Answers / Assnmt 2 / MATH 318 / Fall 2007

[1] The relation R is shown in the digraph:

$$R = \begin{tikzcd} 3 & & 5 \\ 2 \arrow[u] \arrow[urr] & & 4 \arrow[u] \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The pairs incomparable in R^{tr} are: $\{2,4\}$, $\{3,4\}$, $\{3,5\}$ $(*)$

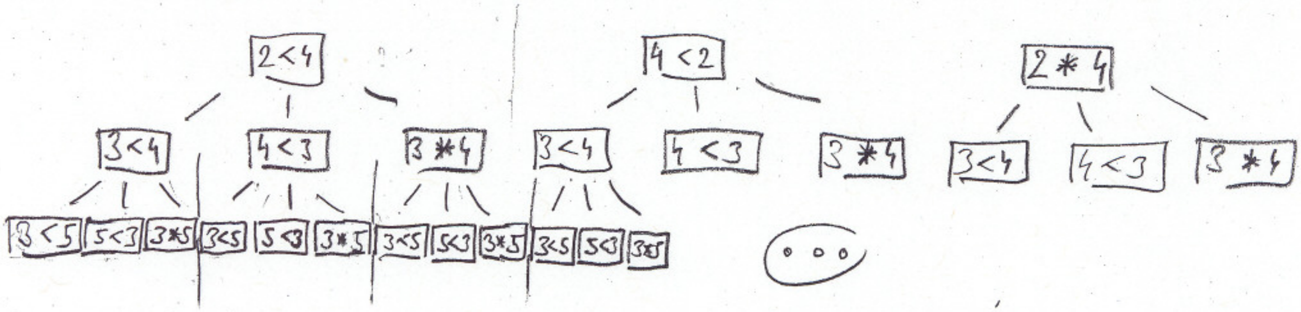
Every irreflexive order on $A=\{1,2,3,4,5\}$ extending R will be of the form $(R\cup\{(a,b),(a',b'),\dots\})^{tr}$ where $\{(a,b),(a',b'),\dots\}$ is a possibly empty set of pairs (a,b) such that (a,b) is one of the three pairs in one of the two possible orders ($(a,b)=(2,4)$ or $(a,b)=(4,2)$, or ...). For the set $\{(a,b),(a',b'),\dots\}$ we have $3\times3\times3=27$ possible choices: for $\{2,4\}$, we decide on $(2,4)$, or on $(4,2)$, or neither, to be put in the set giving us three possibilities; independently we make similar decisions on $\{3,4\}$ and $\{3,5\}$. Let us denote the 3 possible decisions on $\{2,4\}$ by the symbols

$$\boxed{2<4} \qquad \boxed{4<2} \qquad \boxed{2*4}$$

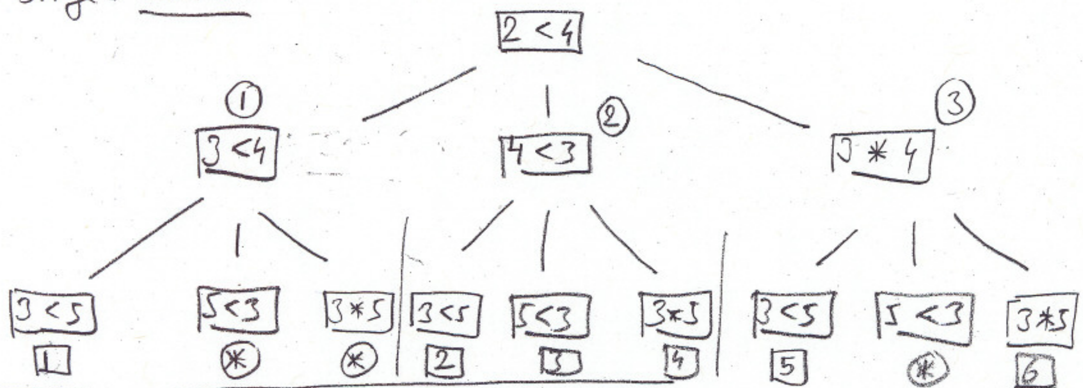
With each decision, once it is accepted, there is a 3-fold possibility on $\{3,4\}$:

$$\boxed{3<4} \qquad \boxed{4<3} \qquad \boxed{3*4}$$

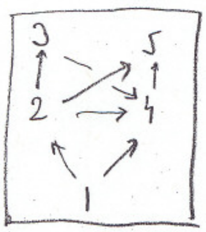
and, finally, similarly for $\{3,5\}$. The composite decisions (on all three of the pairs $(*)$) can be arranged in a tree like:

