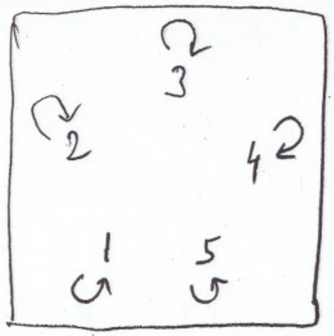
[1] **R_1:**

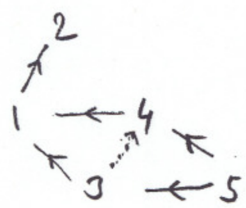


$R_1 = \{(i,i) : i \in A\}$

R_1 is both: reflexive, symmetric, and transitive
and reflexive, antisymmetric, and transitive

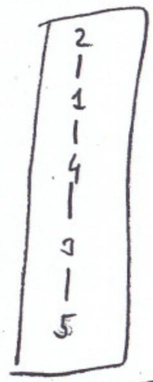
R_2:

S:



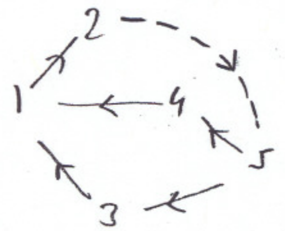
$R_2 := H^{tr}$ for

$H = \{(5,3), (3,4), (4,1), (1,2)\}$



[not the only correct answer]

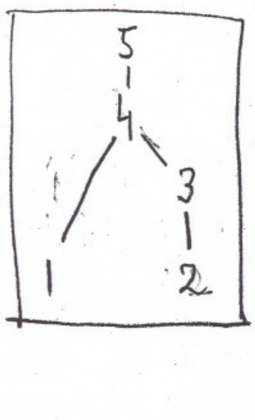
R_3: U is like S, plus $(2,5)$:



U^{tr} is not irreflexive: e.g., $(1,1) \in U^{tr}$. Therefore, there is no irreflexive and transitive relation R such that $U \subset R$.

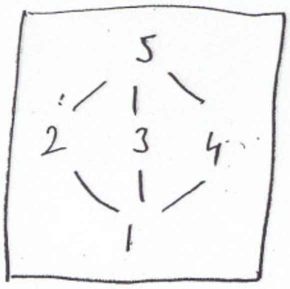
$\therefore$ There is no R_3

R_4:



incomparable pairs: $\{1,2\}$, $\{1,3\}$

R_5 :



$$(2\vee 3)\wedge 4 \;\neq\; (2\wedge 4)\vee(3\wedge 4)$$
$$5\wedge 4 \qquad\qquad 1\vee 1$$
$$4 \;\neq\; 1$$

R_6 : $R_6 = A\times A$: clearly transitive.

But if i, j are arbitrary elements of A, and $S = R_6 - \{(i,j)\}$, then, choosing $k\in A$ such that $k\neq i$, $k\neq j$, we have

$$(i,k)\in S \;,\quad (k,j)\in S$$
$$\text{but } (i,j)\notin S$$

which shows that S is not transitive.

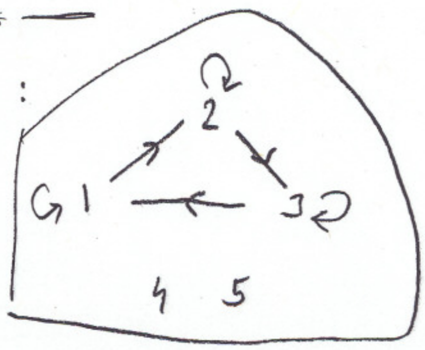
R_7 : no such R_7 : if a relation R is both symmetric and antisymmetric : $\begin{cases} aRb \Rightarrow bRa \\ aRb \;\&\; bRa \Rightarrow a=b \end{cases}$

implies : $aRb \Rightarrow a=b$

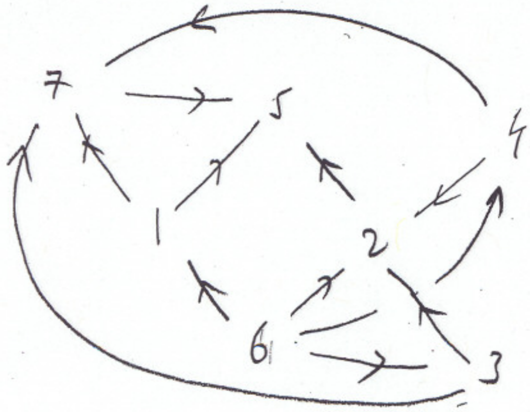
But then : R is transitive :

$$aRb \;\&\; bRc \Rightarrow a=b \;\&\; bRc \Rightarrow aRc.$$

R_6: A minimal example for R_6 :



