MATH 318/Midterm exam/October 18, 2007

[1] x and y denote real numbers. x is defined to be *rational* if $x=\frac{p}{q}$ for some $p, q\in\mathbb{Z}$, $q\neq 0$ ($\mathbb{Z}$ is the set of all positive, negative and zero integers). x is *irrational* if it is not rational.

1)(5 pts) Show that if any two of the reals x, y, $x+y$ are rational, so is the third one.

2)(5 pts) Let R be the relation on $\mathbb{R}$, the set of real numbers, defined by

$$xRy \iff x-y \text{ is rational .}$$

Prove that $(\mathbb{R}, R)$ is an equivalence relation.

3) Assume we have the following information concerning the reals x and y :

(P_1) "At least one of the reals x, y and $x+y$ is irrational"
(P_2) "If x is rational, then either y is rational, or $x+y$ is rational"
(P_3) "If y is rational, then either x is rational, or $x+y$ is rational"
(P_4) "If $x+y$ is rational, then either x is rational, or y is rational"

It follows *easily* that

(C) "All three numbers x, y and $x+y$ are irrational".

Formalize this inference in the form of an entailment in propositional logic, in the following manner. Write

$$\begin{aligned} A &\equiv x \text{ is rational ,}\\ B &\equiv y \text{ is rational ,}\\ C &\equiv x+y \text{ is rational .}\end{aligned}$$

(3.1)(5 pts) Write each of the four assumptions (P_1), (P_2), (P_3) ,(P_4) and the conclusion (C) as Boolean expressions of A, B and C.

(3.2)(5 pts) To the four premisses obtained in (3.1), **add** suitable further premisses (P_5) ,... , all expressing general mathematical facts, and write down the resulting entailment

$$(P_1), (P_2), (P_3), (P_4), (P_5), \ldots \vdash (C) . \qquad (*)$$

This should be done in such a way that (*) becomes a *correct* entailment in propositional logic, and, also, that none of the added (P_5), ... is superfluous.

You are not asked to prove that your answer is correct.
Proving in an informal manner that (C) in fact follows from assumptions (P_1), (P_2), (P_3), (P_4) is *not* asked for.

(3.3)* (for bonus points only) **Show** that the entailment (*) is correct. *Do not use the standard Boolean calculation* (that will earn no marks at all). Use a short argument.

[2] A *binary operation on* a set B is a function $f: B\times B \to B$. (In particular, when f is applied to any ordered pair (x, y) of elements of B, it gives a well-defined element $f(x, y)$ of B again.)

Let f be a binary operation on the set B, and let X be a subset of B. We say that X is closed under f if the following is true:

for every x *and* y,
if both x *and* y *are in* X, *then* $f(x, y)$ *is in* X *as well.*

Fix the binary operation f on B, and let A be the set of all subsets of B that are closed under f. Let R be the subset relation $\subseteq$ restricted to A (for $X, Y \in A$, $(X, Y)\in R \iff X\subseteq Y$). We have defined the *order* (A, R), on the basis of the given B and f.

1) We consider a *special case* for B and f. Let $B=\{0, 1, 2, 3\}$, and let f be the binary operation on B for which $f(x, y)$ is the remainder of $x\cdot y$ when divided by 4.

(For instance: $f(2, 3)=2$, since $2\cdot 3=6$, which, when divided by 4, gives remainder 2. We can also write $f(x, y)=(x\cdot y) \bmod 4$.)

1A)(5 pts) **Display** the operation f in the form of a 4×4 matrix whose (i, j) entry is $f(i, j)$.

1B)(15 pts) **List** all the subsets of $B=\{0, 1, 2, 3\}$ that are closed under the operation f.

1C)(5 pts) **Draw** the Hasse diagram of the order (A, R) in *the special case*.

1D)(5 pts) **List** all pairs $\{X, Y\}$ of elements X and Y of A such that X and Y are incomparable ($X\neq Y$, $X\not\subset Y$ and $Y\not\subset X$), and **determine** $X\wedge Y$, $X\vee Y$ in each case.

2A)(10 pts) *In the general case*, p**rove** that if X and Y are both in A, then so is $X \cap Y$.

2B)* **(for bonus points only)** In the general case, (A, R) is a complete lattice. Briefly state the reasons for this, without going into the details of the proof.

[3] Let A, B, C and D be the following subsets of $\mathbb{R}^2$:

$$A = [y>2]\ ,\ B=[(x-2)^2+y^2<1]\ ,\ C=[(x+2)^2+y^2<1]\ ,\ D=[x^2+y^2<16]\ .$$

1)(10 pts) **Determine**, in the form of Boolean expressions of A, B, C and D , all the atoms of the Boolean subalgebra $\langle A, B, C, D\rangle$ of $(\mathcal{P}(\mathbb{R}^2), \subseteq)$ generated by A, B, C and D .

2)(5 pts) **Determine** the cardinality of $\langle A, B, C, D\rangle$.

3)(5 pts) **Write** each of the generators A, B, C, D as a join of atoms of $\langle A, B, C, D\rangle$.

[4] Let $U \overset{\text{DEF}}{=} (((A\to B)\to C)\to D)$ $V \overset{\text{DEF}}{=} (((D\to C)\to B)\to A)$

$$X \overset{\text{DEF}}{=} U \longrightarrow V\ , \qquad Y \overset{\text{DEF}}{=} V \longrightarrow U\ , \qquad Z \overset{\text{DEF}}{=} X \longrightarrow Y\ .$$

1)(10 pts) **Determine** disjunctive forms (in terms of the letters A, B, C, D) for the Boolean expressions U , V , X , Y , and Z ; make them as short as you can.

2)(5 pts) **Determine** the full disjunctive normal form for Z .

3)(5 pts) Go back to question [3], and let the letters A, B, C, D have the values as specified there. Then Z of the present problem becomes a subset of $\mathbb{R}^2$. *Doing a minimal amount of additional calculation only*, **prove** that, in this case, $Z = \bar{A} \vee D$. (**Hint:** use the atoms of $\langle A, B, C, D\rangle$ and the FDNF of Z .)

Try to be economical! The grading of problem [4] *will take into account how economical your calculations are.*

You may make use of the fact that U *and* V *have the same form, differing only in what order the letters appear in.*