Chapter 2 Binary relations

Section 2.1 Kinds of binary relation

A *binary relation* R *on* a set A is any subset of $A \times A$,

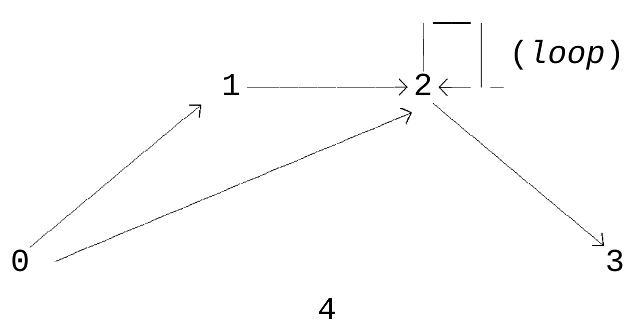
$$R \subseteq A \times A ,$$

that is, any set of ordered pairs of elements of A . For instance, if $f:A \longrightarrow A$ is any function from A to A itself, then $\mathrm{graph}(f)$ is a binary relation on A .

It is remarkable to what extent mathematics is preoccupied with the analysis of binary relations; of course, the reason is that using the abstract idea of a binary relation, we can grasp many mathematical intuitions. In this section, we give an overview of the kinds of binary relation we will look at more closely in the rest of the course.

Instead of saying that R is a binary relation on A , we may write $R \subseteq A \times A$. We also write xRy in place of $(x, y) \in R$.

A general binary relation R on a set A may be visualized as a network of *vertices* standing for the elements of A , and *arcs* (or *arrows*) connecting in definite directions the pairs of vertices that are in the relation R . E.g., the network



represents the binary relation

$$\{(0,1),\ (0,2),\ (1,2),\ (2,2),\ (2,3)\}$$

on the set $\{0, 1, 2, 3, 4\}$. Observe that the data include the set $\{0, 1, 2, 3, 4\}$; the same set of ordered pairs could be considered as a relation on the smaller set $\{0, 1, 2, 3\}$ (but not on any set smaller than the last). The situation is similar to that of functions and their codomains; for us the specification of binary relation includes the specification of its *underlying set*, in the example the set $\{0, 1, 2, 3, 4\}$.

An arbitrary binary relation is also called a *directed graph*, or a *digraph*, in obvious reference to the pictorial representation.

A more robust representation of a binary relation is done through its *adjacency matrix*. If the distinct elements of A are $a_1, a_2, \ldots, a_{n-1}, a_n$, then the adjacency matrix of $R \subseteq A \times A$ is the $n \times n$ matrix whose (i, j)-entry (i^{th} row, j^{th} column) is 1 if $(a_i, a_j) \in R$, 0 otherwise. E.g., the adjacency matrix of the relation in the example is the 5×5-matrix

$$\begin{matrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

here $a_1 = 0$, $a_2 = 1$, etc.

The display of the adjacency matrix depends on the chosen enumeration $a_1, a_2, \ldots$ of the underlying set. If in the example we take, e.g., the reverse enumeration $a_i = 5-i$ ($i = 1, \ldots, 5$), the matrix becomes

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{matrix}$$

From now on, until further notice much later, we will say "relation" instead of "binary relation", simply because we will not have anything to do with relations other than binary ones -- although, as we will eventually see, there are relations that are not binary.

When we look at more relations on the same set, their adjacency matrices are understood to be taken with the same, fixed, enumeration of the underlying set.

Of course, the adjacency matrix is practical only with finite and not too large relations. A closely related representation is via a set in the Cartesian plane provided A is $\mathbb{R}$ or a subset of it; in this case, the relation becomes a region in the plane. E.g., the ordering relation $<$ on $\mathbb{R}$ is represented by the set of points under the line $x = y$ in the Cartesian plane. The graph of a function from $\mathbb{R}$ to $\mathbb{R}$ is another example.

There are certain basic properties of relations. In the list that follows, we give each definition in two forms, one in ordinary words, the second in a symbolic shorthand; we will comment on this shorthand shortly afterwards.