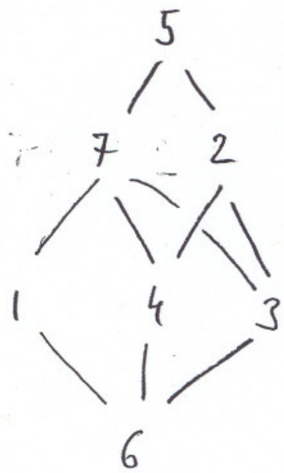
[2] Drawing S: first drawing:



$\boxed{\# S = 12}$

2nd drawing:

H:



(i): $U = \{(6,7), (6,5), (4,5), (3,5)\}$

$\boxed{\# S^{tr} = 16}$

(ii): $H =$

Hasse $= H$ $= \{(6,1), (6,4), (6,3), (1,7), (4,7), (4,2), (3,2), (3,7), (7,5), (2,5)\}$

$\# H = 10$

$H \neq S$; $S = H \,\dot{\cup}\, \{(1,5), (6,2)\}$.

(iii): $3 \vee 4$ does not exist:

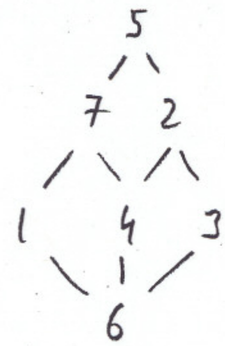
$\{3,4\}\uparrow = \{7,2,5\}$: does not have a least element.

(also: $7 \wedge 2$ does not exist)

(iv):

$(a,b) \overset{def}{=} (3,7)$

H is this:



incomparables:

$(1,4)$:	$1 \wedge 4 = 6$	$1 \vee 4 = 7$
$(1,3)$:	$1 \wedge 3 = 6$	$1 \vee 3 = 7$
$(3,4)$:	$3 \wedge 4 = 6$	$3 \vee 4 = 2$
$(1,2)$:	$1 \wedge 2 = 6$	$1 \vee 2 = 5$
$(3,7)$:	$3 \wedge 7 = 6$	$3 \vee 7 = 5$
$(2,7)$:	$2 \wedge 7 = 4$	$2 \vee 7 = 5$

$\therefore$ $x \wedge y$, $x \vee y$ always exist

$T = 5$, (since $x \wedge y$, $x \vee y$ exist whenever $x \leq y$ or $y \leq x$, in any order.).

Top: $\top = 5$

Bottom: $\perp = 6$.

[3] A Boolean algebra is a

<u>lattice</u>

which is <u>distributive</u>

and in which every element has (at least one) a <u>complement</u>.

Here: a $\boxed{\text{lattice}}$ is an order $(A; \leq)$

in which: a <u>top</u> element $\top$ exists for which

$x \leq \top$ for all $x \in A$;

(ii) a <u>bottom</u> element $\bot$ exists for which

$\bot \leq x$ for all $x \in A$;

(iii) for any x and y in A, a <u>greatest lower bound</u> (meet) $x \wedge y$ of x, y exists in A; the latter is defined by saying that

$x \wedge y \leq x$ and $x \wedge y \leq y$ and for all $z \in A$, if $z \leq x$ & $z \leq y$, then $z \leq x \wedge y$;

(iv) for any x and y in A, a <u>least upper bound</u> (join) $x \vee y$ of x, y exists in A; this is defined by:

$x \vee y \geq x$, $x \vee y \geq y$ and for all $z \in A$, if $z \geq x$ & $z \geq y$, then $z \geq x \vee y$.

The lattice $(A; \leq)$ is $\boxed{\text{distributive}}$ if for all $x, y, z \in A$,

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

In a lattice $(A; \leq)$, $x \in A$, $y \in A$: y is a $\boxed{\text{complement}}$ of x if $x \wedge y = \bot$ and $x \vee y = \top$.

[4] $X = ((A \to (B \vee C)) \wedge (C \to (A \vee B))) \to$

$$\to ((B \to A) \wedge ((A \vee B \to C))$$

(i) $$X = \overline{(\bar A \vee B \vee C)(\bar C \vee A \vee B)} \vee ((\bar B \vee A) \wedge (\underbrace{\overline{A \vee B}}_{\bar A \bar B} \vee C))$$

$$= (A \bar B \bar C \vee \underbrace{C \bar A \bar B) \vee (\bar A \bar B}_{= \bar A \bar B} \vee \bar B C \vee AC)$$