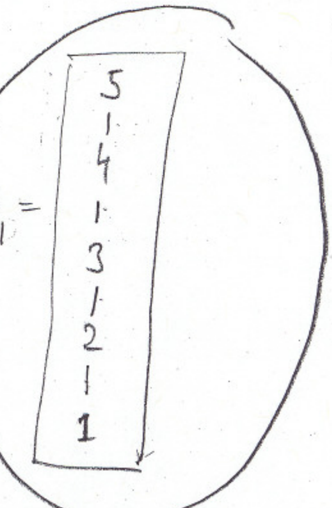


the 27 boxes in the lowest line represent the 27 possibilities mentioned above. However, not all of these give irreflexive orders. One reason may be that the relation defined by the transitive closure is not irreflexive. Another may be that a decision of the form $a * b$ is preempted, since a & b are already comparable at the previous stage. Look at:



At ①, we have: $R_{①} = R \cup \{(2,4),(3,4)\}$:

$R_{①} =$ 

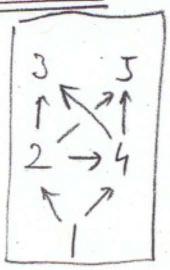
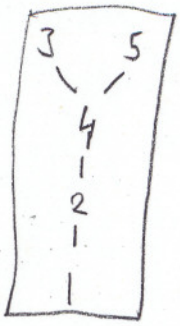
$H_1 =$ 

and for $S_1 = R_{①}^{tr}$, H_1 = Hasse diagram of S_1:

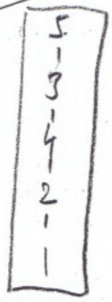
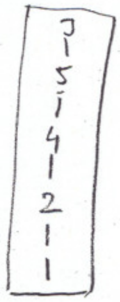
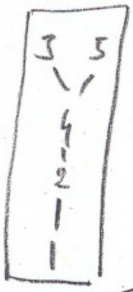
since the pairs $(1,4)$, $(2,4)$, $(2,5)$ can be omitted from R_1, without changing the transitive closure.

This means that the node $\boxed{1}$ in the tree represents S_1,
and the next two nodes marked $\circledast$, $\circledast$ are not allowed.

Looking at node ②: $R_{②} = R \cup \{(2,4), (4,3)\}$;

$R_{②} =$  , $S_{②} = R_{②}^{tr}$ has Hasse $H_{②} =$ 

I can proceed to all three possibilities regarding the pair $\{3,5\}$, and set:

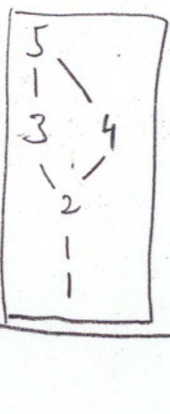
$S_2 = H_2^{tr}$; $H_2 =$  ; $H_3 =$  ; $H_4 = H_{②} =$ 

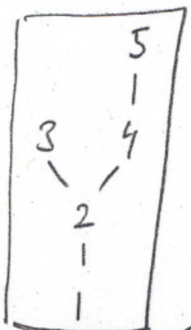
Looking at ③:

$R_{③} = R \cup \{(2,4)\}$, 

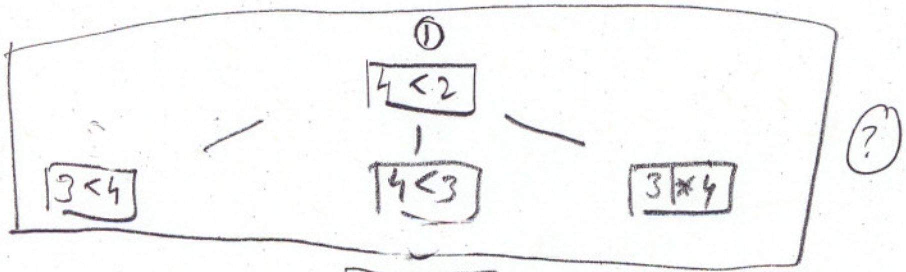
We cannot add $(5,3)$, since then $\boxed{4<3}$ is forced, contradicting $\boxed{3 \ast 4}$

There are two good ones:

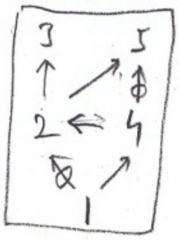
$H_5 =$ 

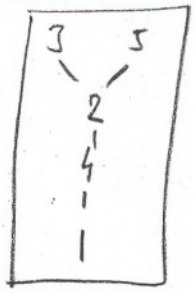
$H_6 =$ 

Next, we start with $\boxed{4<2}$:



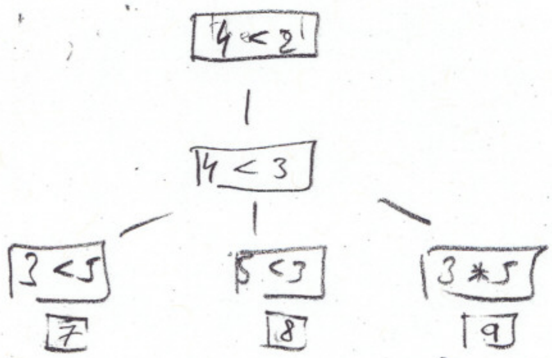
At 1, have:

