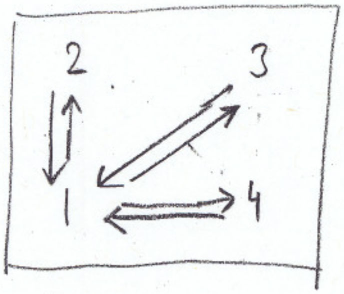
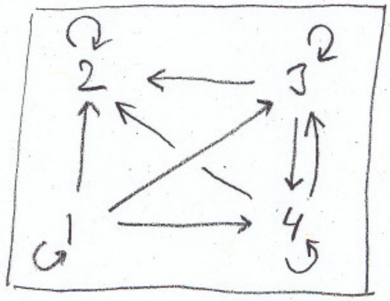
Answers /Assnmt 1/ MATH 318/ Fall 2007

[1] (i)

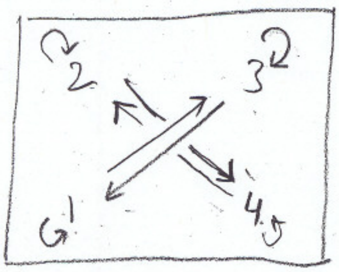
R_1 :



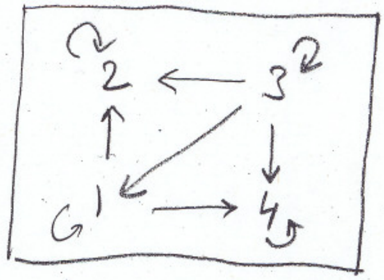
R_2 :



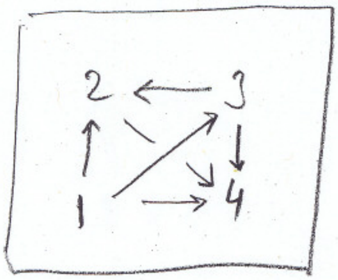
R_3 :



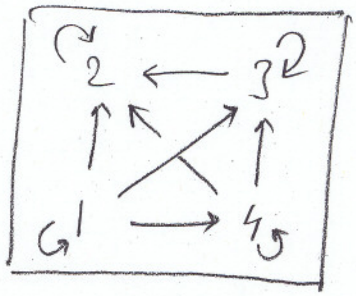
R_4 :



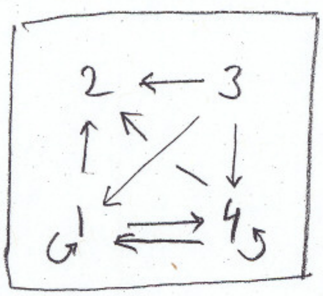
R_5 :



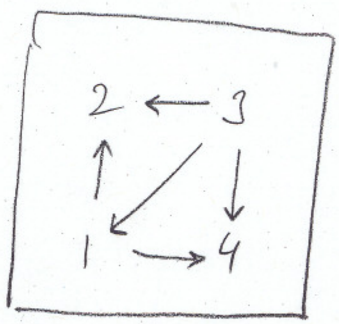
R_6 :



R_7 :



R_8 :



[1] (ii) & (iii):

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
P_1	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\bot$
P_2	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
P_3	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
P_4	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
P_5	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$
P_6	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
P_7	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
P_8	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
Q_1	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\bot$
Q_2	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
Q_3	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$
Q_4	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
Q_5	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
Q_6	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
Q_7	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\top$ = YES

$\bot$ = NO

[2]

R_1:

not reflexive ✓
not transitive:

$$2 R_1 6 R_1 3 \quad \text{but} \quad \text{not} \quad 2 R_1 3$$

symmetric ✓

irreflexive ✓

not antisymmetric:

$$2 R_1 6 \text{ and } 6 R_1 2 \quad \text{but} \quad 2 \neq 6$$

not strictly a.s. ✓

not dichotomous, because

not trichotomous:

$$\neg (2 R_1 3) \ \& \ \neg (2 = 3) \ \& \ \neg (3 R_1 2)$$

R_1 is a graph ("Q_7"):

symmetric & irreflexive

R_2:

not reflexive ✓
not symmetric ✓

(!) transitive : this is a relatively complicated proof

We want to show

$$x R_2 y R_2 z \Rightarrow x R_2 z$$

Proof of transitivity of R_2

Assume $x R_2 y R_2 z$.