(intermediate:

$$(\bar{\bar A} \bar B \bar C \vee \bar{\bar C} \bar A \bar B) \vee (\underbrace{\bar B \bar A \bar B}_{\bar A \bar B} \vee \bar B C \vee \underbrace{A \bar A \bar B}_{\perp} \vee AC))$$

$$= A \bar B \bar C \vee \bar A \bar B \vee \bar B C \vee AC$$

(ii) FIRST ANSWER: (for 2nd: see last line)

$$A \bar B \bar C \vee \bar A \bar B \vee \bar B C \vee AC = \bar B (A \bar C \vee \underbrace{\bar A \vee C}_{= \overline{A \bar C}}) \vee AC =$$

$$= \bar B (\underbrace{A \bar C \vee \overline{A \bar C}}_{U \vee \bar U = T}) \vee AC = \bar B \cdot T \vee AC = AC \vee \bar B.$$

(iii) 1) Based on (ii):

$$X \equiv \boxed{ABC \vee A \bar B C \vee \cancel{A \bar B C} \vee A \bar B \bar C \vee \bar A \bar B C \vee \bar A \bar B \bar C} \quad (*)$$

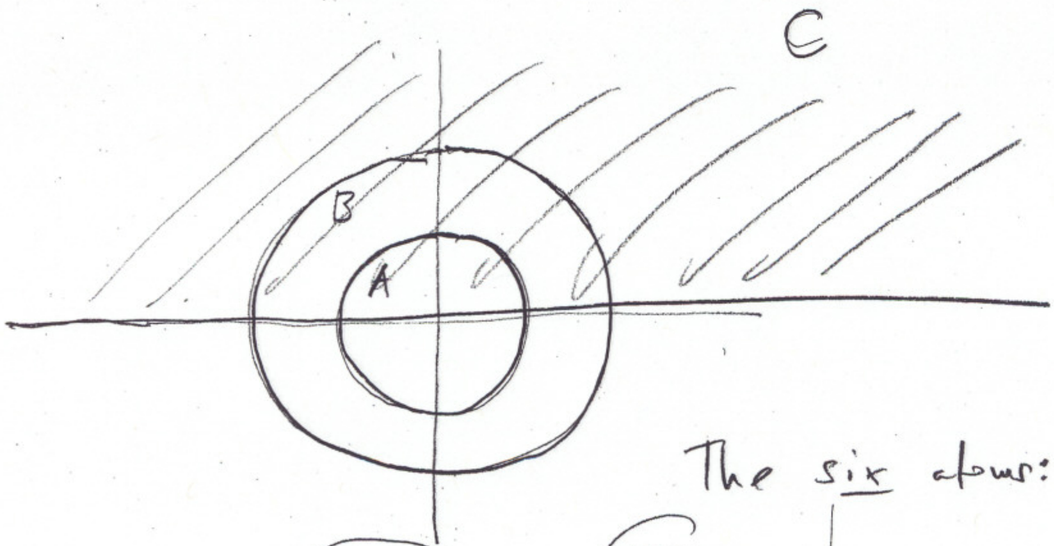
2) Directly: $$\underbrace{A \bar B \bar C}_{1} \vee \underbrace{\bar A \bar B}_{2} \vee \underbrace{\bar B C}_{3} \vee \underbrace{AC}_{4} \equiv \underbrace{A \bar B \bar C}_{1} \vee \underbrace{\bar A \bar B C \vee \bar A \bar B \bar C}_{2}$$

$$\vee \underbrace{A \bar B C \vee \boxed{\bar A \bar B C}}_{3} \vee \underbrace{ABC \vee \boxed{A \bar B C}}_{4} = \text{same as } (*)$$

repeat   repeat

second answer to (ii): the two DNF's in (iii) coincide – that proves it.

[5]

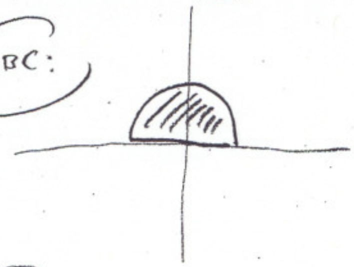


The six atoms:

(i)

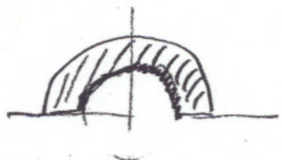
- A
  - AB
    - $ABC \neq \perp$
    - $AB\bar{C} \neq \perp$
  - $A\bar{B} = \perp$
- $\bar{A}$
  - $\bar{A}B$
    - $\bar{A}BC \neq \perp$
    - $\bar{A}B\bar{C} \neq \perp$
  - $\bar{A}\bar{B}$
    - $\bar{A}\bar{B}C \neq \perp$
    - $\bar{A}\bar{B}\bar{C} \neq \perp$

