[1] We consider the following eight properties of relations:

- $P_1$: reflexive
- $P_2$: symmetric
- $P_3$: transitive
- $P_4$: irreflexive
- $P_5$: antisymmetric
- $P_6$: strictly antisymmetric
- $P_7$: dichotomous
- $P_8$: trichotomous

We also look at the seven kinds of relation:

- $Q_1$: preorder
- $Q_2$: equivalence relation
- $Q_3$: reflexive order
- $Q_4$: irreflexive order
- $Q_5$: total reflexive order
- $Q_6$: total irreflexive order
- $Q_7$: graph

(i) Draw the digraphs of the following relations $R_1, \ldots, R_8$, all on the same set $A = \{1, 2, 3, 4\}$, :

$R_1 = \{(3, 1), (1, 3), (1, 4), (1, 2), (4, 1), (2, 1)\}$

$R_2 = \{(3, 3), (3, 4), (3, 2), (1, 3), (1, 1), (1, 4), (1, 2), (4, 3), (4, 4), (4, 2), (2, 2)\}$

$R_3 = \{(3, 3), (3, 1), (1, 3), (1, 1), (4, 4), (4, 2), (2, 4), (2, 2)\}$

$R_4 = \{(3, 3), (3, 1), (3, 4), (3, 2), (1, 1), (1, 4), (1, 2), (4, 4), (2, 2)\}$

$R_5 = \{(3, 4), (3, 2), (1, 3), (1, 4), (1, 2), (2, 4)\}$

$R_6 = \{(3, 3), (3, 2), (1, 3), (1, 1), (1, 4), (1, 2), (4, 3), (4, 4), (4, 2), (2, 2)\}$

$R_7 = \{(3, 1), (3, 4), (3, 2), (1, 1), (1, 4), (1, 2), (4, 1), (4, 4), (4, 2)\}$

$R_8 = \{(3, 1), (3, 4), (3, 2), (1, 4), (1, 2)\}$

(ii) Make an $8 \times 8$ table whose $(i, j)$-entry is YES or NO according to whether or not $R_i$ has the property $P_j$.

(iii) Make a similar $8 \times 7$ table to record which of the relations are and which are not
of each the seven kinds $Q_1, \ldots, Q_7$.

**[2]** Consider the relations listed:

- $R_1$ on $\mathbb{N}$ (remember: "$R$ on $A$" is to say $R \subseteq A \times A$):
  
  $aR_1b \iff a \neq b$ but $a$ and $b$ have a common prime divisor.

- $R_2$ on $\mathbb{N} \times \mathbb{N}$:
  
  $(a, b)R_2(c, d) \iff$ either $a < c$, or ($a = c$ and $b < d$).

- $R_3$ on $\mathbb{Q}^{\neq 0}$ (=the set of all non-zero rational numbers):
  
  $xR_3y \iff x/y$ is an integer

- $R_4$ on $\mathbb{R}^{\geq 0}$ (=the set of all non-negative real numbers):
  
  $xR_4y \iff y - 3x > 0$.

- $R_5$ on $\mathbb{R}$:
  
  $xR_5y \iff (x - y) \in \mathbb{Q}$.

Do the work of problem [1] for these relations. Draw up a $5 \times 8$ table and a $5 \times 7$ table containing the information on the relations as to whether they do or do not have the properties $P_i$, $Q_k$ defined in [1].

**[3]** Remember that relations are sets, namely, sets of ordered pairs. Therefore, the intersection and the union of two relations on a set $A$ are relations on $A$ as well.

Let $P$ be a property of relations $R$. E.g., $P$ could be $P_3$ in [1] above: "$R$ has property $P_3$" means that $R$ is transitive. But also, each $Q_k$ in [1] above is a possible property of relations: e.g., $R$ has property $Q_1$ means that $R$ is a preorder.

We say that property $P$ is "preserved by intersections of relations" if it is true that every time $R$ and $S$ are relations on the same set $A$, both having property $P$, we have that $R \cap S$ also has property $P$. For instance, $P_1$ (reflexivity) is preserved under intersection of relations.

We can talk about a property being preserved by unions of relations in the analogous
The question is:

*which of the properties \( P_1, \ldots, P_9, Q_1, \ldots, Q_7 \) are preserved by intersections of relations, which are not? Which by unions of relations?*

Give a table with two columns and \( 8+7=15 \) rows, the first column for \( R \cap S \), the second for \( R \cup S \), the rows for the \( P_j \) and the \( Q_j \), and the entries containing "yes" or "no". Give brief justifications for the less obvious "yes" answers for the \( P_j \), and counterexamples for the "no" answers for the \( P_j \)'s.