$R \subseteq A \times A$ is *reflexive* if, for all a in A, aRa ;

$$\forall a \in A . aRa \;;$$

symmetric if, for all a, b in A, aRb implies bRa ;

$$\forall a, b \in A .\ aRb \Rightarrow bRa \;;$$

transitive if, for all a, b, c in A, aRb and bRc imply aRc ;

$$\forall a, b, c \in A .\ (aRb \;\&\; bRc) \Longrightarrow aRc \;;$$

irreflexive if, for all a in A, it is *not* the case that aRa ;

$$\forall a \in A .\ \neg(aRa) \;;$$

(observe that this is not merely to say that R is not reflexive);

antisymmetric if, for all a, b in A, aRb and bRa imply that $a=b$;

$$\forall a, b \in A .\ (aRb \;\&\; bRa) \Longrightarrow a=b \;;$$

strictly antisymmetric if, for all a, b in A, aRb implies that it is *not* the case that bRa ;

$$\forall a, b \in A.\ aRb \Longrightarrow \neg(bRa)\ ;$$

$R \subseteq A \times A$ satisfies (the law of) *dichotomy*, or is *dichotomous* if

for all a , b in A , either aRb or bRa (or both) :

$$\forall a, b \in A.\ aRb \text{ or } bRa\ .$$

$R \subseteq A \times A$ satisfies (the law of) *trichotomy*, or is *trichotomous*, if

for all a , b in A , either aRb , or $a=b$, or bRa :

$$\forall a, b \in A.\ aRb \text{ or } a=b \text{ or } bRa\ .$$

Let us comment on the abbreviated, symbolic, statements. Read:

" $\forall a \in A$ "	as	"for all a in A ";
" $\forall a, b \in A$ "	as	"for all a and b in A ";
the symbol &	as	"and";
the symbol $\Longrightarrow$	as	"implies";
the symbol $\neg$	as	"not", "it is not the case that";
the symbol $\vee$	as	"or", "either --, or --, or both".

Reading the symbolic statements according to the key just given, will give you the definitions stated in words. Remember that "or" is *always* to be read as "either --, or --, or both". Note the use of the parentheses in the definitions of "transitive" and "antisymmetric". In both cases, the pair of parentheses and the connective & (and) make a single statement out of two statements, which then becomes the subject of the verb "implies". In the formulation in ordinary language, the use of "imply", the plural form of "implies", has the same effect as the parentheses together with "implies".

The logical symbolism used in the abbreviated statements will itself become an important object of study for us later on.

Exercise 1. Prove that

(a) a relation is strictly antisymmetric if and only if it is antisymmetric and irreflexive;

and

(b) a relation is dichotomous if and only if it is trichotomous and reflexive.

The eight properties listed appear in various combinations.

A *preorder* is a reflexive and transitive relation.

Example 1. Let us consider the set $\mathbb{Z}$, the set of all integers, and let the relation R on $\mathbb{Z}$ be defined as *divisibility*:

aRb if and only if a *divides* b : there is $c\in\mathbb{Z}$ such that $a\cdot c=b$.

Thus, in this example, the instances $2R4$, $2R(-4)$, $(-2)R4$, $(-2)R(-4)$ all hold (why?); also, $0R0$ holds (why?) ; but $4R2$ and $0R1$ do not hold (why?).

The usual notation for " a divides b " is $a\mid b$; we also read $a\mid b$ as " b is *divisible by* a ". Thus,

$$a\mid b \iff \exists c\in\mathbb{Z}.\ a\cdot c=b .$$

Here, we used the logical abbreviation " $\exists c\in\mathbb{Z}$ " to say "there is c in $\mathbb{Z}$ such that ...".

Note that, in case $a\neq 0$, $a\mid b$ if and only if the quotient b/a is itself an integer. (Note the change of the order of the letters a and b from $a\mid b$ to b/a !). On the other hand, $0\mid b$ if and only if $b=0$ (why?).

The relation $\mid$, our R in this example, is a preorder:

for all a, b, c in $\mathbb{Z}$,

$a\mid a$	(reflexivity)
$a\mid b$ and $b\mid c$ imply $a\mid c$	(transitivity)

Exercise 2. Prove the just-stated facts.

An *equivalence relation* is a preorder that is symmetric. In other words, an equivalence relation a relation that is reflexive, symmetric and transitive.

