Mathematical Logic/MATH 318/Fall 2005

Midterm examination

October 17, 2005

All six problems [1], [2], [3], [4], [5] and [6] are worth the same marks.

[1] We let the set A be $A=\{1, 2, 3, 4, 5\}$. We list specifications of relations $R_1, \ldots, R_7$ on the set A . Decide in each case if there is a relation answering the specification. When the answer is "yes", give an example of a relation in question. Give reasons for both positive and negative answers.

R_1 is an equivalence relation and a reflexive order on A at the same time.

R_2 is a total irreflexive order on A such that $S \subset R_2$, where $S=\{(1, 2), (3, 1), (4, 1), (5, 3), (5, 4)\}$.

R_3 is a total irreflexive order on A such that $U \subset R_2$, where $U=\{(1, 2), (3, 1), (2, 5), (4, 1), (5, 3), (5, 4)\}$.

R_4 is the Hasse diagram of an irreflexive order Q on A with the property that there are exactly two incomparable pairs $\{i, j\}$ for Q ($\{i, j\}$ is an *incomparable pair for* Q if $(i, j) \notin Q$, $(j, i) \notin Q$ and $i \neq j$).

R_5 is the Hasse diagram of a lattice on A which is not distributive.

R_6 is a transitive relation on A such that $R_6 \neq \emptyset$, and for any $(i, j) \in R_6$, $R_6 - \{(i, j)\}$ is not transitive.

R_7 is a symmetric and antisymmetric relation on A which is also not transitive.

[2] In all parts of this problem, A is the set $A=\{1, 2, 3, 4, 5, 6, 7\}$, and S is the following relation on the set A:

$$
\begin{aligned}
S = \{ \ & (1, 5), (1, 7) \\
& (2, 5), \\
& (3, 2), (3, 7), \\
& (4, 2), (4, 7) \\
& (6, 1), (6, 2), (6, 3), (6, 4) \\
& (7, 5) \ \} .
\end{aligned}
$$

(i) Give the transitive closure S^{tr} in the form $S^{\mathrm{tr}}=S \cup U$, where U is a suitable set of pairs. (I note that S^{tr} turns out to be irreflexive.)

(ii) Give the Hasse diagram H of S^{tr}, in the usual graphic form, and also as a set of ordered pairs. (Recall that H is a relation on the set A). Is H equal to S?

(iii) Let $\leq\, = S^{\mathrm{tr}} \cup \Delta_A$, the reflexive version of the irreflexive order S^{tr}. Show that $(A; \leq)$ is *not* a lattice.

(iv) Find a single pair (a, b) in H (for H, see (ii)) such that $\hat{H} \overset{\mathrm{def}}{=} H-\{(a, b)\}$ is the Hasse diagram of a lattice. Verify in detail that $\hat{H}$ is indeed the Hasse diagram of a lattice.

Hints: drawing the digraphs of S and S^{tr} will help. Do not hesitate to draw, and possibly redraw, digraphs according to need.

[3] Define the concept of "Boolean algebra".

Remarks: The concept of "order" is *not necessary* to define; it can be used as given. However, every further term used in the definition should be given its own definition in full. Do not use logical formulas in your answer; the definitions should be given in ordinary (mathematical) English.

[4] Let X be the Boolean expression

$$X = ((A \to (B \vee C)) \wedge (C \to (A \vee B))) \longrightarrow ((B \to A) \wedge ((A \vee B) \to C)) \ .$$

(i) Give a disjunctive form for X ; make it as short as you can.

The following two parts (ii), (iii) may be done in either order.

(ii) Show that $X \equiv AC \vee \bar{B}$.

(iii) Give the disjunctive normal form for X .

[5] The subsets A, B, and C of $\mathbb{R}^2$ are given as follows:

$$A = [x^2+y^2<1] , \qquad B = [x^2+y^2<4] , \qquad C = [y>0] \ . \ .$$

(i) Determine the atoms of $\langle A, B, C\rangle$ (the Boolean subalgebra generated by the three sets, of the power-set algebra $(\mathcal{P}(\mathbb{R}^2); \subseteq)$) as Boolean expressions of A, B, C . Also, indicate the atoms graphically, as rough pictures in the Cartesian coordinate system of $\mathbb{R}^2$.

(ii) What is the cardinality (the number of elements) of the Boolean algebra $\langle A, B, C\rangle$?

(iii) For the given values of A , B , and C , the expression X from Problem [3] is a particular element of $\langle A, B, C\rangle$. Write X as a join of atoms of $\langle A, B, C, D\rangle$.

(iv) Draw (roughly) X as a set in the Cartesian coordinate system of $\mathbb{R}^2$.

(v)* **(for bonus marks)** Decide if $\langle A, B, C\rangle = \langle A, B, X\rangle$; justify your answer.

[6] **(i)** By an informal argument, prove that the following inference, involving the natural numbers x and y, is correct.

"Assume that the statements (a), (b) and (c) hold:

(a) If $x>5$, then y is even.

(b) If $y>5$, then x is even.

(c) $x \cdot y$ is odd.

It follows that

(d) $x \cdot y \leq 25$. "

(ii) Turn your argument into a formal proof:

(e) rewrite the statements (a), (b) and (c) as Boolean expressions; use letters denoting suitable ingredients of the statements;

(f) use these Boolean expressions as premisses, and provide additional premisses based on facts you used, maybe implicitly, in (i);

(g) using the standard Boolean calculation, verify the Boolean entailment whose premisses are the ones mentioned in (f), and whose conclusion is the letter standing for (d) .