We have: $x = (a, b)$

$y = (c, d)$

$z = (e, f)$

where $a, \dots, f \in \mathbb{N}$

$x R_2 y$ means $(a,b) R_2 (c,d)$ means

either $a < c$ (Case 1), or ($a = c$ and $b < d$) (Case 2)

$y R_2 z$ means $(c,d) R_2 (e,f)$ means

either $c < e$ (Case 3), or ($c = e$ and $d < f$) (Case 4)

We have four combined cases:

Cases 1 & 3

Cases 2 & 3

Cases 1 & 4

Cases 2 & 4

Cases 1 & 3: $a < c$ and $c < e$: now $a < c < e$, hence $a < e$

therefore $(a,b) R_2 (e,f)$ ✓

$x R_2 z$

Cases 2 & 3: $a = c$ & $b < d$ and $c < e$; $a = c < e$: again, $a < e$

and $x R_2 z$ again ✓

Cases 1 & 4: $a<c$ & $c=e$ & $d<f$: $a<e$

and $x R_2 z$ ✓

Cases 2 & 4: $a=c$ & $b<d$ & $c=e$ & $d<f$:

now $a=c=e$, so $a=e$

and $b<d<f$, so $b<f$

thus $(a,b) R_2 (e,f)$ by the (second alternative):

$a=e$ & $b<f$.

DONE

irreflexive ✓

antisymmetric ✓

strictly antisymmetric ✓ (since antisymmetric & irreflexive)

not dichotomous since not reflexive

! trichotomous: given $x=(a,b)$ and $y=(c,d)$

we have: $a<c$ or $a=c$ or $c<a$

Case 1 Case 2 Case 3

In Case 1: xRy, in Case 3: yRx

In Case 2: we consider the alternatives:

Proof of TRICHOTOMY of R_2

$b < d$ or $b = d$ or $d < b$

Case 4   Case 5   Case 6

In Case 4 (within Case 2): $x R y$ (by the second alternative)

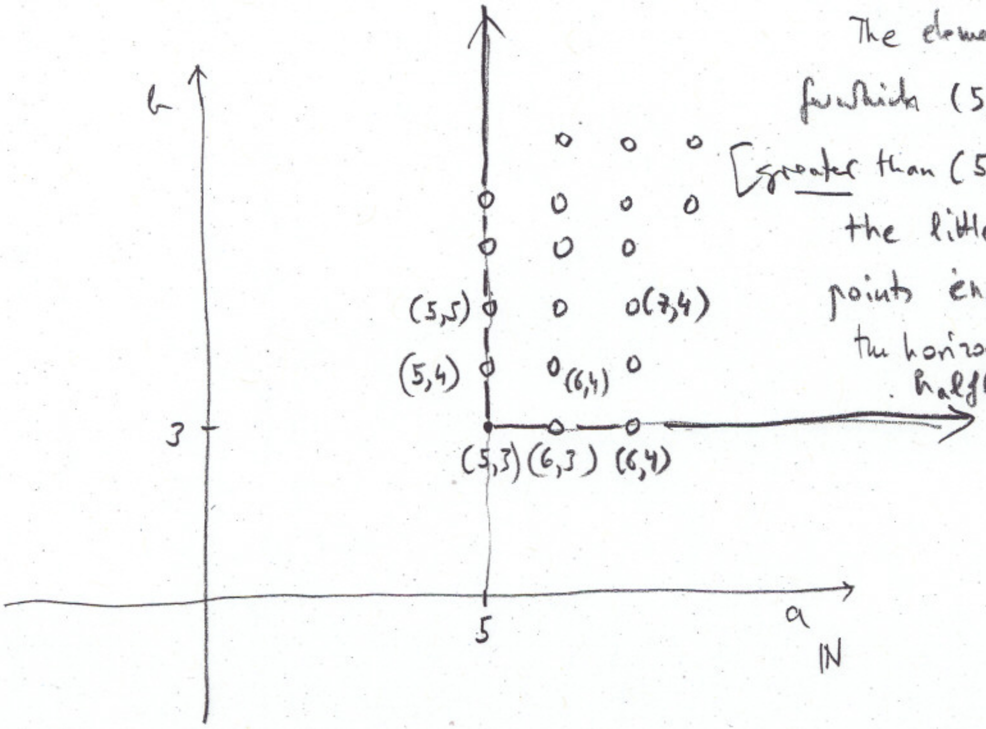
In Case 5: we have $x = (a, b) = (c, d) = y$ now!

In Case 6: $y R x$ ✓   DONE

R_2 is transitive, irreflexive and trichotomous:

it is a total irreflexive order.