$R_{1} =$ 

; $H_{1} =$ 

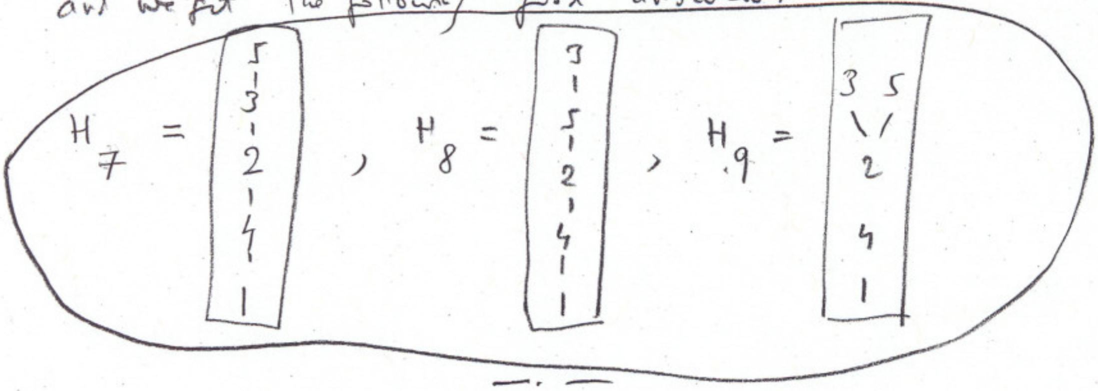
This forces $4<3$; therefore, the choices "$3<4$", "$3*4$" are preempted

Then

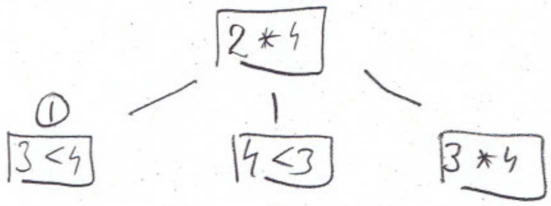


are all OK,

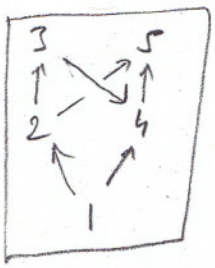
and we get the following good answers:



Next, we start with $\boxed{2 * 4}$:



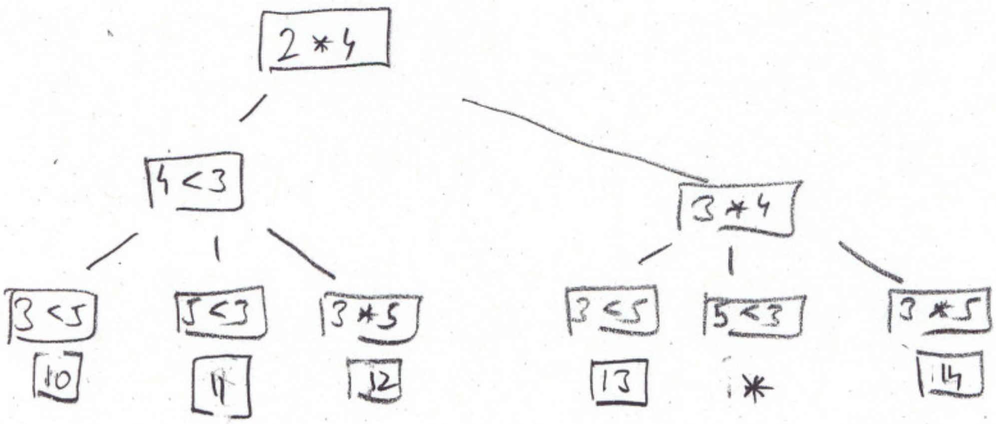
At 1, $\hat{R} = R \cup \{(3,4)\}$:



which forces $2 < 4$

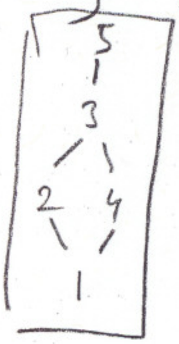
($(2,4)$ belongs to $\hat{R}^{tr}$). This contradicts $\boxed{2 * 4}$, so

1 is out. We continue to

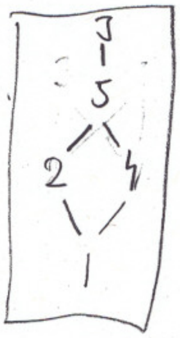


The remaining good answers are

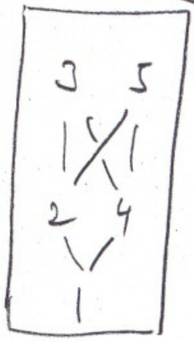
$H_{10} =$



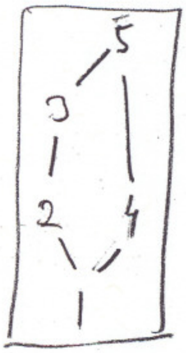
$H_{11} =$



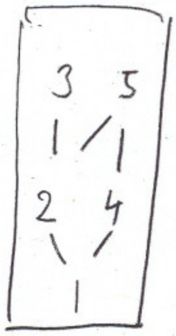
$H_{12} =$



$H_{13} =$



$H_{14} =$



$= R$

The list is $H_1, H_2, \ldots, H_{14}$.

