

**Assignment 8/MATH 247/Winter 2010**  
**Due: Tuesday, March 30**

**[1]** Read the previous posting (nothing is to be handed in for this).

**[2]** Consider the matrices

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 10 & -7 \\ -4 & -11 & 10 \\ -2 & -5 & 5 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

For each of the above matrices as  $A$ , answer the following questions.

- 1) Give the S/N decomposition of  $A$ .
- 2) Calculate the matrix  $e^A$  (the entries of  $e^A$  will be expressions of the number  $e$  itself; these need not be calculated as decimals).
- 3) Calculate  $A^{12}$  (Hint: use the *binomial theorem*:

$$(a+b)^m = \sum_{k=0}^m \binom{m}{k} \cdot a^{m-k} \cdot b^k ;$$

here  $\binom{m}{k} = \frac{m \cdot (m-1) \cdot \dots \cdot (m-k+1)}{k!}$  when  $k > 1$ , and  $\binom{m}{0} = 1$ . The binomial

theorem remains a valid identity if we replace  $a$  and  $b$  by two commuting square matrices. The point is that the expression becomes a short one if the powers of  $b$  are zero for exponents  $r$  and higher, with  $r$  a small number.)

- 4) Give a formula for  $A^m$  for a general positive integer  $m$ .
- 5) Give the general *real* solution of the vector differential equation  $\dot{X} = A \cdot X$ .
- 6) Give the particular solution of the vector differential equation  $\dot{X} = A \cdot X$  satisfying the initial condition

$$X(t=0) = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \text{ when } A \text{ is } 3 \times 3; \text{ and } X(t=0) = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix} \text{ for the last, } 4 \times 4,$$

matrix  $A$ .

**7) Calculate the matrices  $\sin(A)$  and  $\cos(A)$**

**(Hints:** Use Euler's formula  $e^{ix} = \cos(x) + i \cdot \sin(x)$  and its consequence  $e^{-ix} = \cos(x) - i \cdot \sin(x)$  to obtain expressions for  $\cos(x)$  and  $\sin(x)$  in terms  $e^{ix}$  and  $e^{-ix}$ . Then apply the method used for part 2) and calculate  $e^{iA}$  and  $e^{-iA}$ , and finally,  $\sin(A)$  and  $\cos(A)$ .)

**[3] 1)** Consider the following (complicated ...) recursive definition of a sequence  $x_0, x_1, x_2, x_3, \dots$ :

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 5; \\ \text{and for all } k = 0, 1, 2, 3, 4, \dots:$$

$$x_{3k+3} = -x_{3k} + x_{3k+1} + x_{3k+2}, \\ x_{3k+4} = -2x_{3k+1} + x_{3k+2}, \\ x_{3k+5} = -x_{3k+2}.$$

Just to see that you understand this prescription, calculate the value of  $x_7$  directly from the definition. Then, **give** formulas for the *general* terms  $x_{3k}, x_{3k+1}, x_{3k+2}$  for all  $k = 0, 1, 2, 3, 4, \dots$  (**Hints:** this has something to do with the second of the above matrices and its powers. The problem is similar to what was done for recurrence relations in assignment 7.)

**2) Solve the initial value problem**

$$\frac{d^3 x}{(dt)^3} = 3 \frac{d^2 x}{(dt)^2} - 3 \frac{dx}{dt} + x,$$

$$x(0) = 1, \quad \frac{dx}{dt} \Big|_{t=0} = 2, \quad \frac{d^2 x}{(dt)^2} \Big|_{t=0} = -1.$$