Picture of R_2 (as much as it is possible: it is an INFINITE relation)



The elements (c,d) for which $(5,3) R_2 (c,d)$ [greater than $(5,3)$] are the little o's; the points enclosed by the horizontal & vertical halflines starting in $(5,3)$

R_3:

reflexive: $\frac{x}{x} = 1$ is an integer

not symmetric: $\frac{2}{1}$ is an integer, but $\frac{1}{2}$ is not

transitive: $\frac{x}{y}$ is an integer & $\frac{y}{z}$ is an integer

$$\Rightarrow \quad \frac{x}{z} = \underbrace{\frac{x}{y}}_{\text{int}} \cdot \underbrace{\frac{y}{z}}_{\text{int}} \text{ is an integer.}$$

not antisymmetric: $1 R_3 (-1)$ and $(-1) R_3 (1)$, but $-1 \neq 1$

$1/(-1) = -1$ an integer; $(-1)/1 = -1$ an integer

not trichotomous: $\neg(2 R_3 3)$ & $\neg(3 R_3 2)$ & $\neg(2 = 3)$

$\boxed{R_3 \text{ is a preorder}}$

R_4:

not symmetric: $0 R_4 1$ but $\neg(1 R_4 0)$

irreflexive: $x R_4 x$ would mean $x - 3x = -2x > 0$

which is not possible since $x \in \mathbb{R}^{\geq 0}$.

transitive: $x R_4 y R_4 z$ means

$$\frac{y - 3x > 0}{y > 3x} \qquad \& \qquad \frac{z - 3y > 0}{z > 3y}$$

$$\frac{3y > 9x}{z > 9x \geq 3x}$$

$\therefore \ x R_4 z$

not trichotomous: $\neg(1 R_4 2)$ since $2 - 3 \cdot 1 \not> 0$

$\neg(2 R_4 1)$ since $1 - 3 \cdot 2 \neq 0$

and $\neg(1 = 2)$

R_4 is an irreflexive partial order

R_5: reflexive, transitive and symmetric:

an <u>equivalence relation</u>

<u>transitivity</u>: $x R_5 y R_5 z$ means

$x - y \in \mathbb{Q}$, $y - z \in \mathbb{Q}$

$\therefore$ $x - y + y - z = x - z \in \mathbb{Q}$, $\therefore$ $x R_5 z$.

<u>Table</u>:

		R_1	R_2	R_3	R_4	R_5
r	P_1	$\perp$	$\perp$	$\top$	$\perp$	$\top$
s	P_2	$\top$	$\perp$	$\perp$	$\perp$	$\top$
t	P_3	$\perp$	$\top$	$\top$	$\top$	$\top$
i	P_4	$\top$	$\top$	$\perp$	$\top$	$\perp$
a	P_5	$\perp$	$\top$	$\perp$	$\top$	$\perp$
s.a.	P_6	$\perp$	$\top$	$\perp$	$\top$	$\perp$
d	P_7	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
t	P_8	$\perp$	$\top$	$\perp$	$\perp$	$\perp$
p	Q_1	$\perp$	$\perp$	$\top$	$\perp$	$\top$
e	Q_2	$\perp$	$\perp$	$\perp$	$\perp$	$\top$
r.o.	Q_3	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
i.o.	Q_4	$\perp$	$\top$	$\perp$	$\top$	$\perp$
t.r.o.	Q_5	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
t.i.o.	Q_6	$\perp$	$\top$	$\perp$	$\perp$	$\perp$
g	Q_7	$\top$	$\perp$	$\perp$	$\perp$	$\perp$

[3]

	$R \cap S$	$R \cup S$
P_1 reflexive	yes	yes
P_2 symmetric	yes	yes
P_3 transitive	yes	no 1
P_4 irreflexive	yes	yes
P_5 antisymmetric	yes	no 2
P_6 strictly antisymmetric	yes	no 3
P_7 dichotomy	no 4	yes
P_8 trichotomy	no 5	yes

For the Q_k, see below

For the no's:

[1] R: $\boxed{0,\ 1 \to 2}$ S: $\boxed{0 \to 1,\ 2}$ $R \cup S$: $\boxed{0 \to 1 \to 2}$

both transitive

not transitive

[2] R: $\boxed{1 \to 2}$ S: $\boxed{1 \leftarrow 2}$ $R \cup S$: $\boxed{1 \rightleftarrows 2}$

antisymm.

not antisymmetric

[3] same example as for 2

[4]: R: $\boxed{\circlearrowleft 1 \longrightarrow 2 \circlearrowright}$ S: $\boxed{\circlearrowleft 1 \longleftarrow 2 \circlearrowright}$ $R \cap S$: $\boxed{\circlearrowleft 1\ ,\ 2 \circlearrowright}$

both dichotomous

not dichotomous

[5]: same example as for 4

For the yes's: e.g., for transitive:

assume R, S transitive:

$$x(R\cap S)y \ \&\ y(R\cap S)z \Rightarrow xRy \ \&\ xSy \ \&\ yRz \ \&\ ySz$$

$$\Downarrow \qquad \Downarrow$$

$$xRz \ \&\ xSz \quad \Rightarrow x(R\cap S)z.$$

Each Q_k is a conjunction of some P_i's.