$\boxed{[2]}$ (i) : omitted (done in class)

(ii): L_n is a lattice with:

$$\top = n$$
$$\bot = 1$$
$$x \wedge y = \gcd(x,y) \quad \text{———} \quad (3)$$
$$x \vee y = \operatorname{lcm}(x,y) \quad \text{———} \quad (4)$$

proof: By definition, for all $x \in A$, $x \mid n$, which means : $x \leq n$: n is the top element.

Since $1 \mid x$ for any $x \in A$, $1 = \bot$.

If $x, y \in A$, then for $d = \gcd(x,y)$, we have $d \mid x \mid n$, therefore $d \in A$. The condition:

$$\text{for all } u \in A: \quad \frac{u \mid d}{u \mid x \text{ and } u \mid y}$$

is true since it is true for all $u \in \mathbb{N}$ (by Ex2/sec 3.2)

This shows that (3) holds.

Let $x, y \in A$; let $c = \operatorname{lcm}(x,y)$. Since $x \mid n$, $y \mid n$ and c is the least upper bound of $\{x, y\}$ in $(\mathbb{N}, \mid)$ (Ex 2/sec 3.2), it follows that $c \mid n$.

This shows that $\underline{c \in A}$ (the essential point).

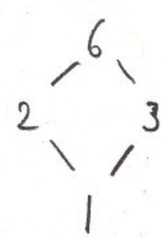
The condition:

$$\text{for all } u \in A: \quad \frac{c \mid u}{x \mid u \text{ and } y \mid u}$$

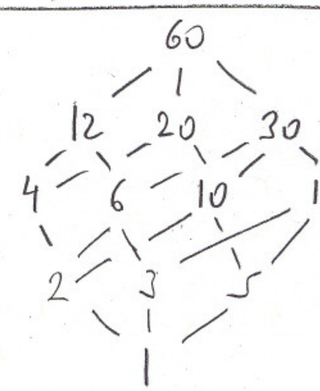
holds since it holds for all $u \in \mathbb{N}$ (by Ex 2/sec 3.2).

$\boxed{[2]}$ (iii):

L_6:



L_{60}:



[2] (iv) We have : $210 = 2 \cdot 3 \cdot 5 \cdot 7$.

With every divisor d of 210, we associate the set of its prime factors. We obtain a function

$$f : A_{210} \longrightarrow \mathcal{P}(\{2,3,5,7\})$$
$$d \longmapsto f(d) = \text{set of prime factors of } d$$

(e.g., $f(30) = f(2 \cdot 3 \cdot 5) = \{2,3,5\}$; $f(1) = \emptyset$)

f is surjective : if $X \subseteq \{2,3,5,7\}$, then we can take $d = \prod X$ = the product of all the members of X, and we clearly get that $f(d) = f(\prod X) = X$.

f is injective: Suppose $f(d) = X$; then, we claim, d must be ΠX. The reason is that d must have each $p \in X$ as a factor — but it can have it as a factor with exponent equal to 1, since $d \mid 210$, and 210 has no repeated prime factor. — Since the equality $f(d) = X$ determines d from X as $d = \Pi X$, f is injective (1-to-1).

f respects the orders: for $d_1, d_2 \in A$

$$d_1 \mid d_2 \iff f(d_1) \subseteq f(d_2)$$

The left-to-right implication is clear; and the reverse implication holds again because

$$d = \Pi f(d).$$

(We have shown that

$$L_{210} \simeq \mathcal{P}(\{2,3,5,7\}, \subseteq)$$

But $\mathcal{P}(\{2,3,5,7\}, \subseteq) \simeq \mathcal{P}(\{1,2,3,4\}, \subseteq)$

is clear.

[2] (vi) The fact that (b) implies (a) is proved in exactly the same way as (i).

For the converse: assume that (a) holds.

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We have an isomorphism

$$f : L_n \xrightarrow{\cong} (\mathcal{P}(B), \subseteq)$$

Let $p \mid n$, with p a prime number; we want to show that $p^2 \nmid n$. Consider $X \underset{\text{def}}{=} f(p)$; $X \subseteq B$. Consider $Y \underset{\text{def}}{=} B - X$. Since f is a bijection, there is $q \in A_n$ such that $f(q) = Y$.

