Section 2.3 Operations on binary relations

Consider the relation R on the set $A=\{0, 1, 2, 3, 4\}$, considered in Section 2.1, with network (digraph) representation and adjacency matrix

[Figure 5 in PDF "Figures 1"]

$$U = [R] = \begin{matrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix},$$

and another one, called S, on the same set A, with network representation and adjacency matrix

[Figure 6 in PDF "Figures 1"]

$$V = [S] = \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}.$$

The product $U \cdot V$ of the two matrices is

$$\begin{matrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \cdot \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} = \begin{matrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}.$$

Here, if we recall the definition of matrix multiplication, the (i, j)-entry is computed as follows. We take $k \in A$, take the product of the (i, k)-entry in U and the (k, j)-entry in D (here we count rows and columns starting with 0 rather than 1), and sum over all k's. Now, for a fixed k, the product of the (i, k)-entry in U and the (k, j)-entry in V is 0 unless both factors are non-zero, that is, unless iRk and kSj both hold, in which case the product is 1. Therefore, the (i, j)-entry in $U \cdot V$ is the same as the number of k's such that both iRk and kSj hold. Put in another way, the (i, j)-entry in $U \cdot V$ is the number of ways we can pass from i to j by first going along an R-arc, then going along an S-arc. If we draw the two relations together, taking care to distinguish the R-arcs from the S-arcs, we get this:

[FIGURE 7 in PDF "Figures 1"]

Now, the (i, j)-entry in $U \cdot V$ is the number of ways we can go from i to j by first going along a solid arc, then going along a dashed one.

With the matrix $U \cdot V$, we may consider the network

[FIGURE 8 in PDF "Figures 1"]

with the numbers on the arcs giving the *weights* with which they appear in the matrix. If we disregard the weights, the resulting relation T on the set A is

[FIGURE 9 in PDF "Figures 1"]

The adjacency matrix of T is the matrix obtained from $U \cdot V$ by changing every positive entry into 1:

$$[T] = \begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \;.$$

What is the relation T? iTj holds just in case there is at least one k such that iRk and kSj; in other words, iTj holds if and only if it is possible to get from i to j by first going along an R-arc, then along an S-arc. We call T the *composite* of the relations R and S, and denote it by $R\circ S$. In general, if

$$R \subseteq A \times A \text{ and } S \subseteq A \times A$$

are relations on the same set A, then the *composite* of R and S (in the said order),

$$R\circ S \subseteq A \times A$$

is the relation on A for which

$$x(R\circ S)z \iff \exists y\in A.\ xRy \ \&\ ySz \;. \tag{1}$$

(Here, as before, we read $\exists y\in A$ as "there exists y in A such that ...").

We can summarize the connection between composition of relations and adjacency matrices as follows. As before, we write $[R]$ for the adjacency matrix of R.

The adjacency matrix of $R\circ S$ is obtained by multiplying the adjacency matrices of R and S, that is, taking $[R]\cdot[S]$, and changing all non-zero entries of the product into 1's. Writing, for any matrix X, $X_!$ for the matrix obtained from X by changing all non-zero entries into 1, we have the equality

$$[R\circ S] = ([R]\cdot[S])_! \;.$$

The composition of relations has the following connection with composition of functions. If $f{:}A\longrightarrow A$ and $g{:}A\longrightarrow A$, then $gf{:}A\longrightarrow A$ is the composite function; and

$$\mathrm{graph}(gf) = \mathrm{graph}(f) \circ \mathrm{graph}(g) \tag{2}$$

(!; for $\mathrm{graph}(f)$, see section 1.3). If we considered relations of the more general form $R \subseteq A \times B$ rather than just $R \subseteq A \times A$, the full extent of functional composition could be reduced to relational composition.

