Assignment 4/MATH318/Fall, 2007
Due: Wednesday, October 31

[1] Below, you find conditions (i) to (ix) for formulas Φ_1, Φ_2, $\ldots$, Φ_9. Determine each formula such that it satisfies the condition *in any order* $(A;\le)$. Each formula should

use the indicated free variables only

and it should

use only $\le$ *and* $=$ *as relations.*

In the later examples, you may use earlier formulas by their names $\Phi_1(-)$, $\Phi_2(-)$, $\Phi_3(-,-)$, with appropriate variables put into the places marked by the blanks, instead of writing out those formulas in full.

(i) $\Phi_1(z) \iff z=\top$

[That is: for any element z in A, $\Phi_1(z)$ is true in the order $(A;\le)$ if and only if z is the top element of $(A;\le)$.]

(ii) $\Phi_2(z) \iff z=\bot$

(iii) $\Phi_3(x,y,z) \iff z=x\wedge y$

(iv) $\Phi_4(x,y,z) \iff z=x\vee y$

(v) $\Phi_5 \iff \top$ exists

[Now, Φ_5 has no free variables; it is a sentence that is true or false given any relation, in particular, any order $(A;\le)$. It should be given so that it is true in $(A;\le)$ if and only if $(A;\le)$ has a top element.]

(vi) $\Phi_6 \iff \bot$ exists

(vii) $\Phi_7(x,y) \iff x\wedge y$ exists

(viii) $\Phi_8(x,y) \iff x\vee y$ exists

(ix) $\Phi_9 \iff (A;\le)$ is a lattice.

[2] Find sentences Σ_{dl} and Σ_{Ba} *using only the relations* $\le$ *and* $=$, such that, for any binary relation $(A;\le)$, we have

$(A;\le)\models\Sigma_{\mathrm{dl}}$ iff $(A;\le)$ is a distributive lattice;

$(A;\le)\models\Sigma_{\mathrm{Ba}}$ iff $(A;\le)$ is a Boolean algebra.

You are encouraged to define the formulas in stages. Use earlier formulas by their names in later formulas.

[3] Consider the formula $\Phi_1 :=: \forall x \exists y (Rxy \wedge \forall z (Rxz \rightarrow z = y))$, and the interpretation $(A_1; R_1)$ in which $A_1 = \{0, 1\}$ and $R_1 = \{(0, 1), (1, 0)\}$. Show that $(A_1; R_1) \vDash \Phi_1$ by exhibiting the detailed calculation of the truth-value tables of all the subformulas of Φ_1, including Φ_1 itself, in $(A_1; R_1)$.

[4] We list some sentences and structures; the only extra-logical primitive is a binary relation R. Sentences:

Φ_1 : the sentence in [3]
$\Phi_2 :=: \forall x \exists y Rxy$, $\qquad \Phi_3 :=: \exists y \forall x Rxy$,
$\Phi_4 :=: \forall x \exists y (Rxy \vee \forall z (z = x \vee Rzx \vee Rzy))$.

Structures (draw the digraphs!):

$M_1 = (A_1; R_1)$: the structure of [3]
$M_2 = (A_2; R_2)$: $A_2 = \{0,1,2,3\}$, $R_2 = \{(1,0),(2,1),(3,0)\}$.
$M_3 = (A_3; R_3)$: $A_3 = \{0,1,2,3\}$, $R_3 = \{(0,1),(0,2),(0,3),(1,2),(2,1)\}$.
$M_4 = (A_4; R_4)$: $A_4 = \{0,1,2,3\}$, $R_4 = \{(0,1),(0,2),(1,2),(2,2),(3,2)\}$.

(i) Decide of each structure M_j and of each sentence Φ_i if the first satisfies the second (there are 16 cases to consider); it is not necessary to give the detailed calculation as in [3].

(ii) Convert each of the sentences Φ_i, $\neg\Phi_i$ $(i = 1, 2, 3, 4)$ to negation normal form (if it is not already in that form).

(iii) Give the Skolem form of each of the eight sentences mentioned in (ii).

(iv) For each pair of values of i and j, $i, j = 1, 2, 3, 4$, one of the statements $M_j \vDash \Phi_i$, $M_j \vDash \neg\Phi_i$ holds as determined in (i). In each case, give some appropriate Skolem functions witnessing the corresponding fact. (If $M_j \vDash \Phi_i$, you'll need the Skolem form of Φ_i determined in (iii); if $M_j \vDash \neg\Phi_i$, the Skolem form of $\neg\Phi_i$.)

[5] Here is a list of sentences Ψ_i, $i = 1, \ldots, 7$. Decide of each if it is logically valid. ("Ψ is logically valid" means Ψ is always true, no matter how we interpret it in a *non-empty* structure.) In case the answer is YES, give an *informal* proof of the fact. In case the answer is NO, provide structure that is a counter-example to the validity of the sentence.

(**Hint:** in all but one of the cases that require a counter-example, a small finite structure will do; in one case, you need an infinite counter-example.)