So, now we have

$$f(p) = X$$

and

$$f(q) = Y$$

But, $X \cup Y = T$. By the isomorphism, it follows that $p \vee q = T$ in L_n, that is,

$$\operatorname{lcm}(p, q) = n. \qquad (1)$$

Also, $X \cap Y = \emptyset$; hence

$$\gcd(p, q) = 1. \qquad (2)$$

By (2), $p \nmid q$. Clearly, $\operatorname{lcm}(p, q) \mid p \cdot q$; therefore, $n \mid p \cdot q$, and $\frac{n}{p} \mid q$. It follows, by $p \nmid q$, that $p \nmid \frac{n}{p}$. This is the same as $p^2 \nmid n$.

QED

$\boxed{[3]}$ (i) The facts that

$$\top = V \quad \text{and} \quad \bot = \{0\}$$

are clear.

To show that, for $X, Y \in \mathrm{Sub}(V)$, we have

$$X \wedge Y = X \cap Y;$$

the main thing to see is that $X \cap Y$ is again a subspace of V. This we learn in Linear Algebra — — and it is not hard to prove. Once we know that $X \cap Y \in \mathrm{Sub}(V)$, what remains is:

$$\text{for all } Z \in \mathrm{Sub}(V) \ : \quad \frac{Z \subseteq X \cap Y}{Z \subseteq X \ \text{ and } \ Z \subseteq Y}$$

— but this is true for all <u>subsets</u> Z of V, because $X \cap Y$ is $X \wedge Y$ in $(\mathcal{P}(V), \subseteq)$.

To show that $X \vee Y = X + Y$, first of all, we show that $X + Y$ is a subspace of V; again, this we learn in Linear Algebra. <u>It remains to show:</u>

$$\text{for all } Z \in \mathrm{Sub}(V) \ : \quad \frac{X + Y \subseteq Z}{X \subseteq Z \ \text{ and } \ Y \subseteq Z} \ : ?$$

Since obviously, $X \subseteq X+Y$ and $Y \subseteq X+Y$, we see that the implication works from top to bottom. Conversely, assume that $X \subseteq Z$ and $Y \subseteq Z$ to conclude that $X+Y \overset{?}{\subseteq} Z$. But any element of $X+Y$ is of the form $x+y$, when $x \in X$ and $y \in Y$. Since $X \subseteq Z$, we have $\underline{x \in Z}$; since $Y \subseteq Z$, we have $\underline{y \in Z}$. Since Z is a subspace of V, it follows that $\underline{x+y \in Z}$. We have shown that any element $x+y$ of $X+Y$ is also an element of Z; this is exactly $X+Y \subseteq Z$.

(ii) We have $U_1 \vee U_2 = U_1 + U_2 = \mathrm{span}(u_1, u_2, u_3) = V = \top$

$\underline{U_1 \wedge U_2} = U_1 \cap U_2 = \underline{\mathrm{span}(u_1)}$: the reason is that U_1, U_2 are both of dimension 2, they are not equal, therefore $U_1 \cap U_2$ can have at most dimension 1; but clearly $\mathrm{span}(u_1) \subseteq U_1 \cap U_2$.

Call: $\boxed{U_4 \underset{\text{def}}{=} \mathrm{span}(u_1)}$; $U_1 \wedge U_2 = U_4$

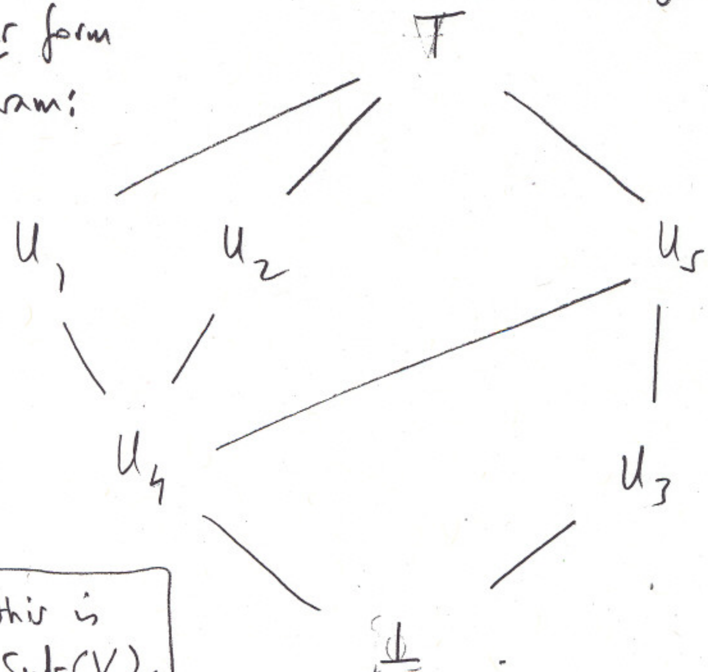
$U_1 \vee U_3 = U_2 \vee U_3 = \top$, $U_1 \wedge U_3 = U_2 \wedge U_3 = \bot = \{0\}$

again, for reasons of dimension:

$U_4 \vee U_3 = \mathrm{span}(u_1, u_4)$; call this $\boxed{U_5 = \mathrm{span}(u_1, u_4)} = U_4 \vee U_3$

From the given U_1, U_2, U_3, we have generated $\top, \bot$ [these would have to be included anyway] and U_4, U_5.