For instance:

$$\boxed{\begin{array}{ccccc} Q_1 & \equiv & P_1 & \& & P_3 \\ \text{preorder} & & \text{reflexive} & \text{and} & \text{transitive} \end{array}}$$

When both P_1 & P_3 are preserved by intersections (unions)

then $Q_1 \equiv P_1 \& P_3$ is also preserved.

Therefore: yes for Q_1 under $R \cap S$, since: yes for P_1 & P_3.

		$R \cap S$	$R \cup S$
Q_1	preorder	yes	$\boxed{\text{NO}}$
Q_2	equivalence	yes	$\boxed{\text{NO}}$
Q_3	reflexive order	yes	$\boxed{\text{NO}}$
Q_4	irref. order	yes	$\boxed{\text{NO}}$
Q_5	total ref. ord.	$\boxed{\text{NO}}$	$\boxed{\text{NO}}$
Q_6	total irr. ord	$\boxed{\text{NO}}$	$\boxed{\text{NO}}$
Q_7	graph	yes	yes

Similarly, for the yes under $R \cap S$.

<u>However</u>: $Q_5 \equiv$ total ref. order $\equiv \underbrace{P_7}_{\text{dich.}} \& \underbrace{P_1 \& P_3 \& P_5}_{\text{ref. order}}$

We know that P_7 is not preserved by intersections ($R \cap S$): see table on p. 9. But, in principle, it is possible that P_7 is preserved by $R \cap S$ if, in addition to P_7, we also assume P_1 & P_3 & P_5 of R and S.

We look at the counterexample for P_7 under $R \cap S$: this is item $\boxed{4}$ on p. 9. We see that both R and S are reflexive orders in that example.

Therefore, this example does show that Q_5 is not preserved by $R \cap S$: both R & S have Q_5, but $R \cap S$ doesn't, in that example.

In the case of Q_6, the example under $\boxed{5}$ on p. 9 is no longer good since in that (which is identical to $\boxed{4}$) R and S are not irreflexive orders.

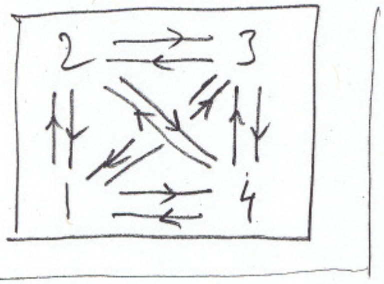
We need another example:

$$R: \boxed{1 \longrightarrow 2} \qquad S: \boxed{1 \longleftarrow 2}$$

For the latter: R & S are both irreflexive total orders, but

$$R \cap S: \boxed{1 \quad 2}$$ is not.

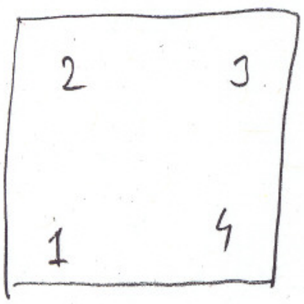
[4] (a)



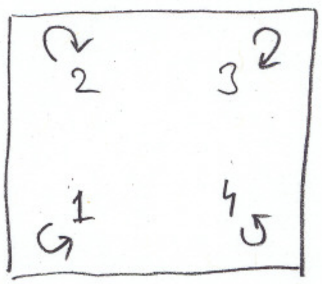
(b) <u>impossible:</u> let x, y be any elements of $\{1,2,3,4\}$.

By dichotomy: xRy or yRx; by symmetry: both: xRy and yRx. We have shown that for all $x, y \in A$, $(x,y) \in R$; $R = A \times A$. But, of course, $R = A \times A$ is <u>transitive</u>!

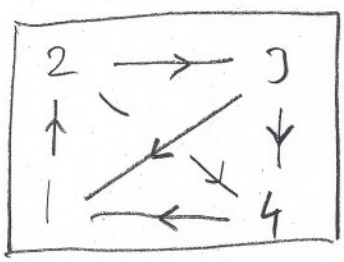
(c)



(d)



(e)



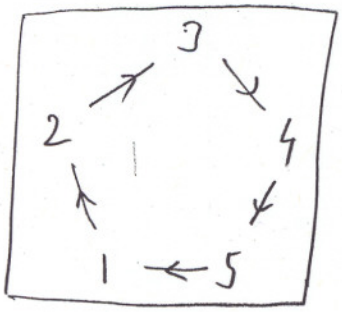
(there are many others for (e))

(f) <u>impossible</u>: we show that if $R \subseteq A \times A$ is both symmetric and antisymmetric, then $\boxed{R \subseteq \Delta_A}$: in other words, $xRy \Rightarrow x=y$. Suppose xRy. By symmetry, also yRx. By antisymmetry, $x=y$ follows. – <u>However</u>, any $R \subseteq \Delta_A$ is transitive: assume xRy & yRz; ∴ $y=z$; and by xRy, <u>xRz</u>.

