Answers/Mdt/MATH318/Fall 2007

[1] **1)** Assume that x and y are rational, to show that $x+y$ is rational. We have $x=\frac{p}{q}$, $y=\frac{r}{s}$, with $p, q, r, s\in\mathbb{Z}$, $q\neq 0$, $s\neq 0$. Then $x+y=\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}$. Since $ps+qr,\ qs\in\mathbb{Z}$, and $qs\neq 0$, we have that $x+y$ is rational.

If $u=\frac{p}{q}$ is rational, then so is $-u=-\frac{p}{q}=\frac{-p}{q}$.

If x and $x+y$ are rational, then so is $(x+y)+(-x)=y$ is rational, by the above.

The same goes for "If x and $x+y$ are rational, then so y."

2) R is reflexive: xRx, since $x-x=0$ is rational ($0=\frac{0}{1}$).
R is symmetric: $xRy\Rightarrow yRx$: if $x-y$ is rational, so is $y-x=-(x-y)$.
R is transitive: $xRy\ \&\ yRz\implies xRz$: Assume $xRy\ \&\ yRz$. $x-y$ and $y-z$ are rational. Then, by 1), $x-y+y-z$ is rational; that is, $x-z$ is rational, that is, xRz .

(3.1)

$(P_1):\neg A\vee\neg B\vee\neg C$
$(P_2):A\longrightarrow(B\vee C)$
$(P_3):B\longrightarrow(A\vee C)$
$(P_4):C\longrightarrow(A\vee B)$
$(C):\ \neg A\wedge\neg B\wedge\neg C$.

Remark In math, "or" is always meant in the non-exclusive sense. That is, when we say "either A or B ", we mean "either A , or B , or both".

(3.2)

$(P_5):(A\wedge B)\longrightarrow C$
$(P_6):(B\wedge C)\longrightarrow A$
$(P_7):(A\wedge C)\longrightarrow B$

The entailment is

$$\bar{A}\vee\bar{B}\vee\bar{C},\ A\longrightarrow(B\vee C),\ B\longrightarrow(A\vee C),\ C\longrightarrow(A\vee B),$$

$$(A\wedge B)\longrightarrow C,\ (B\wedge C)\longrightarrow A,\ (A\wedge C)\longrightarrow B\ \vdash\ \bar{A}\wedge\bar{B}\wedge\bar{C}$$

(3.3)* First, transform in the usual way:

$$\bar{A}\vee\bar{B}\vee\bar{C},\ \bar{A}\vee B\vee C,\ \bar{B}\vee A\vee C,\ \bar{C}\vee A\vee B,\ \overline{A\wedge B}\vee C,\ \overline{B\wedge C}\vee A,\ \overline{A\wedge C}\vee B\ \vdash\ \bar{A}\wedge\bar{B}\wedge\bar{C}\ ;$$

$$(\bar{A}\vee\bar{B}\vee\bar{C})(\bar{A}\vee B\vee C)(\bar{B}\vee A\vee C)(\bar{C}\vee A\vee B)(\bar{A}\vee\bar{B}\vee C)(\bar{B}\vee\bar{C}\vee A)(\bar{A}\vee\bar{C}\vee B)\ \vdash\ \bar{A}\bar{B}\bar{C}\ .$$

The FDNF of $A\vee B\vee C$ has seven terms, all meet-expressions $\overset{(-)}{A}\cdot\overset{(-)}{B}\cdot\overset{(-)}{C}$ such that at least one of the letters is unbarred (only one missing, the term $\bar{A}\bar{B}\bar{C}$). When you take the complement of $A\vee B\vee C$, you get $\bar{A}\bar{B}\bar{C}$. When you take the complement of the FDNF, you get the expression on the left of $\vdash$. Hence, the two expressions on the left and the right of $\vdash$ are in fact identically equal.

If one takes away any one of the premisses, one gets as complement a FDNF which is strictly $< A\vee B\vee C$ for suitable values of A, B, C : therefore, the expression on the left of $\vdash$ is now strictly $>$ than the conclusion; the entailment is not correct.