All these so far form the Hasse diagram:



($\square$)

$\boxed{\text{We claim that this is a sublattice of } \mathrm{Sub}(V).}$

Indeed, the incomparable pairs in this are:

$$(U_1, U_2),\ (U_1, U_3),\ (U_1, U_5),$$
$$(U_2, U_3),\ (U_2, U_5),\ (U_3, U_4)$$

and we have, in $(\mathrm{Sub}(V), \subseteq)$:

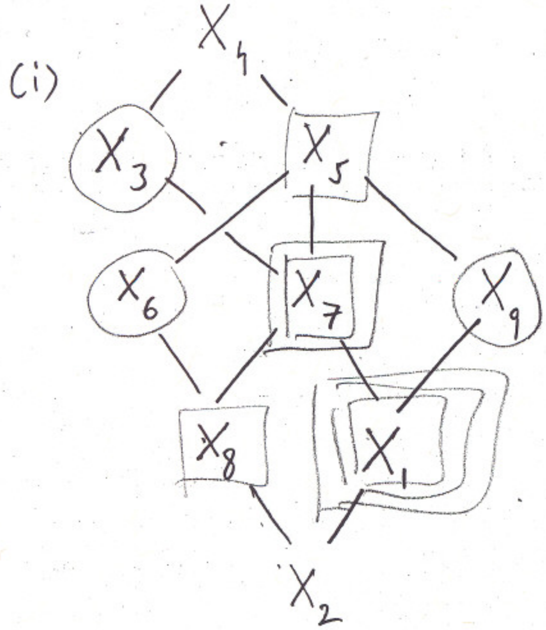
$$U_1 \vee U_2 = \top \qquad U_1 \wedge U_2 = U_4 \quad (\#)$$
$$U_1 \vee U_3 = \top \qquad U_1 \wedge U_3 = \top$$
$$\therefore\ U_1 \vee U_5 = \top \qquad U_1 \wedge U_5 = U_4 \quad (*)$$
$$U_2 \vee U_3 = \top \qquad U_2 \wedge U_3 = \bot$$
$$\therefore\ U_2 \vee U_5 = \top \qquad U_2 \wedge U_5 = U_4 \quad (**)$$
$$U_3 \vee U_4 = U_5 \qquad U_3 \wedge U_4 = \bot$$

The equalities $(*)$, $(**)$ are seen similarly to $(\#)$

($\square$) is closed under $\wedge, \vee, \top, \bot$: it is a sublattice.

[4] $\quad$ F4

(i)



(ii)

$X_8 \wedge X_1 = X_2 = \perp, \quad X_8 \vee X_1 = X_7$

$X_8 \wedge X_9 = X_2, \quad X_8 \vee X_9 = X_5$

$X_1 \wedge X_6 = X_2, \quad X_1 \wedge X_6 = X_5$

$X_6 \wedge X_7 = X_8, \quad X_6 \vee X_7 = X_4 = T$

$X_6 \wedge X_9 = X_2, \quad X_6 \vee X_9 = X_5$

$X_6 \wedge X_3 = X_8, \quad X_6 \vee X_3 = X_4$

$X_7 \wedge X_9 = X_1, \quad X_7 \vee X_9 = X_4$

$X_9 \wedge X_3 = X_1, \quad X_9 \vee X_3 = X_4$

$X_3 \wedge X_5 = X_7, \quad X_3 \vee X_5 = X_4$

(iii)

$$\underbrace{\underbrace{(X_9 \vee X_3)}_{T = X_4} \wedge X_6}_{X_6} \neq \underbrace{\underbrace{(X_9 \wedge X_6)}_{\perp = X_2} \vee \underbrace{(X_3 \wedge X_6)}_{X_8}}_{X_8}$$

$$X_6 \neq X_8 \checkmark$$

Another:

$$\underbrace{\underbrace{(X_7 \vee X_9)}_{T = X_4} \wedge X_6}_{X_6} \neq \underbrace{\underbrace{(X_7 \wedge X_6)}_{X_8} \vee \underbrace{(X_9 \wedge X_6)}_{X_2 = \perp}}_{X_8}$$

$$X_6 \neq X_8$$

(iv)

$$\underbrace{\underbrace{(X_8 \vee X_1)}_{X_7} \wedge X_5}_{X_7} = \underbrace{\underbrace{(X_8 \wedge X_5)}_{X_8} \vee \underbrace{(X_1 \wedge X_5)}_{X_1}}_{X_7}$$

$$X_7 = X_7$$

(v) NO; in this case, for all $i, j = 1, \dots, 8$, $X_i \cap X_j = X_i \wedge X_j$ as going through the first column of (iii).

