Assignment 3/MATH 318/Fall 2007
Due: Friday, October 12

[1] Determine

(i) (preferably short) disjunctive forms

and

(ii) full disjunctive normal forms

for the Boolean expressions X, Y, Z, U, V listed. For (i), make the disjunctive forms as short as you can.

$$X=(A\to B)\to C)\to\bar{B}\bar{C}D\ ,\quad Y=\neg((A\to C)\to(B\to D))\ ,\quad Z=\neg(B\to(\bar{C}\to AB))\ ,$$
$$U=((A\to C)\wedge(B\vee C)\wedge(D\to C))\vee((\neg B)\wedge(A\vee D))\ ,$$
$$V=((\neg A)\wedge B\wedge\neg D)\vee C\vee((A\vee C)\wedge(D\to A)\wedge(B\to C)\wedge(B\to\neg D))\ .$$

[2] Consider the following four subsets of $\mathbb{R}^2$:

$$A=[x^2+(y-3)^2<1]\qquad B=[y>x]\qquad C=[\frac{x^2}{4}+y^2<1]\qquad D=[y<x-3]$$

(i) Determine the atoms of the Boolean subalgebra $\langle A, B, C, D\rangle$ of $(\mathcal{P}(\mathbb{R}^2), \subseteq)$; write the atoms as Boolean expression in terms of the generators A, B, C, D .

(ii) Let X , Y and Z be the elements of $\langle A, B, C, D\rangle$ given by the expressions X , Y and Z in [1], with the present values of A, B, C and D . Using the work in [1](ii) and [2](i), write each of X , Y and Z in the form of a join of atoms of $\langle A, B, C, D\rangle$.

(iii) Decide, using (ii), which of the relations $U<V$ hold, where U and V are any of the three elements X, Y and Z of $\langle A, B, C, D\rangle$.

[3] **(i)(optional)** Let $(P, \le)$ be a Boolean algebra, $(Q, \le)$ a finite subalgebra of $(P, \le)$. Prove that an element $x\in P$ belongs to the subset Q if an only if for all atoms a of $(Q, \le)$, we have $a\wedge x=a$ or $a\wedge x=\bot$.

(ii) Consider the subsets $S=[(y-x)(x-y-3)<0]$ and $T=[(y-x)(x-y-3)\le 0]$ of $\mathbb{R}^2$. Decide if $S\in\langle A, B, C, D\rangle$, $T\in\langle A, B, C, D\rangle$; here, $\langle A, B, C, D\rangle$ is from [2]. (You may use (i)* even if you have not proved it.)

[4] **(i)** Give a Boolean expression in full disjunctive normal form $\Phi(A, B, C, D)$ in the variables A, B, C, D such that, when A, B, C, D range over $\{\top, \bot\}$, $\Phi=\top$ if and only if at least two of the values A, B, C, D are $\top$.

(ii) Rewrite $\Phi(A, B, C, D)$ equivalently with as few applications of Boolean operations as you can.

[5] We have the following assumptions U, V, W, X concerning the real number x :

$U \equiv$ "either $x+\frac{1}{2}$ is an integer, or $x<22$ ";
$V \equiv$ " x is an integer if and only if $x>20$ ";
$W \equiv$ "either x is an integer, or $x+\frac{1}{2}$ is an integer";
$X \equiv$ "if $x+\frac{1}{2}$ is an integer, then $x>20$ ".

(i) Give an informal argument showing that from these assumptions we can infer that $x=21$.

(ii) Write each of the sentences U, V, W, X as a formula in propositional logic, by using letters A, B, C and D for parts of the sentences that cannot be analyzed within propositional logic any further.

(iii) To make the informal inference of part (i) a formal logical inference, we have to supply further premisses, which are statements that we know are true on the basis of our mathematical knowledge. Determine propositional formulas Y and Z in terms of the given A, B, C, D and also $E \equiv$ " $x=21$ " so that Y and Z are mathematically valid, and they are used implicitly in our inference in part (i) of E from the assumptions U, V, W, X .

(iv) Using Boolean algebra, show that the entailment

$$U, V, W, X, Y, Z \vdash E$$

is valid, and that, on the other hand, the entailments

$$U, V, W, X, Y \vdash E ,$$
$$U, V, W, X, Z \vdash E$$

are not valid.

(v) Use "ordinary" logical argumentation (informal mathematical proof) to deduce E from the premisses U, V, W, X, Y, Z .

[6] Let n be any positive integer. Assume that $\ell_1, \ell_2, \ldots, \ell_n$ are n straight lines in the plane $\mathbb{R}^2$ in general position: no three meet in a single point, and no two are parallel to each other. Each line ℓ_i divides the points in the plane into three disjoint sets H_i , K_i and ℓ_i itself: H_i is the set of points on one side of ℓ_i , K_i is the set of points on the other side of ℓ_i . It is a *theorem* that the Boolean subalgebra $\langle H_1, \ldots, H_n, K_1, \ldots, K_n\rangle$ generated by the $2n$ sets indicated has exactly $1+2n^2$ atoms.

(i) Prove the "theorem" stated above for values the values $n=1$, 2 and 3 .

(ii)*(optional) Prove the "theorem" for all $n\in\mathbb{N}-\{0\}$.

[7] Let n be a positive integer. Let $\mathbf{B}$ be a Boolean algebra. For each i and j with $1 \leq i \leq n+1$ and $1 \leq k \leq n$, let P_{ik} be an element of $\mathbf{B}$. It is a *theorem* that

$$\bigwedge_{i=1}^{n+1} \bigvee_{k=1}^{n} P_{ik} \leq \bigvee \{(P_{ik} \wedge P_{jk}) : 1 \leq i < j \leq n+1,\ 1 \leq k \leq n\} .$$

(i) Prove the "theorem" for $n=2$, in the following way (not the best way, by any means!). Write A, B, etc. for the six letters P_{11}, P_{12}, etc.; write out the inequality in with the new notation, and prove it by the usual Boolean calculation.

(ii) Prove the theorem for the case *when* $\mathbf{B}$ *is* $\mathbf{2}$, *the 2-element Boolean algebra* in the following style. Imagine the letters P_{ik} arranged in an $(n+1) \times n$ matrix, P_{ik} in the ith row and the kth column. Express in words, referring to rows and columns of the matrix, what it means to have the left-hand side expression being equal to $\top$ (true). Do the same for the right-hand side expression. Conclude the result.

(iii) Using an appropriate result from the text, conclude from (ii) the "theorem", for all n and all $\mathbf{B}$.