Note the conflict between the notations for composition of functions and that of relations: in (2), there is a reversal of the order of f and g . These matters are conventional, and conventions can be changed. For instance, in the textbook which was listed as recommended reading, composition for relations is defined so that what we wrote as $R \circ S$ is written as $S \circ R$. The textbook's convention would eliminate the conflict with the notation for the composition of functions. On the other hand, the convention adopted here has the advantage that in the defining relation (1), there is no reversal of the order of the variables x and z ; with the opposite convention, (1) would look stranger (see p. 363, Definiton 6 in the textbook). Our notation also has the advantage of meshing well with matrix-operations: $[R \circ S]$ corresponds to $[R] \cdot [S]$ and not to $[S] \cdot [R]$ as it would under the other convention. The conflict with the functional composition could also be handled by changing the convention for the latter, by writing, as some authors do in fact, fg for what we had as gf ; however, this would mean that $(fg)(a)=g(f(a))$; although this also could be remedied by writing $(a)f$ for $f(a)$, which then would make the last thing look like $(a)(fg)=((a)f)g$ ... .

The relationship (2) can be seen as follows: for any $a, c \in A$, we have

$$\begin{aligned}
&(a, c) \in \mathrm{graph}(gf) \\
&\qquad \iff (gf)(a) = c \\
&\qquad \iff g(f(a)) = c \\
&\qquad \iff \text{there is } b \in A \text{ such that } f(a)=b \text{ and } g(b)=c \\
&\qquad \iff \text{there is } b \in A \text{ such that} \\
&\qquad\qquad\qquad (a, b) \in \mathrm{graph}(f) \text{ and } (b, c) \in \mathrm{graph}(g) \\
&\qquad \iff (a, c) \in \mathrm{graph}(f) \circ \mathrm{graph}(g) \ .
\end{aligned}$$

The equivalence of the first line and the last line says that the equality (2) holds (why?).

Assume now that we have a sequence R_0 , R_1 , ... , R_{n-1} of relations, all on the same set A . Then the product

$$[R_0]\cdot[R_1]\cdot\ldots\cdot[R_{n-1}]$$

of the adjacency matrices will have the following significance: its (x, y)-entry will be the number of ways we can pass from the vertex $x \in A$ to the vertex $y \in A$ in exactly n steps by first going along an R_0-arc, then going along an R_1-arc, then on an R_2 one, etc., finally along an R_{n-1}-arc. The matrix

$$([R_0]\cdot[R_1]\cdot\ldots\cdot[R_{n-1}])_!$$

is the adjacency matrix of the relation T on the set A such that xTy holds just in case it is possible to pass from x to y in a manner described in the previous sentence. This relation T is called the *composite* of the relations R_0 , R_1 , ... , R_{n-1} in the given order; denoting the composite by $R_0\circ R_1\circ\ldots\circ R_{n-1}$, we have that

$$[R_0\circ R_1\circ\ldots\circ R_{n-1}] = ([R_0]\cdot[R_1]\cdot\ldots\cdot[R_{n-1}])_! \ .$$

The composite $R_0\circ R_1\circ\ldots\circ R_{n-1}$ is the same as taking the composite of two relations at a time, starting with the R_i , and repeating the process, but taking care that every R_i is taken only once, and their order is respected. E.g.,

$$R_0\circ R_1\circ R_2\circ R_3 = ((R_0\circ R_1)\circ R_2)\circ R_3 = R_0\circ((R_1\circ R_2)\circ R_3) = \ldots .$$

This follows immediately from the fact that the corresponding equalities are true for matrices and their products. This is the same as to say that binary composition of relations is associative:

$$(R\circ S)\circ T = R\circ(S\circ T) ,$$

as a consequence of the associative law for matrix multiplication.