(vi) YES; for instance, $X_9 \vee X_3 = X_4 = \{1,2,3,4,5,6,7,8\}$ but $X_9 \cup X_3 = \{1,2,3,5,8\}$

(vii) $\langle X_3, X_6, X_9 \rangle =$ the whole lattice, since

since

$$\left.\begin{array}{c} X_6 \vee X_9 = X_5 \\ X_3 \wedge X_5 = X_7 \\ X_6 \wedge X_7 = X_8 \\ X_9 \wedge X_7 = X_1 \\ X_4 = \top \\ X_2 = \bot \end{array}\right\}$$

together with the generators, all the elements of the lattice are generated by X_3, X_6, X_9

(viii) There are two distinct answers. The first one was pointed out by one of the students in the class. In any lattice whatever $\langle a, b \rangle = \{a, b, a \wedge b, a \vee b, \top, \bot\}$ for any a & b. Just check, the set shown is closed under the lattice operations. The second answer (that can be applied also in other situation) is based on the following observation: $X = X_3$, X_6, X_7 all have this property: X is not $\top$, it is not $\bot$, and X is not $Y_1 \vee Y_2$ with Y_1, Y_2 both different from X, and X is not $Y_1 \wedge Y_2$ with Y_1, Y_2 both different from X: we may say that X is *irreducible*. Now, if X is irreducible and $X \in \langle Y_1, Y_2, Y_3, \dots \rangle$, then we must have that X is one the Y_i!

It follows that in our case, any generating set for the whole lattice must contain all the (irreducible) elements, in our case, X_3, X_6 and X_9.

[5] (i) Since any $x, y \in A$ are now comparable, $\max(x,y)$, $\min(x,y)$ are always well-defined and in fact, $x \vee y = \max(x,y)$, $x \wedge y = \min(x,y)$.

(These equalities are 'obvious', but can also be verified as follows: we need:

$$\frac{\max(x,y) \leq u}{x \leq u \text{ and } y \leq u} \; ?$$

is true for any u. Case 1. $x \leq y$. Now, $\max(x,y) = y$, and we are looking at

$$\frac{y \leq u}{x \leq u \text{ and } y \leq u}$$

This equivalence is correct since $y \leq u \Rightarrow x \leq u$ by the fact that $x \leq y$.)

However, $\top$ and $\bot$ may not exist in a total order.

For instance, in $(\mathbb{R}, \leq)$ has no bottom or top element.

$\uparrow$ usual order

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[5] (ii) We first prove U is an order:

U is reflexive: $(a,b)\,U\,(a,b)$?

Yes because $(a,b)\,U\,(a,b) \overset{\text{def}}{\iff} \underbrace{a R a}_{\text{true}} \ \& \ \underbrace{b S b}_{\text{true}}$

since R is reflexive   since S is reflexive

U is transitive: assume $(a_1,b_1)\,U\,(a_2,b_2)\,U\,(a_3,b_3)$.

We have: $a_1 R a_2 R a_3$ and $b_1 S b_2 S b_3$; since R & S are transitive: $a_1 R a_3$ and $b_1 S b_3$; which means

$$(a_1,b_1)\,U\,(a_3,b_3).$$

U is antisymmetric: assume $(a_1,b_1)\,U\,(a_2,b_2)\,U\,(a_1,b_1)$

Hence: $a_1 R a_2 R a_1$, $b_1 S b_2 S b_1$; by the antisymmetry of R & S, $a_1 = a_2$ & $b_1 = b_2$; which means $(a_1,b_1) = (a_2,b_2)$; this is what was to be proved.

Next, we prove the lattice properties:

$\top$ exists in $(C;U)$: in fact, $\top = (\top_R, \top_S)$

$\bot$ — " — : — " —, $\bot = (\bot_R, \bot_S)$

[obvious...]

$x \wedge y$ exists in $(C;U)$: $x = (a_1,b_1)$, $y = (a_2,b_2)$

I claim that $x \wedge y = (a_1 \wedge a_2, b_1 \wedge b_2)$. ?

For this, need:

$$\frac{u \le x \wedge y}{u \le x \ \& \ u \le y}$$ if and only if ?

We wrote $\le$ for all three of R, S, U!

$u = (a_3, b_3)$;

$$\frac{(a_3, b_3) \le (a_1 \wedge a_2,\ b_1 \wedge b_2)}{(a_3, b_3) \le (a_1, b_1) \ \& \ (a_3, b_3) \le (a_2, b_2)} \quad ??$$

Yes because

$$\begin{array}{c}
(a_3, b_3) \le (a_1 \wedge a_2,\ b_1 \wedge b_2) \\
\hline
a_3 \le a_1 \wedge a_2 \quad \& \quad b_3 \le b_1 \wedge b_2 \\
\hline
a_3 \le a_1 \ \& \ a_3 \le a_2 \quad \& \quad b_3 \le b_1 \ \& \ b_3 \le b_2 \\
\hline
a_3 \le a_1 \ \& \ b_3 \le b_1 \quad \& \quad a_3 \le a_2 \ \& \ b_3 \le b_2 \\
\hline
(a_3, b_3) \le (a_1, b_1) \quad \& \quad (a_3, b_3) \le (a_2, b_2)
\end{array}$$