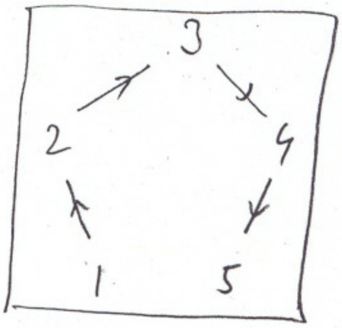
(g) <u>impossible</u>: either $1R2$ or $2R1$ (trichotomy);
by symmetry, <u>both</u>: $1R2$ <u>and</u> $2R1$;
by transitivity: $1R1$, contradicting irreflexivity.

[5] (i) $n=5$: [watch out for the scrambling!]

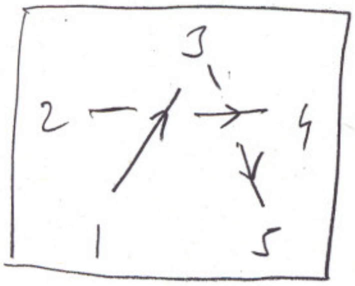
R_3 :



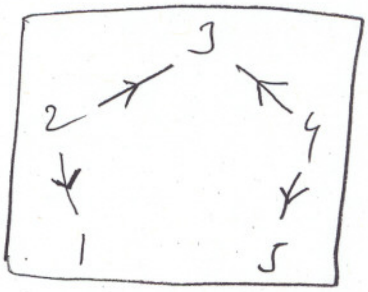
R_1 :



R_2 :



R_4 :



(ii) (general n) $R_3^{tr} = A \times A$; $\underbrace{|R_3^{tr}|}_{\substack{\text{no. of elements (pairs)} \\ \text{in the set } R_1^{tr}}} = n^2$

$R_1^{tr} = \{(i,j) \in A \times A : i<j\}$; $|R_1^{tr}| = \dfrac{n(n-1)}{2}$

$<$: usual order of integers

$R_2^{tr} = \{(i,j) \in A \times A : i<j \text{ and both } i \text{ and } j \text{ are even}\} = X$

$\cup \{(i,j) \in A \times A : i<j \text{ and both } i \text{ and } j \text{ are odd}\} = Y$

For n even, $n = 2k$, the even numbers in $\{1, \dots, 2k\}$ are: $2, 4, \dots, 2k$; their number is: k

the odd numbers in $\{1, \dots, 2k\}$ are: $1, 3, \dots, 2k-1$; their number is: k

Therefore, $|X| = \frac{k(k-1)}{2} = |Y|$

and $|R_2^{tr}| = |X| + |Y| = k(k-1)$

For n odd, $n = 2k+1$: the even numbers in $\{1, \ldots, 2k+1\}$ are: $2, 4, \ldots, 2k$; their number is: k

the odd numbers in $\{1, \ldots, 2k+1\}$ are: $1, 3, \ldots, 2k+1$; their number is: $k+1$

Therefore, $|X| = \frac{k(k-1)}{2}$, $|Y| = \frac{(k+1)k}{2}$

and $|R_3^{tr}| = |X| + |Y| = \frac{k(k-1) + (k+1)k}{2} = k^2$

Conclusion: $\begin{cases} n = 2k & \Rightarrow |R_3^{tr}| = k(k-1) \\ n = 2k+1 & \Rightarrow |R_3^{tr}| = k^2 \end{cases}$

$R_4^{tr} = R_4$; that is, R_4 is transitive (already)

Reason: if $a R b$, then a is even and b is odd. Therefore, $x R y$ & $y R z$ can never be true; since y should then be both odd (by $x R y$), and even (by $y R z$).

[6] ALL VARIABLES DENOTE NONNEGATIVE INTEGERS

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(i) Reflexive: $\overset{?}{} \; aEa$

aEa is true since $a | a^1$ and $a | a^1$;

$i = j = 1$ work

Symmetric: obvious since a & b "play the same role" in the definition.

We may also say: if the pair (i, j) witnesses aEb, then the reversed pair (j, i) witnesses bEa.

Transitive: $aEbEc \overset{?}{\Longrightarrow} aEc$

Assume aEb and bEc.

Therefore: there are i and j such that $\boxed{b | a^i}$ (1) and $\boxed{a | b^j}$ (2)

and there are $\underbrace{k \text{ and } l}_{!}$ such that $\boxed{c | b^k}$ (3) and $\boxed{b | c^l}$ (4)

not the same i & j!