[2] **1A)** The table of the operation f :

	y=0	1	2	3
x=0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

1B) Listing the sets closed under f :

#=0 : $\emptyset$:*OK*

#=1 : $\{0\}$:*OK*, $\{1\}$:*OK*, $\{2\}$: *notOK* ($f(2,2)=0$), $\{3\}$:*notOK*

#=2 : $\{0,1\}$:*OK*, $\{0,2\}$:*OK*, $\{0,3\}$:*notOK* ($f(3,3)=1$) ,
$\{1,2\}$: *notOK* , $\{1,3\}$: *OK*(!), $\{2,3\}$:*notOK*

#=3 : $\{0,1,2\}$:*OK*, $\{0,1,3\}$:*OK*, $\{0,2,3\}$:*notOK*, $\{1,2,3\}$:*notOK*.

#=4 : $\{0,1,2,3\}$:*OK*

List of A : $A=\{X_1=\emptyset, X_2=\{0\}, X_3=\{1\}, X_4=\{0,1\}, X_5=\{0,2\}, X_6=\{1,3\}, X_7=\{0,1,2\}, X_8=\{0,1,3\}, X_9=\{0,1,2,3\}\}$.

1C) Hasse:

$$
\begin{tikzcd}
 & & x_9 & & \\
 & x_7 \arrow[ur] & & x_8 \arrow[ul] & \\
x_5 \arrow[ur] & & x_4 \arrow[ul] \arrow[ur] & & x_6 \arrow[ul] \\
 & x_2 \arrow[ul] \arrow[ur] & & x_3 \arrow[ul] \arrow[ur] & \\
 & & x_1 \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

1D) Incomparables and their meets and joins:

x_2, x_3 : $x_2 \vee x_3 = x_4$, $x_2 \wedge x_3 = x_1$
x_2, x_6 : $x_2 \vee x_6 = x_8$, $x_2 \wedge x_6 = x_1$
x_3, x_5 : $x_3 \vee x_5 = x_7$, $x_3 \wedge x_5 = x_1$

x_5, x_4 : $x_5 \vee x_4 = x_7$, $x_5 \wedge x_4 = x_2$
x_5, x_6 : $x_5 \vee x_6 = x_9$, $x_5 \wedge x_6 = x_1$
x_4, x_6 : $x_4 \vee x_6 = x_8$, $x_4 \wedge x_6 = x_3$

x_5, x_8 : $x_5 \vee x_8 = x_9$, $x_5 \wedge x_8 = x_2$
x_6, x_7 : $x_6 \vee x_7 = x_9$, $x_6 \wedge x_7 = x_3$
x_7, x_8 : $x_7 \vee x_8 = x_9$, $x_7 \wedge x_8 = x_4$

2A) Assume that X , Y are in A , to show that $X \cap Y$ is in $\underset{?}{A}$. We need to show that $X \cap Y$ is closed under f . Let $x, y \in X \cap Y$, to show that $f(x, y) \in X \cap Y$. Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq X$, we have that $x, y \in X$ and $x, y \in Y$. Since both X and Y are closed under f , we have that $f(x, y) \in X$ and $f(x, y) \in Y$. Therefore, $f(x, y) \in X \cap Y$, which was to be shown

2B)* By a similar argument, we can show that if $\mathcal{X}$ is a non-empty subset of A , then $\bigcap \mathcal{X}$ is again in A . It follows that, in this case, the meet $\bigwedge \mathcal{X}$ exists in A , and it is equal to $\bigcap \mathcal{X}$. For $\mathcal{X} = \emptyset$, $\bigwedge \emptyset (= \top) = B$. Since $\bigwedge \mathcal{X}$ exists for any $\mathcal{X} \subseteq$ A , (A,$\subseteq$) is a complete lattice.