ABC:

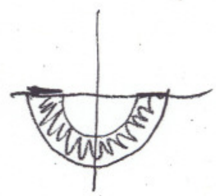


$AB\bar{C}$:

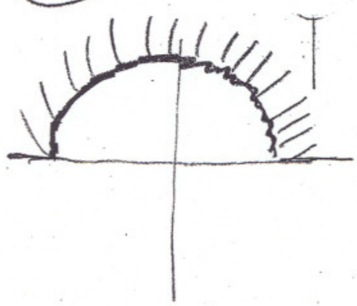
$\bar{A}BC$:



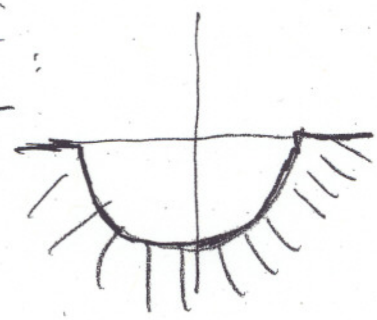
$\bar{A}B\bar{C}$:



$\bar{A}\bar{B}C$:



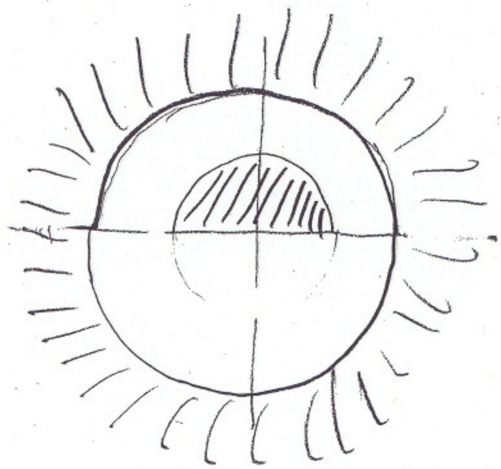
$\bar{A}\bar{B}\bar{C}$:



(ii) $2^6 = 64$

(iii) $X = AC \vee \bar{B} = ABC \vee \quad \vee \quad \bar{A}\bar{B}C \vee \bar{A}\bar{B}\bar{C}$

(iv)



(v)* Start the atomic analysis of (A, B, X), when

$X = AC \vee \bar{B}$

A — AB — …

A — $\boxed{A\bar{B} = \bot}$

$\bar{A}$ — $\bar{A}B$ — $\underline{\bar{A}BX} = \bar{A}B(AC \vee \bar{B}) = \bot \vee \underline{\bot = \bot}$

!

…

Even if all other five of the six elementary conjunctions avoiding $A\bar{B}$ are $\neq \bot$, the number of atoms in $\langle A, B, X \rangle$ must be ≤ 5, not $= 6$.

$\langle A, B, C \rangle \neq \langle A, B, X \rangle$.

[6] (i) Since $x \cdot y$ is odd (by (c)), both x and y are odd. Since $x > 5$ implies that y is even, (by (a)), it follows that $x \leq 5$. Since $y > 5$ implies that x is even (by (b)), it follows that $y \leq 5$. Since $x \leq 5$, $y \leq 5$, it follows that $x \cdot y \leq 25$.

(ii) $A :=: x > 5$ $B :=:$ y is even

$C :=: y > 5$ $D :=:$ x is even

$E :=:$ $x \cdot y$ is odd $F :=: x \cdot y \leq 25$.

(a) : $A \to B$

(b) : $C \to D$

(c) : E

((d) : F)

additional premisses (common knowledge)) :

(5): $E \to ((\neg B) \wedge (\neg D))$: "If $x \cdot y$ is odd, y is not even and x is not even"

(6): $((\neg A) \wedge (\neg C)) \to F$: "If $x \not> 5$ and $y \not> 5$, then $x \cdot y \leq 25$"

WANT: (a), (b), (c), (5), (6) $\vdash$ F:

$$A \to B,\ C \to D,\ E,\ E \to ((\neg B) \wedge (\neg D)),\ ((\neg A) \wedge (\neg C)) \to F \overset{?}{\vdash} F$$

Do: $(\bar{A} \vee B)(\bar{C} \vee D)\, E\, \underbrace{(\bar{E} \vee \bar{B}\bar{D})}\, (\underbrace{\overline{\bar{A}\bar{C}} \vee F}) \overset{?}{\vdash} F$

$E\bar{B}\bar{D}$, $\quad A \vee C \vee F$

$\bar{C} E \bar{B} \bar{D}$

$\bar{A} \bar{C} E \bar{B} \bar{D}$

$\bar{A} \bar{C} E \bar{B} \bar{D} F \leq F$ RIGHT!