Assignment 7/MATH 247/Winter, 2010
Due: Friday, March 19

Powers of a square matrix

Given a square matrix A, its powers A^k for large, or even arbitrary, integer exponents k can be calculated by diagonalizing A -- if that is possible (!). Namely, if $A = P \cdot B \cdot P^{-1}$, then

$$A^k = A \cdot A \cdot \ldots \cdot A = P \cdot B \cdot P^{-1} \cdot P \cdot B \cdot P^{-1} \cdot \ldots \cdot P \cdot B \cdot P^{-1} =$$

$$= P \cdot B \cdot B \cdot \ldots \cdot B \cdot P^{-1} = P \cdot B^k \cdot P^{-1};$$

and if $B = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $B^k = \mathrm{diag}((\lambda_1)^k, (\lambda_2)^k, \ldots, (\lambda_n)^k)$.

Example: Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\mathrm{char}_A(\lambda) = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$; thus the eigenvalues of A are $\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{1+4} = \frac{1}{2}(1 \pm \sqrt{5})$;

eigenvectors : $P_1 = \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}$, $P_2 = \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}$;

and with $P = \begin{pmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix}$, $P^{-1} = -\frac{1}{4}\begin{pmatrix} 1-\sqrt{5} & -2 \\ -1-\sqrt{5} & 2 \end{pmatrix}$,

$$A = P \cdot diag(\lambda_1, \lambda_2) \cdot P^{-1} = P \cdot \frac{1}{2}\begin{pmatrix} 1+\sqrt{5} & 0 \\ 0 & 1-\sqrt{5} \end{pmatrix} \cdot P^{-1};$$

and finally,

$$A^k = P \cdot diag(\lambda_1{}^k, \lambda_2{}^k) \cdot P^{-1} = P \cdot \frac{1}{2^k}\begin{pmatrix} (1+\sqrt{5})^k & 0 \\ 0 & (1-\sqrt{5})^k \end{pmatrix} \cdot P^{-1}$$

$$= \begin{pmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix} \cdot \frac{1}{2^k}\begin{pmatrix} (1+\sqrt{5})^k & 0 \\ 0 & (1-\sqrt{5})^k \end{pmatrix} \cdot \left(-\frac{1}{4\sqrt{5}}\right) \cdot \begin{pmatrix} 1-\sqrt{5} & -2 \\ -1-\sqrt{5} & 2 \end{pmatrix} =$$

$$= -\frac{1}{2^{k+2}\sqrt{5}} \begin{pmatrix} 2 \cdot (1+\sqrt{5})^k & 2 \cdot (1-\sqrt{5})^k \\ (1+\sqrt{5})^{k+1} & (1-\sqrt{5})^{k+1} \end{pmatrix} \cdot \begin{pmatrix} 1-\sqrt{5} & -2 \\ -1-\sqrt{5} & 2 \end{pmatrix}$$

$$= \frac{1}{2^{k+1}\sqrt{5}} \begin{pmatrix} 4(a^{k-1} - b^{k-1}) & 2(a^k - b^k) \\ 2(a^k - b^k) & a^{k+1} - b^{k+1} \end{pmatrix}.$$

where we used the abbreviations $a = 1 + \sqrt{5}, \quad b = 1 - \sqrt{5}$.

(We also used that $(1+\sqrt{5})(1-\sqrt{5}) = -4$; and so

$$(1+\sqrt{5})^k \cdot (1-\sqrt{5}) = (1+\sqrt{5})^{k-1} \cdot (1+\sqrt{5})(1-\sqrt{5}) = (1+\sqrt{5})^{k-1} \cdot (-4)\,;$$

and similar further equalities.)

Fibonacci numbers and recurrence equations

The Fibonacci numbers $f_1, \ f_2, \ f_3, \ \dots \ f_n, \ \dots$ are defined as follows: the first two terms are defined as $f_1 = 1, \ f_2 = 1$, and the terms $f_3, \ f_4, \ \dots$ are defined by the *recursive* formula

$$f_{n+2} = f_n + f_{n+1} \,. \qquad (1)$$

We derive an explicit formula for f_n as follows. Let us write X_n for the vector $\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$.