But: $b | a^i$ (1) implies $\boxed{b^k | a^{ik}}$ (5): because

$b | a^i$ means $bx = a^i$, $\therefore b^k x^k = a^{ik}$, $\therefore b^k | a^{ik}$

(3) & (5): $c | b^k | a^{ik} \Rightarrow c | a^{ik}$

Similarly: $a | c^{jl}$

This shows that the pair (ik, jl) witnesses aEc.

(i): DONE

(ii) $\boxed{\text{First}}$, assume that $a \neq 0$, $b \neq 0$.

Then: (*) there is i such that $b \mid a^i$

$\iff$

all prime factors of b are prime factors of a

<u>because</u>: <u>first</u>, the implication $\Rightarrow$:

if $b \mid a^i$ and $\underbrace{p}_{\text{prime}} \mid b$, then $p \mid a^i$, and

this is only possible if p is a prime factor of a.

<u>second</u>, $\Leftarrow$: if all prime factors of b are factors of a, then we can take i = the largest exponent of a prime factor in b; then all prime factors in a^i will have exponents at least as high as those in b, which means that $b \mid a^i$.

Applying (*) with the roles of a and b reversed, we have that: if $a \neq 0$, $b \neq 0$, then

$a E b \iff a$ and b have the same prime factors

$\boxed{\text{Second}}$: if, say, $a = 0$, then $a \mid b^j$ implies that $b = 0$.

Thus, $a E b$ can hold when one of the a, b is zero only if <u>both</u> are zero. And we do have that $0 E 0$.

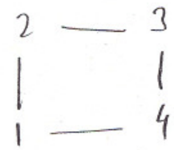
<u>End of proof of (ii)</u>

For (iii), see page 1

[6](iii) $\{\{0\}, \{1\}, \{2,4,8,16\}, \{3,9\}, \{5\}, \{7\}, \{6,12,18\}, \{13\}, \{14\}, \{15\}, \{17\}, \{19\}\}$

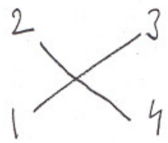
[7]

R :



$$[R] = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

S :



$$[S] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

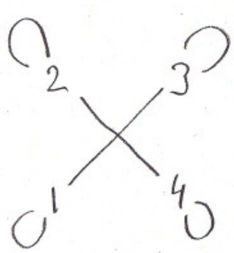
$RS = R$:

$$[RS] = [R][S] = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = [R]$$

$SR = R$

$$[SR] = [S][R] = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

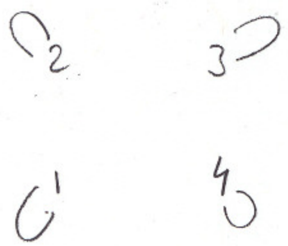
$R^{\circ 2}$:



$$[R]^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}, \quad [R^{\circ 2}] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$R^{\circ 3} = R$:

$$[R^{\circ 2}][R] = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}, \quad [R^{\circ 3}] = [R]$$

$S^{\circ 2} = \Delta_A$: 

$$[S][S] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

All the distinct powers of R: $\Delta_A = (R^{\circ 0})$, R and $R^{\circ 2}$

$$\left(R^{\circ n} = \begin{cases} R & \text{if } n = 2k+1 \quad (k \in \mathbb{N}) \\ R^{\circ 2} & \text{if } n = 2k \quad (k \in \mathbb{N}, k > 0) \\ \Delta_A & \text{if } n = 0 \end{cases} \right)$$

All the distinct powers of S: Δ_A and S

$$\left(S^{\circ n} = \begin{cases} \Delta_A & \text{if } n \text{ is even} \\ S & \text{if } n \text{ is odd} \end{cases} \right)$$

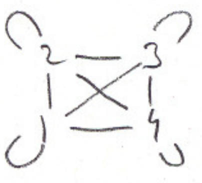
Possibilities for $R^{\circ m} \circ S^{\circ n}$:

$$\left.\begin{matrix} \Delta_A \\ R \\ R^{\circ 2} \end{matrix}\right\} \left\{\begin{matrix} \Delta_A \\ S \end{matrix}\right. = \left\{\begin{matrix} \Delta_A \\ R \\ R^{\circ 2} \end{matrix}\right. \text{ and } S$$

we saw above : $\underline{R \circ S = R}$; so $\underline{R^{\circ 2} \circ S} = R \circ R \circ S = R \circ R = \underline{R^{\circ 2}}$

The same possibilities for $S^{\circ n} \circ R^{\circ m}$.