This is a rather simple, but very important kind of relation; in the next section, we'll give a detailed discussion of it.

Example 2. The most basic kind of example for an equivalence relation is the *equality relation* on any set A , which is given by the subset $\{(a, a) \mid a \in A\}$ of $A \times A$. The notation for this relation is Δ_A ; although equality of mathematical objects is understood independently of what sets they belong to, still, for each set A , there is a distinct relation Δ_A of equality on A . This is an instance of our understanding of the concept of relation as one that includes the underlying set in its specification.

Of course, we write $a=b$ for " a equals b ". Then, the fact that, for any set A , Δ_A is an equivalence relation, is expressed by the familiar facts that, for any mathematical objects a , b and c whatsoever, we have

$$a = a \qquad \text{(reflexivity)}$$
$$a = b \text{ and } b = c \text{ imply } a = c \qquad \text{(transitivity)}$$
$$a = b \text{ implies } b = a \qquad \text{(symmetry)}$$

A *partial order*, or more simply, an *order*, is a preorder is that is also antisymmetric.

Example 3. The most important example for an order is the *subset-relation* on $\mathcal{P}(A)$, the set of all subsets of A , for an arbitrary set A . That is, now the relation $R \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ is

$$R = \{(X, Y) \in \mathcal{P}(A) \times \mathcal{P}(A) : X \subseteq Y\} .$$

Once again, just as in the case of equality, the relation $\subseteq$ is defined uniformly for any two arguments X and Y as long as they are sets, independently of the underlying set $\mathcal{P}(A)$.

However, still, we have as many distinct relations as there are sets A ; for each A , the underlying set of the relation is $\mathcal{P}(A)$, not A .

When we want to refer to the subset-relation on $\mathcal{P}(A)$ unambiguously, we may write $\subseteq_{\mathcal{P}(A)}$, although this notation is not commonly used. Also, we may simply write $\subseteq$, in which case the underlying set $\mathcal{P}(A)$ should be inferred from the context.

In section 1.2, we already discussed the transitivity and antisymmetry properties of $\subseteq$. The reflexivity of $\subseteq$ is just the obvious fact that every set is a subset of itself.

An *irreflexive* (or *strict*) *order* is a transitive and irreflexive relation. Thus, an irreflexive order is *not* an order, since in the definition of "order", we required reflexivity, not its opposite, irreflexivity! This is a case of using an adjective ("irreflexive") that does not specify something more narrowly than the noun following it, as it would be expected normally. The term indicates that we are talking about something that is "like an order, except for the fact that it is irreflexive". By the way, we may and sometimes do say "reflexive order" for what we called "order" before, to contrast the two notions more descriptively.

Example 4. The proper-subset-relation $\subset$ on $\mathcal{P}(A)$, for any set A . That is,

$$\subset \; = \; \{(X, Y)\in\mathcal{P}(A)\times\mathcal{P}(A) : X\subset Y\} .$$

is, for any set A , an irreflexive order.

Note that, by Exercise 1.(a), every irreflexive order is strictly antisymmetric.

In an important sense, reflexive orders and irreflexive orders "amount to the same thing". This is explained in the following exercise, Exercise 3. Roughly speaking, what happens is that having any reflexive order, we may pass to a corresponding irreflexive order, the *irreflexive version* of the given reflexive order, without losing any information; and also vice versa, we have a *reflexive version* of any irreflexive order. For instance, the irreflexive version of $\subseteq$ is $\subset$, and the reflexive version of $\subset$ is $\subseteq$.

Exercise 3. **(i)** Let R be a (reflexive) order on the set A. Define the relation S on A by:

$$aSb \underset{\text{def}}{\Longleftrightarrow} aRb \;\&\; a\neq b\;.$$

S is called the *irreflexive version* of R. Show that S is an irreflexive order on A.

(ii) Let S be an irreflexive order on the set A. Define the relation R on A by:

$$aRb \underset{\text{def}}{\Longleftrightarrow} aSb \text{ or } a=b\;.$$

R is called the *reflexive version* of S. Show that R is a (reflexive) order on A.

(iii) For a given R as in (i), let us write $R^{\#}$ for the irreflexive version of R: the S defined in (i). For a given S as in (ii), let us write S^{*} for the reflexive version of S: the R defined as in (ii). Prove that $(R^{\#})^{*}=R$ and $(S^{*})^{\#}=S$.