$\Psi_1 :=: \forall x \exists y Rxy \rightarrow \exists y \forall x Rxy$

$\Psi_2 :=: \exists y \forall x Rxy \rightarrow \forall x \exists y Rxy$

$$\Psi_3 :=: (\forall x\forall y(Rxy\to\neg Ryx) \wedge \forall x\exists yRxy \wedge \forall y\exists xRxy) \longrightarrow \forall x\forall y(Rxy\vee x=y\vee Ryx)$$

$$\Psi_4 :=: \forall x\exists yRxy \longrightarrow \forall x\exists y\exists z(Rxy\wedge Ryz)$$

$$\Psi_5 := (\forall xRxx \wedge \forall x\forall y(Rxy\to Ryx) \wedge \forall x\forall y\forall z((Rxy\wedge Ryz)\to Rxz)) \longrightarrow \forall x\forall y\forall z(Rxy \vee Rxz \vee Ryz)$$

$$\Psi_6 :=: (\forall x\forall y(Rxy\to Ryx) \wedge \forall x\forall y((Rxy \wedge Ryx)\to x=y)) \longrightarrow \forall x\forall y\forall z((Rxy\wedge Ryz)\to Rxz))$$

$$\Psi_7 :=: (\forall x\forall y\forall z((Rxy\wedge Ryz)\to Rxz) \wedge \forall x\neg Rxx) \longrightarrow \exists x\forall y\neg Rxy$$

[6] Show that the (long) sentence given below is satisfiable, by giving a (small) finite (non-empty) structure $(A; R)$ satisfying it:

$$\forall x\exists yRxy \wedge \forall x\forall y(Rxy\to\neg Ryx) \wedge \forall xyuv((Rxy \wedge Ruv)\longrightarrow(x=u\longleftrightarrow y=v)) \wedge \exists x\exists y(\neg Rxy \wedge \neg Ryx \wedge \neg x=y) .$$

Support your answer by "skolemizing": providing (1) the Skolem normal form of the sentence, and (2) particular Skolem functions on the structure $(A; R)$.

[7] We consider the following five sentences in predicate logic:

$$\Phi_1 :=: \forall x\forall z[Rxz\longrightarrow\exists y(Rxy\wedge Ryz)]$$
$$\Phi_2 :=: \forall x\exists y[Rxy \wedge \forall z(Rxz\to(y=z \vee Ryz)]$$
$$\Phi_3 :=: \exists x\forall y[x=y \vee Rxy]$$
$$\Phi_4 :=: \exists x\forall y[x=y \vee Ryx]$$
$$\Phi_5 :=: \exists x(\exists yRyx \wedge \forall y(Ryx\longrightarrow\exists z(Ryz \wedge Rzx)) .$$

We consider the following six universes, all of them subsets of $\mathbb{R}$, the set of all reals:
$U_1=\mathbb{N}$, $U_2=\mathbb{Z}$, $U_3=\mathbb{Q}$, $U_4=\mathbb{R}$, $U_5=[0, 1] = \{x\in\mathbb{R}:0\le x\le 1\}$, $U_6=\{\frac{n}{n+1}:n\in\mathbb{N}\}\cup\{1\}$.
We let R_i be the ordinary <-relation on the reals restricted to U_i : $R_i={<}\restriction U_i$, for $i=1, ..., 6$. Give a 5×6 table containing the information on whether or not $(U_i; R_i)\models\Phi_j$ for all $i=1, ..., 6$ and $j=1, ...5$. Support **four** of your answers by "skolemizing" as in [4]; they should involve Φ_1 and Φ_2 and their negations like in $(U_i; R_i)\models\Phi_j$ and $(U_i; R_i)\models\neg\Phi_j$ for $j=1$ and $j=2$, with optional i's.

[8](optional) **(i)** In what follows, the universe (the range of the variables) is $\mathbb{R}$, the set of all reals. Let $f:\mathbb{R}\to\mathbb{R}$ any function (defined on all of $\mathbb{R}$). Express continuity of f (if that is the case) in predicate logic as follows. Use the primitives $P(-)$, $D(-, -, -)$ defined by

$P(u) \equiv u$ is positive $\qquad D(x, y, u) \equiv |x-y| < u$,

and write down a sentence Φ such that $(\mathbb{R}; P, D, f) \models \Phi$ if and only if f is continuous (everywhere). (In other words: "specify the concept of continuous function on $\mathbb{R}$ on the basis of the given predicates P and D ".). Make sure Φ is a "natural" sentence, directly expressing continuity.

(ii) Give the NNF of both Φ and $\neg\Phi$.

(iii) Provide Skolem function(s) witnessing the facts that **(a)** the exponential function $f = \exp$ ($f(x) = e^{x}$) is continuous, and **(b)** the step-function $f = s$ defined by

$$s(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is not continuous ($(\mathbb{R}; P, D, \exp) \models \Phi$, $(\mathbb{R}; P, D, s) \models \neg\Phi$).