Thus, $X_1 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ X_2 = \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}, \ \dots, \ X_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}, \ X_{n+1} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$.

Let A be the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Next, note that the equation (1) gives the vector/matrix equation

$$X_{n+1} = A \cdot X_n\,; \qquad (2)$$

indeed, the first entry in the product is $0 \cdot f_n + 1 \cdot f_{n+1}$, which is the same as f_{n+1}, the first entry in X_{n+1}; the second entry in the product is $1 \cdot f_n + 1 \cdot f_{n+1}$, which, by the recursion equation (1), is equal to f_{n+2}, the second entry in X_{n+1}.

Therefore, we have

$$X_2 = A \cdot X_1 \ ,$$

$$X_3 = A \cdot X_2 = A \cdot A \cdot X_1 = A^2 \cdot X_1 \ ,$$

$$X_4 = A \cdot X_3 = A \cdot A^2 \cdot X_1 = A^3 \cdot X_1 \ ,$$

and in general

$$X_n = A^{n-1} \cdot X_1 \qquad (n = 1, 2, 3, ...) \, .$$

We can use the explicit formula for A^{n-1} derived above to get:

$$X_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \frac{1}{2^n \sqrt{5}} \begin{pmatrix} 4(a^{n-2} - b^{n-2}) & 2(a^{n-1} - b^{n-1}) \\ 2(a^{n-1} - b^{n-1}) & a^n - b^n \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which gives

$$f_n = \frac{1}{2^{n-1}\sqrt{5}} \left(2(a^{n-2} - b^{n-2}) + a^{n-1} - b^{n-1} \right)$$

This can be rewritten as

$$f_n = \frac{1}{2^{n-1}\sqrt{5}} \left(a^{n-2}(2+a) - b^{n-2}(2+b) \right) \text{ ; and}$$

$$f_n = \frac{1}{2^{n-1}\sqrt{5}} \left[(1+\sqrt{5})^{n-2}(3+\sqrt{5}) - (1-\sqrt{5})^{n-2}(3-\sqrt{5}) \right].$$

This is not a very simple formula; in particular, it gives the integer f_n in terms of irrational quantities such as $\sqrt{5}$. But, for large values of n, it is a useful way to get f_n, at least approximately; the original recursive formula requires performing $n-2$ additions to arrive at f_n.

A similar method will apply to any recurrence relation

$$x_{k+n} \;=\; a_{n-1} \cdot x_{k+n-1} \;+\; a_{n-2} \cdot x_{k+n-2} \;+\; ... \;+\; a_0 \cdot x_k \, .$$

provided a certain matrix has distinct eigenvalues; see problem [1] below.

[1] **1)** Consider the matrices $A = \begin{pmatrix} 3 & -1 & -1 \\ -3 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$.

Give formulas for A^n and B^n for a general positive integer n.

2) Consider the following *recurrence relation*:

$$x_{n+3} = 6x_{n+2} - 11x_{n+1} + 6x_n \qquad (3)$$

This defines a sequence $x_1, x_2, x_3, x_4, x_5, x_6, \ldots$ in which the first three terms x_1, x_2, x_3 are unspecified, and every other term is given by the formula (3) in terms of the three previous terms. Use the method applied above for the Fibonacci sequence, and **give** an explicit formula for x_n, valid for $n \geq 4$, in terms of x_1, x_2, x_3. (The result will be simpler than the one for the Fibonacci sequence).