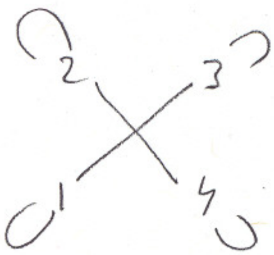
$R^{tr} = R \cup R^{\circ 2} \cup R^{\circ 3} \cup \dots = R \cup R^{\circ 2}$:



$$[R^{tr}] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{(total relation)}$$

$R^{r/tr} = R^{tr}$

$S^{tr} = S \cup S^{\circ 2} \cup S^{\circ 3} \cup \cdots = S \cup \Delta_A$:



$$[S^{tr}] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$S^{r/tr} = S^{tr}$

R has one connected component: $\{1,2,3,4\}$

S has two connected components: $\{1,3\}$ and $\{2,4\}$

[8] Ex1(p.31): (a) strictly antisymmetric $\Longleftrightarrow$ antisymmetric & irreflexive:

(a,1) Suppose $R \subseteq A \times A$ is strictly antisymmetric; we show: it is $\underbrace{\text{antisymmetric}}_{(1)}$ & $\underbrace{\text{irreflexive}}_{(2)}$.

(1): this says: $[aRb \ \& \ bRa \Rightarrow a=b]$ for all $a,b \in A$;
The assumption says: "aRb & bRa" is impossible; that is "aRb & bRa" is always FALSE ($\perp$). But then we have $\perp \Rightarrow a=b$, which is T (TRUE) ($\perp \Rightarrow$ any $=$ T)

(2): We have $aRb \Rightarrow \neg(bRa)$ for any a & b in A.
Put $a=b$, any element in A. Get: $aRa \Rightarrow \neg(aRa)$, is TRUE
But this means that we must have: aRa is FALSE; otherwise, aRa is TRUE, $\neg(aRa)$ is FALSE, and $aRa \Rightarrow \neg(aRa)$ is $T \Rightarrow \perp$, which is $\perp$ — not the case! — To say that aRa is FALSE for arbitrary $a \in A$ is IRREFLEXIVITY.

(a.2) Suppose $R \subseteq A \times A$ is ANTISYMMETRIC and IRREFLEXIVE; we show it is STRICTLY ANTISYMMETRIC.

Assume aRb, to show $\neg(bRa)$. If we had, contrary to the assertion, bRa, then, by ANTISYMMETRY:

$$aRb \ \& \ bRa \Rightarrow a = b$$

we would have $a = b$. However, aRb and $a = b$ says that aRa, which is impossible by IRREFLEXIVE property of R. We conclude that we cannot have bRa; so we have $\neg(bRa)$ as promised.

(b) dichotomous $\Longleftrightarrow$ trichotomous and reflexive:

OMITTED.

[9] Ex 2 (p. 33):

$a|a$: because $a|a \Longleftrightarrow \exists c \in \mathbb{Z}.\ ac = a$ and we can take $c = 1$.

$\underbrace{a|b \ \& \ b|c}_{\text{assume these}} \Rightarrow a|c$:

assume these; so there are x and $y \in \mathbb{Z}$ such that $b = ax$ and $c = by$; then $c = by = (ax)y = a(xy)$. This implies that $a|c$, since $xy \in \mathbb{Z}$.

[10] : Ex 3 (p. 35):

?: (i) S is irreflexive : by definition [$aSb \Rightarrow a \neq b$]

S is transitive:

$$\underbrace{aSb}_{aRb \ \& \ a\neq b} \ \& \ \underbrace{bSc}_{bRc \ \& \ b\neq c} \stackrel{?}{\Longrightarrow} \underbrace{aSc}_{aRc \ \& \ a\neq c}$$

by R being transitive.

It remains to see: $a \neq c$. But if $a = c$ then aRb gives cRb; and since we have bRc, by the antisymmetry of R, $b = c$ follows, in contradiction to the assumed $b \neq c$.

(ii) R is reflexive : by definition [$a = b \Rightarrow aRb$]

R is transitive:

$$\underbrace{aRb}_{\boxed{aSb \text{ or } a=b}} \ \& \ \underbrace{bRc}_{\boxed{bSc \text{ or } b=c}} \stackrel{?}{\Longrightarrow} \underbrace{aRc}_{aSc \text{ or } a=c}$$

Case 1. aSb and bSc. Now, aSc by S being transitive

Case 2. aSb and $b=c$. Now, aSc is automatic.

Case 3. $a=b$ and bSc. aSc follows

Case 4. $a=b$ and $b=c$. Now, $a=c$ follows

But, the two boxes imply that one of the four cases must hold. DONE.

(iii) For $(R^\#)^* = R$ (now, R is a reflexive order):

$$\begin{aligned}(a,b)\in(R^\#)^* &\iff (a,b)\in R^\# \text{ or } a=b\\ &\iff \underbrace{(aRb \text{ and } a\neq b)} \text{ or } \underbrace{a=b}\\ &\overset{?}{\iff} aRb\end{aligned}$$

1. Assume aRb. Then, either $a\neq b$, in which case $\underbrace{aRb \text{ and } a\neq b}$; or $a=b$, in which case $\underbrace{a=b}$. DONE.