[3] **1)**

$AB \quad = \perp$

A

$A\bar{B}C \quad = \perp$

$A\bar{B} = A \neq \perp$

$A\bar{B}\bar{C} = A \neq \perp$

$A\bar{B}\bar{C}D = AD \neq \perp \ : \ A\bar{B}\bar{C}D = AD \ : atom$

$A\bar{B}\bar{C}\bar{D} = A\bar{D} \neq \perp \ : \ A\bar{B}\bar{C}\bar{D} = A\bar{D} \ : atom$

$\bar{A}BC = \perp$

$\bar{A}B \neq \perp$

$\bar{A}B\bar{C} = B \neq \perp$

$\bar{A}B\bar{C}D = B \neq \perp \ : \ \bar{A}B\bar{C}D = B : atom$

$\bar{A}B\bar{C}\bar{D} \quad = \perp$

$\bar{A}$

$\bar{A}\bar{B}CD = C \neq \perp \ : \ \bar{A}\bar{B}CD = C : atom$

$\bar{A}\bar{B}C = C \neq \perp$

$\bar{A}\bar{B}C\bar{D} \quad = \perp$

$\bar{A}\bar{B} \neq \perp$

$\bar{A}\bar{B}\bar{C}D \quad \neq \perp \ : \ \bar{A}\bar{B}\bar{C}D \quad : atom$

$\bar{A}\bar{B}\bar{C} \neq \perp$

$\bar{A}\bar{B}\bar{C}\bar{D} \quad \neq \perp \ : \ \bar{A}\bar{B}\bar{C}\bar{D} \quad : atom$

2) $\#\langle A, B, C, D\rangle = 2^{number\ of\ atoms} = 2^6 = 64$.

3) $A = AD \vee A\bar{D}$

$B = B$

$C = C$

$D = B \vee C \vee AD \vee \bar{A}\bar{B}\bar{C}D$.

[4] **1)** $U = (((A \to B) \to C) \to D) = \overline{\overline{\bar{A} \vee B} \ \vee \ C} \ \vee D = (\bar{A} \vee B) \wedge \bar{C} \vee D$

$= (\bar{A} \vee B)\bar{C} \vee D$;

$U = \bar{A}\bar{C} \vee B\bar{C} \vee D$.

$V = (((D \to C) \to B) \to A) \ = \ \bar{D}\bar{B} \vee C\bar{B} \vee A$

$V = \bar{B}\bar{D} \vee \bar{B}C \vee A$

(this is obtained from the previous result by exchanging A, B, C, D for D, C, B, A , respectively).

$$X = U \longrightarrow V = (\bar{A}\bar{C} \vee B\bar{C} \vee D) \longrightarrow (\bar{B}\bar{D} \vee \bar{B}C \vee A) =$$

$$= \overline{(\bar{A}\bar{C} \vee B\bar{C} \vee D)} \vee (\bar{B}\bar{D} \vee \bar{B}C \vee A) = (A \vee C)(\bar{B} \vee C)\bar{D} \vee \bar{B}\bar{D} \vee \bar{B}C \vee A$$

$$= \underbrace{A\bar{B}\bar{D}}_{} \vee \underbrace{\bar{B}C\bar{D}}_{} \vee \underbrace{AC\bar{D}}_{} \vee C\bar{D} \vee \bar{B}\bar{D} \vee \bar{B}C \vee A = C\bar{D} \vee \bar{B}\bar{D} \vee \bar{B}C \vee A$$

absorbed into
A *and* $\bar{B}\bar{D}$

$$X = C\bar{D} \vee \bar{B}\bar{D} \vee \bar{B}C \vee A \ .$$

$$Y = V \longrightarrow U = B\bar{A} \vee \bar{C}\bar{A} \vee \bar{C}B \vee D = \bar{A}B \vee \bar{A}\bar{C} \vee B\bar{C} \vee D$$

from the previous result, by exchanging A, B, C, D for D, C, B, A, respectively.