Addition of adjacency matrices have a connection to *union* of relations. If R and S are relations on the same set A, then they are subsets of the set $A \times A$, and their union, $R \cup S$, is again a subset of the same set, that is, $R \cup S$ is a relation on A. The union of the relations on $[5)$ called R and S above is given by the network and the adjacency matrix

[FIGURE 10 in PDF "Figures 1']

$$\begin{matrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}$$

It is clear that, in general,

$$[R \cup S] = ([R] + [S])_!$$

(why?). More generally, if the R_i's are binary relations on the same set, then $\bigcup_{i \varepsilon I} R_i$ is again one, and

$$[\bigcup_{i \varepsilon I} R_i] = (\sum_{i \varepsilon I} [R_i])_! \ .$$

We have the distributive laws connecting union and composition:

$$(R \cup S) \circ T = (R \circ T) \cup (S \circ T) \ ,$$

$$T \circ (R \cup S) = (T \circ R) \cup (T \circ S) \ .$$

These may be verified directly from the definitions, or also by using the adjacency matrices, and the corresponding laws for matrix operations. When doing the latter, one uses the facts that for an adjacency matrix X, $X_! = X$, and that in general $(X \cdot Y)_! = (X_! \cdot Y_!)_!$ and $(X+Y)_! = (X_! + Y_!)_!$.

Let us now consider the composites of a relation $R \subseteq A \times A$ with itself, that is, the relations $R^{\circ 2} = R \circ R$, $R^{\circ 3} = R \circ R \circ R$, $...$, $R^{\circ n} = R \circ R \circ ... \circ R$ (n factors) ;

$R^{\circ(n+1)} = R^{\circ n} \circ R$. We have that

$xR^{\circ n}y$ iff there is an *R-path of length n from x to y* ,

where an *R-path from x to y* is a sequence $\langle a_0, a_1, ..., a_n \rangle$ of elements of A such that $a_0=x$, $a_n=y$ and, for each $i<n$, $a_i R a_{i+1}$; n is the *length* of the path $\langle a_i \rangle_{i \le n}$. In other words, a path consists of arcs starting at x , connecting to each other head to tail, and ending in y ; the length of the path is the number of arcs (rather than the number of vertices) involved. For the relation U given by the network and by the adjacency matrix

[FIGURE 11 in PDF "Figures 1"]

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}$$

the powers U have the following representations:

$U^{\circ 2}$ [FIGURE 12 in PDF "Figures 1"]

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{matrix}$$

$U^{\circ 3}$ [FIGURE 13 in PDF "Figures 1"]

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$$

$U^{\circ 4}$ [FIGURE 14 in PDF "Figures 1"]

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}$$

$U^{\circ 5}$ [FIGURE 15 in PDF "Figures 1"]

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{matrix}$$

We see that $U^{\circ 5} = U^{\circ 2}$, from which of course it follows that $U^{\circ 6} = U^{\circ 3}$, $U^{\circ 7} = U^{\circ 4}$, $U^{\circ 8} = U^{\circ 2}$; in general,

$$U^{\circ 3n} = U^{\circ 3},\ U^{\circ (3n+1)} = U^{\circ 4},\ U^{\circ (3n+2)} = U^{\circ 2}\ (n = 1, 2, \ldots).$$

Counting U itself as $U^{\circ 1}$, all the distinct powers are $U^{\circ i}$ for $i = 1, 2, 3, 4$, and there is a periodicity with period 3, starting with $U^{\circ 2}$ (and not with $U^{\circ 1}$).

It is clear that for any relation R on a *finite* set A, there can be only finitely many distinct powers of R, since there are altogether only finitely many relations on A. More particularly, the number of binary relation on A is $|\mathcal{P}(A\times A)| = 2^{|A\times A|} = 2^{(|A|^2)}$; there cannot be more than this number of powers of R. Hence, the sequence of the powers R, $R^{\circ 2}$, $R^{\circ 3}$, ... will be periodic, with a period starting at some power (which may not be easily predicted in general).

Now, consider the union of all the powers of R:

$$R^{\mathrm{tr}} \underset{\mathrm{def}}{=} R \cup R^{\circ 2} \cup R^{\circ 3} \cup \ldots = \bigcup_{i=1}^{\infty} R^{\circ i}.$$

R^{tr} is the relation on A for which x and y are in this relation if there is an R-path of any (positive, finite) length from x to y. In other words, $xR^{\mathrm{tr}}y$ holds just in case one can reach y from x along R-arcs; at least one arc has to be involved in going from x to y.

