1 H_2 \ell_3$ and $\ell_1 H_2 K_3$)

Since each of the five old atoms give rise to three new atoms, the total number of new atoms is: # of old atoms + $2 \cdot 5 = 9 + 10 = 19 = 1 + 2 \cdot 3^2$ ✓

[6](ii)*: similar to what was said for $n=3$; the details are omitted.

[7] (i)

$A = P_{11}$, $B = P_{12}$, $C = P_{21}$, $D = P_{22}$
$E = P_{31}$, $F = P_{32}$

We get :

$$(A \vee B)(C \vee D)(E \vee F) \overset{?}{\leq} AC \vee AE \vee CE \vee BD \vee BF \vee DF$$

LHS :

$$(AC \vee BC \vee AD \vee BD)(E \vee F) = ACE \vee BCE \vee ADE \vee BDE \vee ACF \vee BCF \vee ADF \vee BDF$$

$$\left.\begin{array}{l} ACE \leq AC \leq \text{RHS} \\ BCE \leq CE \leq \text{RHS} \\ ADE \leq AE \leq \text{RHS} \\ BDE \leq BD \leq \text{RHS} \\ ACF \leq AC \leq \text{RHS} \\ BCF \leq BF \leq \text{RHS} \\ ADF \leq DF \leq \text{RHS} \\ BDF \leq BD \leq \text{RHS} \end{array}\right\} \Rightarrow \text{LHS} \leq \text{RHS} \checkmark$$

— · —

[7] (ii) It is good to imagine the P_{ik} arranged in the shape of a matrix, i denoting row-index, k denoting column-index. Thus, there are $n+1$ rows and n columns: one more row than columns.
For $n=3$, this would look like this:

$$\begin{matrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \\ P_{41} & P_{42} & P_{43} \end{matrix}$$

Let's prove the inequality in 2. first. This amounts to saying that if the LHS is T, then so is the RHS.
Assume LHS is T. That means that

in every row (for every i)
there is at least one entry (there is k)
which is $=$ T.

Imagine having marked, in each row, one P_{ik} which is $=$ T. Since there are more rows than columns (one more), there must exist two distinct rows, say row-i and row-j, $1 \le i < j \le n+1$, such that the marked P_{ik_1} and P_{jk_2} are in the same column, that is, $k_1 = k_2 = k$. We have proved that

there are: $1 \leq i < j \leq n+1$ such that

and $1 \leq k \leq n$

such that both P_{ik} and P_{jk} are true.

That means exactly that the RHS is true.

Having proved the inequality in $\mathbf{2}$, we refer to the theorem on p. 129 of the text which says that

<u>any identity that holds in $\mathbf{2}$, holds in any Boolean algebra whatsoever</u>

In our case, we have an inequality

$$s(P_{11}, \dots, P_{n+1,n}) \leq t(P_{11}, \dots, P_{n+1,n})$$

but the inequality $s \leq t$ is equivalent to the <u>equality</u> $s \wedge t = s$.