2. Assume (aRb and $a\neq b$) or $a=b$, to show aRb. If the first alternative holds, aRb is true. But if $a=b$, then too, aRb, since R is reflexive. DONE

For $(S^*)^\# = S$:

$$\begin{aligned}(a,b)\in(S^*)^\# &\iff (a,b)\in S^* \text{ and } a\neq b\\ &\iff (aSb \text{ or } a=b) \text{ and } a\neq b\\ &\overset{?}{\iff} aSb\end{aligned}$$

1. Assume aSb. Then $a\neq b$ follows, since S is irreflexive. So, (aSb or $a=b$) and $a\neq b$ follows.
2. Assume: (aSb or $a=b$) and $a\neq b$. But then $a=b$ is impossible, so aSb follows. DONE.

[11] Ex 4 (p. 35):

a) Suppose R is a reflexive total order. We need to show (after Ex 3) that $R^{\#}$ has trichotomy:

$$aR^{\#}b \quad \text{or} \quad a=b \quad \text{or} \quad bR^{\#}a \qquad (1)$$

Assume $a \neq b$, to show $aR^{\#}b$ or $bR^{\#}a$. By dichotomy for R, either aRb, or bRa. Since $a \neq b$, in the first case $aR^{\#}b$, in the second $bR^{\#}a$. We have shown:

$a \neq b$ implies: either $aR^{\#}b$, or $bR^{\#}a$.

This is the same as (1).

b) Suppose S is an irreflexive total order. After Ex 3, what remains to show is: S^* has dichotomy.

So, let $a, b \in A$. By trichotomy for S, we have:

either aSb — in which case aS^*b

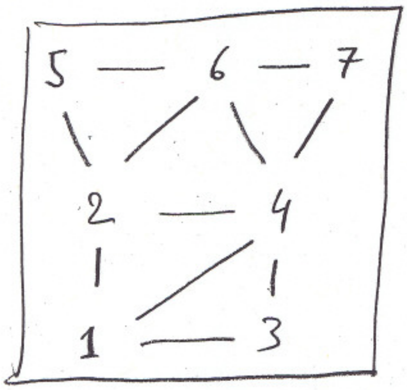
or $a=b$ — in which case also, aS^*b (as well as bS^*a)

or bSa — in which case bS^*b.

In any case, either aS^*b, or bS^*a. DONE.

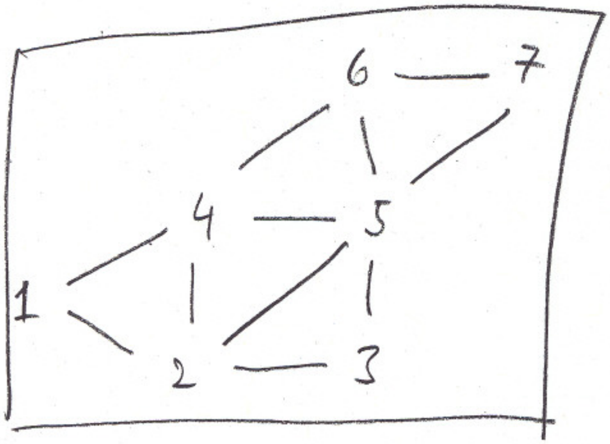
[12] G_2 :

!



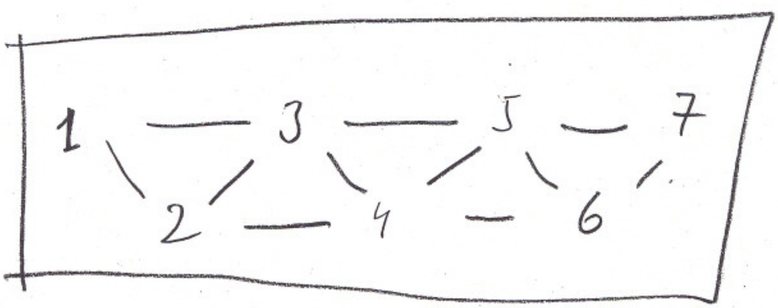
1 : 3
2 : 4
3 : 2
4 : 5
5 : 2
6 : 4
7 : 2

G_1 :



1 : 2
2 : 4
3 : 2
4 : 4
5 : 5
6 : 3
7 : 2

G_3 :



1 : 2
2 : 3
3 : 4
4 : 4
5 : 4
6 : 3
7 : 2

$G_2 \cong G_1$ by the isomorphism

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 1 & 2 & 3 \end{pmatrix}$$

$G_3 \ncong G_1$ ($G_3 \ncong G_2$) : G_3 has no triangle with three adjoining triangles on its three sides; G_1 and G_2 each have one : 246 in G_1, 245 in G_2