U^{tr} in case of the last example, called U, is given by the network and the adjacency matrix

[FIGURE 16 in PDF "Figures 1"]

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{matrix} .$$

Here, the circle containing the three vertices 0, 1 and 2 symbolizes that within that circle, everything is in the relation with everything else; moreover, each remaining element, in this case 3 or 4, is related to every one in the circle in the same way; that is, 3 is in the relation with every one in the circle, but those in the circle are not in the relation with 3 the other way around, etc.

If the underlying set A of R is finite, in particular, it has n elements, then in the definition of R^{tr} above, we may take just the first n terms, and get the same result, R^{tr} :

$$R^{tr} = R \cup R^{\circ 2} \cup R^{\circ 3} \cup \ldots \cup R^{\circ n} = \bigcup_{i=1}^{n} R^{\circ i} ,$$

The reason is that if there is a (finite, positive-length) R-path from x to y, then there is one with at most n arcs. This is because if one has an R-path with more than n arcs, then there must be two different arcs in the path that end in the same element; one can cut out the part of the path between the repeated elements and thereby shorten the path; one can do this as long as one has a path with more than one n arcs; eventually, one must end up with a path with at most n arcs, and with the same starting and finishing vertices as the original path, namely, x and y.

Returning to a general binary relation R on a set A, R^{tr} is called the *transitive closure* of R. The reason for the name is the fact that

R^{tr} *is the least transitive relation on* A *containing* R.

Recall from Section 2.1 that a relation S is *transitive* if xSy and ySz imply that xSz ; note that

$$S \textit{ is transitive} \iff S^{\circ 2} \subseteq S$$

(why ?).

R^{tr} is transitive, since $xR^{\mathrm{tr}}y$, $yR^{\mathrm{tr}}z$ mean that there is an R-path from x to y , and one from y to z ; putting those two paths together, one gets a path from x to z , showing that $xR^{\mathrm{tr}}z$. We could have argued in this way too:

$$R^{\mathrm{tr}} \circ R^{\mathrm{tr}} = \bigcup_{i=1}^{\infty} R^{\circ i} \circ \bigcup_{j=1}^{\infty} R^{\circ j} = (R \cup R^{\circ 2} \cup \ldots) \circ (R \cup R^{\circ 2} \cup \ldots)$$

$$= R \circ (R \cup R^{\circ 2} \cup \ldots) \cup R^{\circ 2} \circ (R \cup R^{\circ 2} \cup \ldots) \cup \ldots$$

by the distributive law;

$$= R^{\circ 2} \cup R^{\circ 3} \cup \ldots \cup R^{\circ 3} \cup R^{\circ 4} \cup \ldots$$

$$= R^{\circ 2} \cup R^{\circ 3} \cup R^{\circ 4} \cup \ldots$$

$$\subseteq R^{\mathrm{tr}} \quad .$$

To say that R^{tr} is the *least* transitive relation containing R means that whenever S is transitive and $R \subseteq S$, then $R^{\mathrm{tr}} \subseteq S$. To see this, note that, in general,

if $R \subseteq R'$ and $S \subseteq S'$, then $R \circ S \subseteq R' \circ S'$.

So, in our case, $R \circ R \subseteq S \circ S \subseteq S$, the last containment by the transitivity of S . By induction, $R^{\circ n} \subseteq S$ for all $n \geq 1$: from $R^{\circ n} \subseteq S$ it follows that $R^{\circ(n+1)} = R^{\circ n} \circ R \subseteq S \circ S \subseteq S$. Since $R^{\circ n} \subseteq S$ for all $n \geq 1$, the union of the the $R^{\circ n}$ is also contained in S , and this means that $R^{\text{tr}} \subseteq S$, as we claimed.