3) Let $f(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \ldots - a_0$ be a polynomial, with n *distinct* roots $\lambda_1, \lambda_2, \ldots \lambda_n$. Consider the recurrence relation

$$x_{k+n} = a_{n-1} \cdot x_{k+n-1} + a_{n-2} \cdot x_{k+n-2} + \ldots + a_0 \cdot x_k . \qquad (4)$$

Prove that **a)** for every $i = 1, 2, \ldots n$, the sequence $x_1 = 1, x_2 = \lambda_i, \ldots x_k = (\lambda_i)^k, \ldots$ is a solution to the recurrence equation (4) ; and

b) for every sequence $\{x_k\}_{k=1,2,\ldots}$ satisfying (4), we have constants $c_1, c_2, \ldots c_n$ such that

$$x_k = c_1(\lambda_1)^k + c_2(\lambda_2)^k + \ldots + c_n(\lambda_n)^k \qquad (5)$$

for all $k = 1, 2, \ldots$. (In other words, (5) gives the *general solution* for (4)).

c) **Prove** that the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}$$

is the given polynomial $f(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - ... - a_0$.

Hint: c) can be shown directly, or by first showing that the vector $\Lambda_i = \begin{pmatrix} 1 \\ \lambda_i \\ (\lambda_i)^2 \\ \vdots \\ (\lambda_i)^{n-1} \end{pmatrix}$ is an eigenvector of A for the eigenvalue λ_i. The assertion is true for any polynomial $f(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - ... - a_0$, without assuming that it has n distinct roots.

d) Use the result in b) to **derive** the explicit formula for the general term of the Fibonacci sequence discussed above; and also for the problem in part 2).

4) Let $A = (a_{ij})^{n\times n}$ be an upper triangular matrix: $a_{ij} = 0$ whenever $i > j$, and also assume that the diagonal elements are all the same: $a_{ii} = a$ for all i. **Prove** that A is never diagonalizable unless it is already diagonal: $a_{ij} = 0$ whenever $i \neq j$. (This at least tells us that there are many non-diagonalizable matrices!)

[2] **Give** the general *real* solution of each of the following systems of differential equations:

1) $\dot{X} = A \cdot X$

2) $\dot{X} = B \cdot X$

3) $\dot{X} = C \cdot X$.

Here, A and B are the matrices given in [1] 1); and $C = \begin{pmatrix} 3 & -2 & 0 & 2 \\ 4 & -3 & 0 & 2 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 2 & -1 \end{pmatrix}$.

(Of course, in 1) and 2), X is a 3-vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, of functions of t:

$x = x(t),\ y = y(t),\ z = z(t)$; and in 3),

$$X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix};\quad x = x(t),\ y = y(t),\ z = z(t),\ w = w(t)\,.)$$

[3] 1) **Determine** the general real solution of the following system of second order differential equations:

$$\ddot{x} = x + 2y, \qquad \ddot{y} = 4x - y\,. \qquad (6)$$

2) **Find** a solution $\begin{pmatrix} x \\ y \end{pmatrix}$ of the system (6) that satisfies the initial conditions

$x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 1$.

[4] For all parts 1) to 4), A is a (real) symmetric $n \times n$ matrix, with (not necessarily distinct) eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$, with corresponding orthogonal system of eigenvectors $P_1,\ P_2,\ \ldots\ P_n$.

1) **Prove** that

$$A(X) = \sum_{i=1}^{n} \lambda_i \cdot \frac{\langle X, P_i \rangle}{\langle P_i, P_i \rangle} \cdot P_i$$

(Of course, $\langle X, Y \rangle$ denotes the dot-product $X^{tr} \cdot Y$.)

2) Suppose that each $\lambda_i \neq 0$. **Prove** that

$$A^{-1}(X) \;=\; \sum_{i=1}^{n} \frac{\langle X, P_i \rangle}{\lambda_i \cdot \langle P_i, P_i \rangle} \cdot P_i \qquad\qquad (\, X \in \mathbb{R}^n \,).$$

3) Assume that $|\lambda_i| > 1$ for all $i = 1, \ldots, n$. **Prove** that for every non-zero vector $X \in \mathbb{R}^n$ we have that $\|X\| < \|AX\|$.

4) Assume that there is i such that $|\lambda_i| \geq 1$ and there is j such that $|\lambda_j| \leq 1$. **Prove** that there is a non-zero vector $X \in \mathbb{R}^n$ such that $\|X\| = \|AX\|$.

5) For the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$, **find** a non-zero vector $X \in \mathbb{R}^n$ such that $\|X\| = \|AX\|$.