MATH 318 / FALL, 2007

Examples of entailments in predicate logic

Example 1

(*) "All horses are animals. Therefore, all heads of horses are heads of animals."

Formalization: $Hx \equiv$ "x is a horse"

$Ax \equiv$ "x is an animal"

$Hxy \equiv$ "x is the head of y"

Under this: "All horses are animals" $\equiv \forall x(Hx \to Ax)$

"x is the head of a horse" $\equiv \exists y(Hy \wedge Hxy)$

"x is the head of an animal" $\equiv \exists y(Ay \wedge Hxy)$

"All heads of horses are heads of animals" $\equiv$ "For all x, if x is the head of a horse, then x is the head of an animal"

$\equiv \forall x[\exists y(Hy \wedge Hxy) \to \exists y(Ay \wedge Hxy)]$

The entailment (*) becomes:

$$\forall x(Hx \to Ax) \vdash \forall x[(\exists y(Hy \wedge Hxy)) \to \exists y(Ay \wedge Hxy)]$$

$\boxed{1}$ Informal proof: "Assume premiss: $\forall x\,(Hx \to Ax)$.
To show conclusion, let x be arbitrary, and assume
$\boxed{2}$ antecedent of implication: $\exists y\,(Hy \wedge Hxy)$, to prove consequent: $\underline{\exists y(Ay \wedge Hxy)}$.
$\boxed{3}$ Let y be such that $Hy \wedge Hxy$.
$\boxed{4}$ Apply premiss for y as x, to conclude that $Hy \to Ay$.
Since Hy holds, we have Ay. On the other hand, we also have Hxy.
$\boxed{5},\boxed{6}$ So, we have $Ay \wedge Hxy$; and thus $\exists y\,(Ay \wedge Hxy)$. Done."

↑ These numbers refer to the lines of the formal deduction below

Formal deduction:

1	1	$\forall x(Hx \to Ax)$	P
2	2	$\exists y(Hy \wedge Hxy)$	P
3	3	$Hy \wedge Hxy$	P
1	4	$Hy \to Ay$	US: 1 (y for x)
1, 3	5	$Ay \wedge Hxy$	T: 3, 4
1, 3	6	$\exists y(Ay \wedge Hxy)$	EG: 5
1, 2	7	$\exists y(Ay \wedge Hxy)$	ES: 6 (2&3) y is not free in $\boxed{1}$, $\boxed{6}$
1	8	$\exists y(Hy \wedge Hxy) \longrightarrow \exists y(Ay \wedge Hxy)$	D: 7
1	9	$\forall x(\exists y(Hy \wedge Hxy) \to \exists y(Ay \wedge Hxy))$	UG: 8

Explanations: Line 9 is the desired entailment (see line 1).

Line 5: let's rewrite line 3 as $A \wedge B$, and line 4 as $A \to C$. Then line 5 is $A \wedge C$. Rule T, is applied because

$$\underbrace{A \wedge B}_{3}, A \to C \vdash A \wedge C$$

Boolean entailment; "obviously correct"

(Boolean verification: $(A \wedge B)(\bar{A} \vee C) \overset{?}{\leq} A \wedge C$

$AB(\bar{A} \vee C) = ABC$; $ABC \leq A$ and $ABC \leq C$; so, $ABC \leq A \wedge C$)

Line 7: the application of ES is this:

$$\frac{\forall x(Hx \to Ax),\quad Hy \wedge Hxy \;\vdash\; \exists y(Ay \wedge Hxy)}{\forall x(Hx \to Ax),\quad \exists y(Hy \wedge Hxy) \;\vdash\; \exists y(Ay \wedge Hxy)}$$

y is not free in

Line 8: The rule D: $\dfrac{\Phi, \Psi \vdash \Gamma.}{\Phi \vdash \Psi \to \Gamma}$

In our case: $\dfrac{\boxed{1}, \boxed{2} \vdash \boxed{7}}{\boxed{1} \vdash \boxed{2} \to \boxed{7}}$ ← line 7 / ← line 8

$\boxed{\text{Example 2}}$ $\forall x \forall y\,(Rxy \vee Ryx) \vdash \forall x \forall y\, \exists z\,(Rxz \wedge Ryz)$

Informal proof:

$\boxed{1}$ → assuming premiss,

fix x and y, to find z such that $\overset{?}{\boxed{Rxz \wedge Ryz.}}$

Applying premiss, we have

$\boxed{4}$ → either Rxy (Case1)

$\boxed{10}$ → or Ryx (Case2)

$\boxed{7}$ → In Case1, we again apply premiss to get $Ryy \vee Ryy$ which is Ryy.

$\boxed{8}$, $\boxed{9}$ → But then, in Case1, $Rxy \wedge Ryy$, and we can choose $z = y$.

$\boxed{13}$ → In Case2, similarly, we can choose $z = x$. Done.