Recall that the relation R on A is *reflexive* if xRx for all $x \in A$. Denoting the *equality relation* $\{(x, x\} \mid x \in A\}$ on A by Δ_A ,

$$\Delta_A = \{(x, x\} \mid x \in A\}$$

we have that

$$R \textit{ is reflexive} \iff \Delta_A \subseteq R .$$

For an arbitrary relation R , the *reflexive/transitive closure* $R^{\text{r/tr}}$ is $\Delta_A \cup R^{\text{tr}}$, or if we agree on the convention that $R^{\circ 0} = \Delta_A$, then $R^{\text{r/tr}} = \bigcup_{i=0}^{\infty} R^{\circ i}$. We may say that $xR^{\text{r/tr}}y$ iff there is an R-path of possibly zero length from x to y ; an R-path of *zero length* is just a vertex, without any arc. $R^{\text{r/tr}}$ is the least reflexive and transitive relation on A containing R . If R is reflexive, then $R^{\text{r/tr}} = R^{\text{tr}}$.

Since reflexive and transitive relations are the same as preorders (see Section 2.1), the reflexive/transitive closure could also be called the *preorder closure*.

Note that

$$\Delta_A \circ R = R \circ \Delta_A = R$$

for all relations R on A .

Now, let us introduce another operation on binary relations on a fixed set A , that of taking the *converse* of a relation. The converse of R , denoted by R^* , is $\{(x, y) \mid (y, x) \in R\}$; in other words,

$$xR^{*}y \iff yRx \ .$$

In terms of adjacency matrices, this means taking the *transpose* of the matrix:

$$[R^{*}] = [R]^{*} \ ;$$

we write X^{*} for the transpose of the matrix X (usually, X^{*} would mean the conjugate transpose, but our matrices are all real, so X^{*} is in fact the transpose). E.g., the converse of the relation U considered above is

[FIGURE 17 in PDF "Figures 1"]

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} \ .$$

There are certain simple laws on the converse and the other operations:

$$R^{**} = R \ ,$$

$$(R \circ S)^{*} = S^{*} \circ R^{*}$$

(note the reversal of the order!),

$$(R \cup S)^{*} = R^{*} \cup S^{*} \ .$$

These correspond to similar laws on matrices; e.g., $(X \cdot Y)^{*} = Y^{*} \cdot X^{*}$.

Let us note that all three operations: $\circ$, $\cup$ and $*$ are monotonic, i.e., they are compatible with $\subseteq$; for $\circ$ this was stated above; for $\cup$ this is to say that

$$R \subseteq R' \ \ \& \ \ S \subseteq S' \quad \Longrightarrow \quad R \cup S \ \subseteq \ R' \cup S' \ ;$$

for $*$:

$$R \subseteq S \implies R^* \subseteq S^* .$$

Recall from the last section that the relation R on A is *symmetric* if xRy implies yRx for all $x, y \in A$; note that

$$R \text{ is symmetric} \iff R^* = R .$$

If R is symmetric, then so is $R^{\mathrm{r/tr}}$:

$$(R^{\mathrm{r/tr}})^* = (\bigcup_{i=0}^{\infty} R^{\circ i})^* = \bigcup_{i=0}^{\infty} (R^{\circ i})^* = \bigcup_{i=0}^{\infty} R^{*\circ i} = \bigcup_{i=0}^{\infty} R^{\circ i} = R^{\mathrm{r/tr}} .$$

Since $R^{\mathrm{r/tr}}$ is always reflexive and transitive, we get that in case R is symmetric, $R^{\mathrm{r/tr}}$ is an equivalence relation.

What is the meaning of $R^{\mathrm{r/tr}}$ for a symmetric irreflexive relation R ? As we said in Section 2.1, a symmetric relation R may be considered as an undirected graph, with edges between two (different) vertices just in case the pair of the vertices is in the relation. We have that $xR^{\mathrm{r/tr}}y$ just in case there is a (possibly zero-length) path between x and y ; now, there is no need for reference to a direction of the path. In other words, $xR^{\mathrm{r/tr}}y$ iff x and y are connected by a path in the graph. The equivalence classes of $R^{\mathrm{r/tr}}$ are the *connected components* of the graph R ; every vertex is in precisely one connected component, and two vertices are in the same connected component iff they are connected by a path.