Example 5. For any A, if R is the subset-relation $\subseteq$ on $\mathcal{P}(A)$, then $R^{\#}$ is the proper-subset-relation $\subset$. For S the proper-subset-relation $\subset$, S^{*} is $\subseteq$.

A *total* (or *linear*) *order* is an order satisfying dichotomy. An *irreflexive total order* is an irreflexive order satisfying trichotomy.

Example 6. The ordinary "less-than-or-equal-to" relation, $\leq$, on $\mathbb{R}$, the set of all real numbers, is the primary example for a (reflexive) total order. Its irreflexive version is $<$, the strict "less-than" relation on $\mathbb{R}$, is the main example for an irreflexive total order.

Exercise 4. Show that if $R\subseteq A\times A$ is a reflexive total order on A, then $R^{\#}\subseteq A\times A$ is an irreflexive total order on A; and if $S\subseteq A\times A$ is an irreflexive total order on A, then $S^{*}\subseteq A\times A$ is a reflexive total order on A.

What is usually called a *graph* is nothing but a symmetric and irreflexive relation. Graphs are represented by networks of vertices and undirected *edges*; irreflexivity is the condition that no edge should come from and go to the same vertex.

If R and S are relations on the same set A, that is, R and S are both subsets of $A\times A$, we say that R is a *subrelation* of S, or that S is an *extension* of R if $R\subseteq S$. There is a largest relation $A\times A$ and a smallest one, the empty relation $\emptyset$, among the relations on A; every relation on A is a subrelation of $A\times A$, and an extension of $\emptyset$.

Let $B\subseteq A$, R a relation on A. The *restriction of* R *to* B, denoted by $R\restriction B$, is $R\cap(B\times B)$; this is a relation on B. In other words, we have

$$x(R|B)y \iff xRy$$

for all $x, y\in B$. E.g., if $\leq$ is the ordinary ordering relation on $\mathbb{R}$, then $\leq\restriction\mathbb{N}$ is the ordinary ordering relation on $\mathbb{N}$. It is a simple but fundamental observation that all the properties of relations considered above are inherited from any relation to any of its restrictions. E.g., if R is transitive, then so is $R\restriction B$; or, if R is a total ordering, so is $R\restriction B$.

Subrelations and restrictions should be carefully distinguished; whereas a relation and its subrelation have the same underlying set, restrictions to proper subsets have different underlying sets. Moreover, a restriction of a total ordering is always a total ordering; but a subrelation of a total ordering is not necessarily a total ordering.

Finally in this section, let us discuss a very important idea related to relations, an idea that is at the heart of what we may call the *abstractness* of the theory of relations. This is the idea of *isomorphism.*

From now on, we will write $(A;R)$ for the relation R on A, to emphasize that the data for the relation include the underlying set A.

Let $(A;R)$, $(B;S)$ be two (binary) relations. An *isomorphism from* $(A;R)$ *to* $(B;S)$ is a *bijective* mapping (function) $f:A\longrightarrow B$ from A to B such that

$$uRv \text{ iff } f(u)Sf(v) \text{ for all } u, v \in A . \tag{1}$$

In words, f is an isomorphism from $(A; R)$ to $(B; S)$ if f is a one-to-one correspondence of the elements of A with the elements of B so that if a pair of elements in A are in the relation R, then the corresponding (under f) elements in B are in the relation S, and vice versa.

The symbolic expression $f:(A; R) \xrightarrow{\cong} (B; S)$ indicates that f is an isomorphism from $(A; R)$ to $(B; S)$.

We say that $A; R)$, $(B; S)$ are *isomorphic*, in symbols $(A; R) \cong (B; S)$, if there exists at least one isomorphism from $(A; R)$ to $(B; S)$.

Let us give some examples.

Let $A=\{0, 1, 2, 3, 4\}$, $B=\{7, 8, 9, 10, 11\}$. The relations $(A; R)$, $(B; S)$ represented by the digraphs

[For drawings, see FIGURE 1 in PDF "Figures 1"]

are isomorphic, since the mapping $f = \binom{0\ 1\ 2\ 3\ 4}{7\ 8\ 9\ 10\ 11}$ is a bijective map $f:A \to B$, and condition (1) above is satisfied. This is seen by the fact that the two digraphs look the same except for the names of the corresponding nodes; in particular, the two digraphs are mapped exactly onto each other by our function f. We may say that one of the relations is an *isomorphic copy* of the other.