Formal deduction:

1	1	$\forall x \forall y (Rxy \vee Ryx)$	P
1	2	$\forall y (Rxy \vee Ryx)$	US:1
1	3	$Rxy \vee Ryx$	US:2
4	4	Rxy	P
1	5	$\forall x \forall z (Rxz \vee Rzx)$	CBV:1
1	6	$\forall z (Ryz \vee Rzy)$	US:5
1	7	$Ryy \vee Ryy$	US:6 (y for z)
1,4	8	$Rxy \wedge Ryy$	T: 4, 7
1,4	9	$\exists z (Rxz \wedge Ryz)$	EG:8 (y for z)
10	10	Ryx	P
1	11	$Rxx \vee Rxx$	US: 2 (x for y)
1,10	12	$Rxx \vee Ryx$	T: 10,11
1,10	13	$\exists z (Rxz \wedge Ryz)$	EG: 12 (x for z)
1	14	$\exists z (Rxz \wedge Ryz)$	AC: 9, 13

Explanations:

$\boxed{\text{Line 5}}$: CBV : Change of Bound Variables

We changed y in $\boxed{\forall x \forall y\,(Rxy \vee Ryx)}$ to z, and got

$$\forall x \forall z\,(Rxz \vee Rzx)$$

Why? Because, we want to get $Ryy \vee Ryy$ from $\boxed{1}$. However, doing US, we cannot substitute y for x in $\forall y\,(Rxy \vee Ryx)$, because this would be an illegal substitution: the new occurrences of y (in place of x) would become bound. On the other hand, if we use $\forall x \forall z\,(Rxz \vee Rzx)$ instead of $\forall x \forall y\,(Rxy \vee Ryx)$, there is no problem with substituting y for x in $\forall z\,(Rxz \vee Rzx)$.

$\boxed{\text{Line 14}}$: Rule AC (Argument by Cases):

$$\frac{A, \boxed{B} \vdash D \qquad A, C \vdash D}{A, B \vee C \vdash D}$$

In our application, this was:

$$\frac{\boxed{1}, \boxed{4} \vdash \boxed{9} \qquad \boxed{1}, \boxed{10} \vdash \boxed{9}}{\boxed{1}, \underbrace{\boxed{4} \vee \boxed{10}}_{\boxed{3}} \vdash \underset{\boxed{14}}{\underset{\parallel}{\boxed{9}}}}$$

$$\boxed{1}, \boxed{3} \vdash \boxed{14}$$

But also, we had

$$\boxed{1} \vdash \boxed{3}$$

so we can take away $\boxed{3}$, and get

$$\boxed{1} \vdash \boxed{14} \quad \text{which is line 14.}$$

$\boxed{\text{Example 3}}$ $\forall x \exists y (x = fy),\ \forall x \exists y R(fx, y) \vdash \forall x \exists y R(x, fy)$

Informal argument:

We assume the premisses.

We want, for any given x, to find y such that $R(x, fy)$.

Take x as given, and apply premiss no. 1
to get z such that $x = fz$

Apply premiss no. 2 to z as x,
to get w such that $R(fz, w)$.

Since $x = fz$, we have $R(x, w)$.

Finally, apply premiss no. 1 again, now to w,
to find y such that $w = fy$.

Since $R(x, w)$, we have $R(x, fy)$. DONE.

Formal deduction:

1	1	$\forall x \exists y (x = fy)$	P
2	2	$\forall x \exists y R(fx, y)$	P
1	3	$\exists y (x = fy)$	US:1
1	4	$\exists z (x = fz)$	CBV:3
5	5	$x = fz$	P
2	6	$\exists y R(fz, y)$	US: 2 (z for x)
2	7	$\exists w R(fz, w)$	CBV:6
8	8	$R(fz, w)$	P
5,8	9	$R(x, w)$	E : 5,8 ← $\boxed{\text{Rule of Equality}}$
1	10	$\exists y (w = fy)$	US: 1 (w for x)
11	11	$w = fy$	P

5, 8, 11	12	$R(x, fy)$	E : 9, 11	← $\boxed{\text{Rule of Equality}}$
5, 8, 11	13	$\exists y R(x, fy)$	EG : 12	
5, 8, 1 $\boxed{\text{trading } \boxed{11} \text{ for } \boxed{10}}$	14	$\exists y R(x, fy)$	ES : 13 (active variable : y : not free in $\boxed{5}$, $\boxed{8}$, $\boxed{13}$)	
5, 2, 1 $\boxed{8}$ for $\boxed{7}$	15	$\exists y R(x, fy)$	ES : 14 (active var.: w : not free in $\boxed{5}$, $\boxed{1}$, $\boxed{14}$)	
1, 2 $\boxed{5}$ for $\boxed{4}$	16	$\exists y R(x, fy)$	ES : 15 (active : z : not free in $\boxed{1}$, $\boxed{2}$, $\boxed{15}$)	
1, 2	17	$\forall x \exists y R(x, fy)$	UG : 16 (x not free in 1, 2	

Explanations: $\boxed{\text{Line 17}}$: the desired entailment

$\boxed{\text{Lines 9 \& 12}}$: applications of the Rule of Equality,

which say that "any term can be replaced by an equal term".

Formally:

$$\boxed{t_1 = t_2,\ \Phi[t_1] \vdash \Phi[t_2]} \quad \boxed{t_2 = t_1,\ \Phi[t_1] \vdash \Phi[t_2]}$$

In $\boxed{\text{Line 9}}$: 5 asserts $x = fz$; therefore in $\boxed{8}$, which is $R(fz, w)$, I can replace fz by x, to get $R(x, w)$

Since I used 5 & 8, and the premiss-column for them contains 5, and 8, I have to put 5, 8 into premiss column in 9

In $\boxed{\text{Line 12}}$: we applied

$$\boxed{\underbrace{w = fy}_{\boxed{11}},\ \underbrace{R(x, w)}_{\boxed{9}} \vdash \underbrace{R(x, fy)}_{\boxed{12}}}$$

Since in $\boxed{\text{Line 9}}$, the premisses are 5, 8, in $\boxed{\text{Line 11}}$, it is 11, in Line 12, the premisses are 5, 8, 11