Assignment 2/MATH 318/Fall 2007
Due: Friday, September 28

[1] **(i)** Let $A=\{1, 2, 3, 4, 5\}$, and $R = \{(1,2), (1,4), (2,3), (2,5), (4,5)\}$. List the Hasse diagrams all irreflexive orders S on the set A for which $R \subseteq S$. Give a systematic listing, and give arguments for your list being complete.

[**Instructions:** Give a list of all the pairs of elements that are incomparable in R^{tr}. Any S as in the question will be determined by deciding of each pair in the list whether it is comparable in S, and if yes, in which way. This will give you an immediate upper estimate on how many S's there are. However, as expected, those decisions are not independent of each other. To obtain a complete listing, proceed in the form of a "decision tree", starting with the three possibilities for the first pair (if $\{a, b\}$ is the pair, the three possibilities are: 1st: aSb, 2nd: bSa, 3rd: neither aSb, nor bSa), and branching from each to possible decisions on the second pair, etc. Finally, draw the Hasse diagrams. This is also helpful in eliminating impossible answers: when attempting to draw the Hasse diagram, you find that the proposed relation is not an irreflexive order. I note that there are exactly 14 different S's (note that $S=R^{\mathrm{tr}}$ is one of them).]

(ii) Which ones of the Hasse diagrams obtained in (i) are Hasse diagrams of lattices? Justify your answers.

[2] **(i)** Verify example **L2** in section 3.1, p. 83; that is, prove that the order $(\mathbb{N}, |)$ is a lattice, with lattice operations given in example **L2**.

(ii) Let n be a positive integer. Let $A_n = \{d\in\mathbb{N}: d|n\}$. Using (i), prove that $L_n = (A_n; |\restriction A)$ is a lattice. Identify the operations $\top$, $\bot$, $\wedge$ and $\vee$ in the lattice L_n. ($|\restriction A_n$ is the relation of divisibility restricted to the set A_n)

(iii) Draw the Hasse diagram of the lattices L_6 and L_{60}.

(iv) Prove that the lattice L_{210} and the subset lattice $\mathcal{P}(\{1, 2, 3, 4\}, \subseteq)$ are isomorphic to each other.

(v) Prove that for any $n\in\mathbb{N}-\{0\}$, the following two conditions are equivalent:
(a) there is a (finite) set B such that L_n is isomorphic to the subset lattice $(\mathcal{P}(B), \subseteq)$
(b) n is square-free: $p^2|n$ implies that $p=1$.

[3] **(i)** Prove the assertions made in example **L4**, section 3.2 (p. 84). That is, prove that $(\mathrm{Sub}(V), \subseteq)$, the (partially) ordered set of all subspaces of a vector space V, is a lattice, and the lattice operations are as specified in the Notes.

(ii) Let (u_1, u_2, u_3) be a basis of $V=\mathbb{R}^3$, and let $u_4\in V$ be a vector that is linearly independent from both (u_1, u_2) and (u_1, u_3). Let $U_1=\mathrm{span}(u_1, u_2)$,

$U_2=\mathrm{span}(u_1, u_3)$ and $U_3=\mathrm{span}(u_4)$. Determine the Hasse diagram of the sublattice $\langle U_1, U_2, U_3\rangle$ of $(\mathrm{Sub}(V), \subseteq)$ generated by U_1, U_2, U_3. Justify your answer.

[This example needs a bit of linear algebra. A numerical example is $u_1=e_1=(1, 0, 0)$, $u_2=e_2=(0, 1, 0)$, $u_3=e_3=(0, 0, 1)$, and $u_4=(1, 1, 1)$.]

[4] We let $B=\{i\in\mathbb{N}:1\leq i\leq 8\}$, and

$X_1=\{1\}$, $X_2=\emptyset$, $X_3=\{1, 2, 3, 8\}$, $X_4=\{1, 2, 3, 4, 5, 6, 7, 8\}$,
$X_5=\{1, 2, 4, 5, 6, 8\}$, $X_6=\{4, 8\}$, $X_7=\{1, 2, 8\}$, $X_8=\{8\}$, $X_9=\{1, 5\}$.

Let $A=\{X_k:1\leq k\leq 9\}$, and let the order R be defined on A by:

$$(X_k, X_\ell)\in R \iff X_k\subseteq X_\ell$$

(that is, R is just the subset relation restricted to the sets in A).

(i) Draw the Hasse diagram of the order (A, R).
(ii) Verify that (A, R) is a lattice, by giving $x\wedge y$ and $x\vee y$ for all $x, y\in A$; to save writing, disregard the pairs (x, y) for which xRy; also, do not list both (x, y) and (y, x).
(iii) Find some particular $x, y, z\in A$ for which $(x\vee y)\wedge z\neq(x\wedge z)\vee(y\wedge z)$.
(iv) Find some particular $x, y, z\in A$ for which $(x\vee y)\wedge z=(x\wedge z)\vee(y\wedge z)$, and $z\nleq x$, $z\nleq y$.
(v) Are there $x, y\in A$ for which $x\wedge y\neq x\cap y$?
(vi) Are there $x, y\in A$ for which $x\vee y\neq x\cup y$?
(vii) Find three elements that together generate the lattice (A, R).
(viii) Give a reason why, in this case, fewer than three elements cannot generate the lattice.

[5] **(i)** Let $(A; \leq)$ be a reflexive *total* order. Which of the lattice operations $\top$, $\bot$, $\wedge$ and $\vee$ are fully defined, and which may not be defined, in the order $(A; \leq)$? Give reasons for all your answers.

[**Note:** in any order $(A; \leq)$, if x and y are comparable, that is, either $x\leq y$ or $y\leq x$ (or both), we use the notation $\max(x, y)$ for the *greater* one of x and y, and $\min(x, y)$ for the *smaller* one of x and y. If $x=y$, then $\max(x, y)=\min(x, y)=x=y$. If, however, x and y are incomparable (not comparable), then $\max(x, y)$, $\min(x, y)$ are not defined.]

(ii) Let $L_1=(A; R)$ and $L_2=(B:S)$ be lattices. We define another lattice $L_1\times L_2$, the *product* of lattices L_1, L_2, as follows. $L_1\times L_2 = (C; U)$ where $C=A\times B$, and the relation U is defined by

$$(a_1, b_1)U(a_2, b_2) \overset{\mathrm{def}}{\iff} a_1Ra_2 \;\&\; b_1Sb_2 .$$

Prove that $L_1 \times L_2$, that is, the relation $(C; U)$, is indeed a lattice.

(iii) Let $L=(A; R)$ where $A=\{1, 2, 3\}$, and R the ordinary $\leq$ on numbers restricted to A. Draw the Hasse diagram of the lattice $L \times L$.

[6] Let B be any set, and $f: B \to B$ a function. We say of a subset X of B that it is *closed under* f if, whenever $x \in X$, we also have $f(x) \in X$. Let A be the set of all subsets of B that are closed under f:

$$A \underset{\mathrm{def}}{=} \{ X \subseteq B : \forall x (x \in X \longrightarrow f(x) \in X) \} .$$

(i) Consider the case when

$$B=\{1, 2, 3, 4, 5, 6\} \text{ and } f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 2 & 1 & 6 & 6 \end{pmatrix} .$$

Draw the Hasse diagram of the order $(A, \subseteq)$ $(= (A, \subseteq \upharpoonright A)$.

(ii) In the general case, prove that $(A, \subseteq)$ is a sublattice of $(\mathcal{P}(B); \subseteq)$.

Also do: **Exercise 3** (p. 79) and **Exercise 5** (p. 81).