The notion of isomorphism being rather clear, it may come as a surprise that it may be *hard* to decide of two given relations $(A; R)$, $(B; S)$ are isomorphic or not. Let us note two further examples.

The relations, in fact *graphs*, $(A; R)$, $(A; S)$, both on the same set

$A=\{1, 2, 3, 4, 5, 6, 7\}$, given as $R=P\cup P^*$ (P^* means the *converse* of P: take all pairs in P in reverse order; see also later in section 2.3; $P\cup P^*$ is the least *symmetric* relation containing P), and $S=Q\cup Q^*$, where

$$P=\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 1)\}$$

$$Q=\{(1, 3), (3, 5), (5, 7), (7, 2), (2, 4), (4, 6), (6, 1)\}$$

[For drawings, see FIGURE 2 in PDF "Figures 1"]

are isomorphic, by the isomorphism $\begin{pmatrix}1&2&3&4&5&6&7\\1&3&5&7&2&4&6\end{pmatrix}$. There are other isomorphisms from $(A; R)$ to $(A; S)$ as well, e.g., $\begin{pmatrix}1&2&3&4&5&6&7\\2&4&6&1&3&5&7\end{pmatrix}$.

This example shows that a (possibly superficial) difference in the shape of the drawings of the digraphs does not necessarily mean the absence of an isomorphism.

On the other hand, the next two graphs $(A; R)$, $(A; S)$, both on the same set $A=\{1, 2, 3, 4, 5, 6, 7\}$, are *not* isomorphic.

$R=P\cup P^*$:

$$P=\{(1, 4), (1, 6), (1, 7), (2, 3), (2, 5), (2, 7),\\ (3, 5), (3, 7), (4, 6), (4, 7), (5, 7), (6, 7)\} .$$

$S=Q\cup Q^*$:

$$Q=\{(1, 2), (1, 3), (1, 7), (2, 4), (2, 7), (3, 6),\\ (3, 7), (4, 5), (4, 7), (5, 6), (5, 7), (6, 7)\} .$$

[For drawings, see FIGURE 3 in PDF "Figures 1"]

This fact is not quite obvious; we will return to it below shortly.

The main point about the notion of isomorphism is that if two relations are isomorphic, we may consider them essentially the same as far as mathematically interesting properties are concerned. More precisely, any *mathematically interesting* property of a relation, if present with one relation, it is also present with any other that is isomorphic to the one. For instance, if $(A; R)$ has any one of our eight basic properties (reflexive, transitive, etc.), then any $(B; S)$ that is isomorphic to $(A; R)$ also has the same property. It follows that if $(A; R)$ is, e.g., a total order, and $(A; R) \cong (B; S)$, then also $(B; S)$ is a total order. We may express this by saying that in mathematics, we are interested only in properties of relations that are *invariant under isomorphism.* This is an expression of the fact that mathematics is interested in *abstract* properties, the latter being identified with ones that are invariant under isomorphism.

Let us return to our last example of a pair of relations $(A; R)$ and $(A; S)$, ones that we said were not isomorphic. A property that the first has but the second does not, is that

"there is a unique vertex which is related to exactly six other vertices; moreover, the longest non-self-intersecting path starting from this vertex is of length 3".

Indeed, this is true of the first graph; the unique vertex with the property mentioned is the centrally located one; and if we start out of it along edges, after at most four edges, we will meet a vertex we met before. However, the second graph does not satisfy this property. It still has a unique vertex in relation with six others, again the central one, but it is possible to go from it along a non-self-intersecting path of length 6.

It is more or less clear, and in fact it is not hard to prove rigorously, that the last-stated property in quotes is *invariant* under isomorphism: if true of one relation, will be true of any isomorphic copy as well. Therefore, in our example, $(A; R)$ and $(B; S)$ cannot be isomorphic, since said property holds of one, but not the other. We may conclude that the the difference in the drawings of the two relations is not superficial, but *essential*.