For Z, first $\bar{X}$

$$\bar{X} = \overline{(C\bar{D} \vee \bar{B}\bar{D} \vee \bar{B}C \vee A)} = (\bar{C} \vee D)(B \vee D)(B \vee \bar{C})\bar{A} =$$

$$(B\bar{C} \vee \underbrace{BD}_{abs} \vee \underbrace{\bar{C}D}_{abs} \vee D)(B \vee \bar{C})\bar{A} = (B\bar{C} \vee D)(B \vee \bar{C})\bar{A} = (B\bar{C} \vee BD \vee \underbrace{B\bar{C}}_{rep} \vee \bar{C}D)\bar{A} =$$

$$\bar{X} = \bar{A}B\bar{C} \vee \bar{A}BD \vee \bar{A}\bar{C}D \ .$$

$$Z = X \longrightarrow Y = \bar{X} \vee Y = \underbrace{\bar{A}B\bar{C}}_{abs} \vee \underbrace{\bar{A}BD}_{abs} \vee \underbrace{\bar{A}\bar{C}D}_{abs} \vee \bar{A}B \vee \bar{A}\bar{C} \vee B\bar{C} \vee D$$

$$Z = \bar{A}B \vee \bar{A}\bar{C} \vee B\bar{C} \vee D \ .$$

Better for Z : $Z = (U \to V) \longrightarrow (V \to U) = \overline{\bar{U} \vee V} \vee (\bar{V} \vee U) = U\bar{V} \vee \bar{V} \vee U = \bar{V} \vee U = V \to U =$
Y (!) ;

$$Z = \bar{A}B \vee \bar{A}\bar{C} \vee B\bar{C} \vee D \ .$$

2) $\bar{A}B = \bar{A}BCD \vee \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}D \vee \bar{A}B\bar{C}\bar{D}$.

From $\bar{A}\bar{C}$, we need only those $\bar{A}\overset{(-)}{B}\bar{C}\overset{(-)}{D}$ which do not appear in $\bar{A}B$; there are two of those:

$$\bar{A}B \vee \bar{A}\bar{C} = \underline{\bar{A}BCD \vee \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}D \vee \bar{A}B\bar{C}\bar{D}} \vee \underline{\bar{A}\bar{B}\bar{C}D \vee \bar{A}\bar{B}\bar{C}\bar{D}} .$$

From $B\bar{C}$, we need only those $\overset{(-)}{A}B\bar{C}\overset{(-)}{D}$ which do not appear before; there are two of those:

$$\bar{A}B \vee \bar{A}\bar{C} \vee B\bar{C} =$$

$$= \underline{\bar{A}BCD \vee \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}D \vee \bar{A}B\bar{C}\bar{D}} \vee \underline{\bar{A}\bar{B}\bar{C}D \vee \bar{A}\bar{B}\bar{C}\bar{D}} \vee \underline{AB\bar{C}D \vee AB\bar{C}\bar{D}}$$

Finally, from D, from the eight $\overset{(-)}{A}\overset{(-)}{B}\overset{(-)}{C}D$, there are four ones not appearing before:

$$\bar{A}B \vee \bar{A}\bar{C} \vee B\bar{C} \vee D =$$

$$= \underline{\bar{A}BCD \vee \bar{A}BC\bar{D} \vee \bar{A}B\bar{C}D \vee \bar{A}B\bar{C}\bar{D}} \vee \underline{\bar{A}\bar{B}\bar{C}D \vee \bar{A}\bar{B}\bar{C}\bar{D}} \vee \underline{AB\bar{C}D \vee AB\bar{C}\bar{D}} \vee$$

$$\underline{ABCD \vee A\bar{B}CD \vee A\bar{B}\bar{C}D \vee \bar{A}\bar{B}CD}$$

3) In the FDNF expression of Z, the only atom of $\langle A, B, C, D\rangle$ that is missing is the second one, $A\bar{B}\bar{C}\bar{D} = A\bar{D}$. This means that Z is the complement of $A\bar{D}$, $Z=\bar{A}\vee D$.