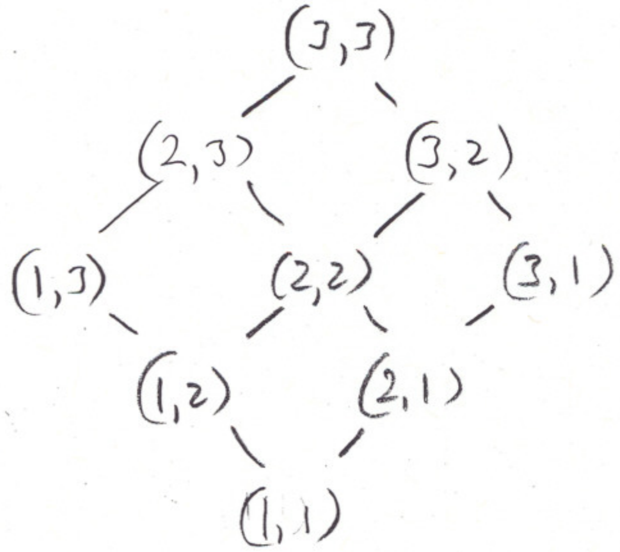
def of $\le_U$ in terms of $\le_R$ and $\le_S$

the $\wedge$-property in (A, R) and (B, S) known!

All horizontal lines mean: if and only if

$\boxed{[5](ii) \text{ finished}}$

[5] (iii)



[6] (i): omitted [later...]

[6] (ii) We have to show that

$$\top \in A \qquad (1)?$$
$$\perp \in A \qquad (2)?$$
$$X, Y \in A \Rightarrow X \wedge Y \in A \qquad (3)?$$
$$X, Y \in A \Rightarrow X \vee Y \in A \qquad (4)?$$

here all operations : $\top, \perp, \wedge, \vee$ are meant in the sense of the lattice $(\mathcal{P}(B); \subseteq)$.

<u>We know:</u> $\top = B$, $\perp = \emptyset$, $X \wedge Y = X \cap Y$, $X \vee Y = X \cap Y$.

(1) means: $\top = B \in A$. Is this true?

It means: $x \in B \Rightarrow f(x) \in B$. (?)

Certainly true since $f: B \to B$ is a function from B to B

(2): $\underbrace{\underbrace{x \in \emptyset}_{\perp} \Rightarrow f(x) \in \emptyset}_{\top}$? ✓

(3) : assuming $X, Y \in A$, we want

$x \in X \cap Y \Rightarrow f(x) \in X \cap Y$ (?) ✓

Assume $x \in X \cap Y$. Then $x \in X$ and $x \in Y$. Since $X \in A$, we conclude that $f(x) \in X$; similarly, $f(y) \in Y$. $\therefore f(x) \in X \cap Y$

(4) : assuming $X, Y \in A$, we want : $X \cup Y \in A$ (?)

that is :

$$x \in X \cup Y \Rightarrow f(x) \in X \cup Y \quad (?)$$

Assume $x \in X \cup Y$. Then : either $x \in X$, or $x \in Y$. In the first case $f(x) \in X$ since $X \in A$; $f(x) \in X \cup Y$ follows. In the second case, $f(x) \in Y$ since $Y \in A$; $f(x) \in X \cup Y$ again follows. In both cases, $f(x) \in X \cup Y$. (?) is proved

— . —

$\boxed{\text{Exercise 3, p. 79}}$

$$\bigvee X \overset{?}{=} \bigwedge (X\uparrow)$$

Following the hint in the text:

(1) Assume $a = \bigwedge(X\uparrow)$ exists. Then:

for every $x \in X$, x is a lower bound of $X\uparrow$;

therefore, $x \leq a$. We have shown that

for all $x \in X$: $x \leq a$

which means that $a \in X\uparrow$.

Also, a is the least element of $X\uparrow$, since

a is a lower bound of $X\uparrow$, thus $a \leq y$

for all $y \in X\uparrow$. We have shown:

$$a = \bigvee X$$

(2) Assume $b = \bigvee X$ exists. Then b is the least

element of $X\uparrow$; $b \leq y$ for all $y \in X\uparrow$:

b is a lower bound of $X\uparrow$.

Moreover, if u is any lower bound of $X\uparrow$, then, since $b \in X\uparrow$, we have $u \leq b$; this shows that b is the greatest lower bound of $X\uparrow$:

$$b = \bigwedge (X\uparrow)$$

— · —

The other equality $\bigwedge X = \bigvee (X\downarrow)$ is very similar. (reverse $\leq$ everywhere).

Exercise 5 (p. 81): omitted (later...)