All six problems [1], [2], [3], [4], [5] and [6] are worth the same marks.

[1] We let the set $A$ be $A=\{1, 2, 3, 4, 5\}$. We list specifications of relations $R_1, \ldots, R_7$ on the set $A$. Decide in each case if there is a relation answering the specification. When the answer is "yes", give an example of a relation in question. Give reasons for both positive and negative answers.

$R_1$ is an equivalence relation and a reflexive order on $A$ at the same time.

$R_2$ is a total irreflexive order on $A$ such that $S \subseteq R_2$, where $S=\{(1, 2), (3, 1), (4, 1), (5, 3), (5, 4)\}$.

$R_3$ is a total irreflexive order on $A$ such that $U \subseteq R_2$, where $U=\{(1, 2), (3, 1), (2, 5), (4, 1), (5, 3), (5, 4)\}$.

$R_4$ is the Hasse diagram of an irreflexive order $Q$ on $A$ with the property that there are exactly two incomparable pairs $\{i, j\}$ for $Q$ ( $\{i, j\}$ is an incomparable pair for $Q$ if $\langle i, j \rangle \in Q$, $\langle j, i \rangle \notin Q$ and $i \neq j$).

$R_5$ is the Hasse diagram of a lattice on $A$ which is not distributive.

$R_6$ is a transitive relation on $A$ such that $R_6 \neq \emptyset$, and for any $\langle i, j \rangle \in R_6$, $R_6 - \{\langle i, j \rangle\}$ is not transitive.

$R_7$ is a symmetric and antisymmetric relation on $A$ which is also not transitive.
In all parts of this problem, \( A \) is the set \( A = \{ 1, 2, 3, 4, 5, 6, 7 \} \), and \( S \) is the following relation on the set \( A \):

\[
S = \{ (1, 5), (1, 7), (2, 5), (3, 2), (3, 7), (4, 2), (4, 7), (6, 1), (6, 2), (6, 3), (6, 4), (7, 5) \}.
\]

(i) Give the transitive closure \( S^\text{tr} \) in the form \( S^\text{tr} = S \cup U \), where \( U \) is a suitable set of pairs. (I note that \( S^\text{tr} \) turns out to be irreflexive.)

(ii) Give the Hasse diagram \( H \) of \( S^\text{tr} \), in the usual graphic form, and also as a set of ordered pairs. (Recall that \( H \) is a relation on the set \( A \).) Is \( H \) equal to \( S \)?

(iii) Let \( \leq = S^\text{tr} \cup A \), the reflexive version of the irreflexive order \( S^\text{tr} \). Show that \( (A; \leq) \) is not a lattice.

(iv) Find a single pair \( (a, b) \) in \( H \) (for \( H \), see (ii)) such that \( \hat{H} \) is the Hasse diagram of a lattice. Verify in detail that \( \hat{H} \) is indeed the Hasse diagram of a lattice.

Hints: drawing the digraphs of \( S \) and \( S^\text{tr} \) will help. Do not hesitate to draw, and possibly redraw, digraphs according to need.
Define the concept of "Boolean algebra".

**Remarks:** The concept of "order" is *not necessary* to define; it can be used as given. However, every further term used in the definition should be given its own definition in full. Do not use logical formulas in your answer; the definitions should be given in ordinary (mathematical) English.

Let \( X \) be the Boolean expression

\[
X = ( (A \rightarrow (B \lor C)) \land (C \rightarrow (A \lor B)) ) \rightarrow (B \rightarrow A) \land ((A \lor B) \rightarrow C)
\]

(i) Give a disjunctive form for \( X \); make it as short as you can.

The following two parts (ii), (iii) may be done in either order.

(ii) Show that \( X \equiv AC \lor \bar{B} \).

(iii) Give the disjunctive normal form for \( X \).

The subsets \( A, B, \) and \( C \) of \( \mathbb{R}^2 \) are given as follows:

\[
A = [x^2 + y^2 < 1], \quad B = [x^2 + y^2 < 4], \quad C = [y > 0]
\]

(i) Determine the atoms of \( \langle A, B, C \rangle \) (the Boolean subalgebra generated by the three sets, of the power-set algebra \( (\mathcal{P}(\mathbb{R}^2); \subseteq) \)) as Boolean expressions of \( A, B, C \). Also, indicate the atoms graphically, as rough pictures in the Cartesian coordinate system of \( \mathbb{R}^2 \).

(ii) What is the cardinality (the number of elements) of the Boolean algebra \( \langle A, B, C \rangle \)?

(iii) For the given values of \( A, B, \) and \( C \), the expression \( X \) from Problem [3] is a particular element of \( \langle A, B, C \rangle \). Write \( X \) as a join of atoms of \( \langle A, B, C, D \rangle \).

(iv) Draw (roughly) \( X \) as a set in the Cartesian coordinate system of \( \mathbb{R}^2 \).

(v) *(for bonus marks)* Decide if \( \langle A, B \rangle = \langle A, B, X \rangle \); justify your answer.
By an informal argument, prove that the following inference, involving the natural numbers $x$ and $y$, is correct.

"Assume that the statements (a), (b) and (c) hold:

(a) If $x > 5$, then $y$ is even.
(b) If $y > 5$, then $x$ is even.
(c) $x \cdot y$ is odd.

It follows that
(d) $x \cdot y \leq 25$.

Turn your argument into a formal proof:

(e) rewrite the statements (a), (b) and (c) as Boolean expressions; use letters denoting suitable ingredients of the statements;

(f) use these Boolean expressions as premisses, and provide additional premisses based on facts you used, maybe implicitly, in (i);

(g) using the standard Boolean calculation, verify the Boolean entailment whose premisses are the ones mentioned in (f), and whose conclusion is the letter standing for (d).