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[4] Let \( A = \{1, 2, 3, 4\} \). For each of the combinations (a) to (g) of properties below, give, if possible, an example of a relation \( R \), by drawing a digraph, on the set \( A \) satisfying it -- or else, if that is not possible, explain why that is so.

(a) symmetric, irreflexive and trichotomous;
(b) symmetric, dichotomous and *not* transitive;
(c) symmetric, antisymmetric and irreflexive;
(d) symmetric, antisymmetric and reflexive;
(e) strictly antisymmetric, trichotomous and *not* transitive;
(f) symmetric, antisymmetric and *not* transitive.
(g) symmetric, irreflexive, transitive and trichotomous;

---

[5] Let \( n \) be a fixed positive integer, and let \( A = \{i \in \mathbb{N} : 1 \leq i \leq n\} \). We list some relations on the set \( A \):

\[
R_1 = \{(i, i+1) : i \in A \land i+1 \in A\}
\]

\[
R_2 = \{(i, i+2) : i \in A \land i+2 \in A\}
\]

\[
R_3 = \{(i, i+1) : i \in A \land i+1 \in A\} \cup \{(n, 1)\}
\]

\[
R_4 = \{(2i, 2i+1) : 2i \in A \land 2i+1 \in A\} \cup \{(2i, 2i-1) : 2i \in A \land 2i-1 \in A\}.
\]

(i) Draw digraphs for each of the above when \( n=5 \).

(ii) Describe the transitive closure \( R_j^{\text{tr}} \) of each the above relations, for general \( n \), in a simple way, and determine the number of elements (which are ordered pairs) in each.
Define the relation $E$ on $\mathbb{N}$ by the condition

$$aEb \iff \text{there are integers } i \text{ and } j \text{ such that } b \mid ai \text{ and } a \mid bj.$$ 

(i) Prove that $E$ is an equivalence relation on $\mathbb{N}$.

(ii) Show that $aEb$ holds if and only if either $a=b=0$, or $a$ and $b$ have the same prime factors (none if $a=b=1$).

(iii) Let $A=\{i \in \mathbb{N}: i < 20\}$. Give the partition $A/ (E \uparrow A)$ corresponding to the equivalence $E \uparrow A$ on $A$ (for $E \uparrow A$, see "restriction": p.36, line 8).

[Note: Recall that $x^0 = 1$, even for $x=0$. For partitions, see Section 2.2, in particular, p. 42].

Consider the following relations $R$ and $S$ on the set $A=\{1, 2, 3, 4\}$:

$$R:\begin{array}{c c c}
1 & 2 & 3 \\
\hline
4 & & \\
\end{array}$$

$$S:\begin{array}{c c c}
1 & 2 & 3 \\
\hline
4 & & \\
\end{array}$$

(An edge without an arrow-head is equivalent to the two arrows in both directions between the two vertices.) Give the network ("digraph" without arrow-heads) as well as the adjacency matrix of each of the relations of the form $R^m \circ S^n$, $S^n \circ R^m$, for all $m, n \in \mathbb{N}$ (there are just a few distinct ones among of these). Also give $R^{\uparrow r}$, $S^{\uparrow r}$, $R^{r/\uparrow r}$, $S^{r/\uparrow r}$ in both ways.

Exercise 1 (p.31),

Exercise 2 (p.33)

Exercise 3 (p.35).

Exercise 4 (p.35).
Remark: When doing any one of the above exercises, collected in [9] to [11], you may use results appearing anywhere earlier in the text; in particular, any of the earlier exercises. For instance, in doing Exercise 4(p.35), you may use the results of Exercise 3(p.35) even if you did not do the latter.

[12] We have three graphs $G_i = S_i \cup S_i^*$, $i = 1, 2, 3$, all on the set $\{1, 2, 3, 4, 5, 6, 7\}$ of vertices:

$S_1 = \{ (1, 2), (1, 4), (2, 3), (2, 4), (2, 5), (3, 5), (4, 5), (4, 6), (5, 6), (5, 7), (6, 7) \}$

$S_2 = \{ (1, 2), (1, 3), (1, 4), (2, 4), (2, 5), (2, 6), (3, 4), (4, 6), (4, 7), (5, 6), (6, 7) \}$

$S_3 = \{ (1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6), (5, 7), (6, 7) \}$.

(i) Draw networks (edges without arrow-heads) of the three graphs in such a way that the (straight) edges do not cross each other (in these cases this is possible; of course, this is not always possible.)

(ii) Two of the graphs are isomorphic to each other; which ones are they? Give an isomorphism between them.

(iii) Show that one of the graphs is not isomorphic to either of the other two.