Assignment 4/MATH 247/Winter 2010
Due: Tuesday, February 9

First, you find, in Part 1, a summary of some of the material discussed in class, and also found in Chapter 6 of the text.

The problems to solve are in Part 2, starting on page 5.

Part 1: *Matrix representation and change of basis: the special case for operators.*

A *linear operator* is a linear mapping whose domain and codomain are the same space:

$$T : V \to V .$$

(Linear operators are the most important, but of course, not the only type, of linear mapping, which has the general form $T : U \to V$, with possibly different vector spaces U and V.)

1. Coordinate vectors

Let V be a vector space, $\mathcal{U} = (u_1, ..., u_n)$ a basis of V; $\dim(V) = n$. For an arbitrary vector w in V, the *coordinate vector of* w *relative to* $\mathcal{U}$, in notation $[w]_{\mathcal{U}}$, is the column vector

$$[w]_{\mathcal{U}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

for which $w = x_1 \cdot u_1 + ... + x_n \cdot u_n = \sum_{i=1}^{n} x_i \cdot u_i$.

Fact 1 ("Coordinates are linear functions")***:***

$$\text{If } w = \sum_{j=1}^{k} a_j \cdot w_j = a_1 \cdot w_1 + ... + a_k \cdot w_k \text{ , then } [w]_{\mathcal{U}} = \sum_{j=1}^{k} a_j \cdot [w_j]_{\mathcal{U}} .$$

Proof: Write $[w_j]_{\mathcal{U}} = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{bmatrix}$. Then $w_j = \sum_{i=1}^{n} x_{ij} \cdot u_i$, and for $w = \sum_{j=1}^{k} a_j \cdot w_j$, we have

$w = \sum_{j=1}^{k} a_j \cdot \sum_{i=1}^{n} x_{ij} \cdot u_i = \sum_{i=1}^{n} (\sum_{j=1}^{k} a_j \cdot x_{ij}) \cdot u_i$. For $[w]_{\mathcal{U}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, this means $x_i = \sum_{j=1}^{k} a_j \cdot x_{ij}$.

And this is the same as $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{k} a_j \cdot \begin{bmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{bmatrix}$, that is, $[w]_{\mathcal{U}} = \sum_{j=1}^{k} a_j \cdot [w_j]_{\mathcal{U}}$.

2. Matrix representation

Let $T : V \to V$ be a linear operator on V, $\mathcal{U} = (u_1, \ldots, u_n)$ a basis of V; $\dim(V) = n$. The *matrix of* T *relative to* the basis $\mathcal{U}$, in notation $[T]_{\mathcal{U}}$, is the $n \times n$ matrix defined as

$$[T]_{\mathcal{U}} = [\ [T(u_1)]_{\mathcal{U}}, \ldots, [T(u_n)]_{\mathcal{U}}\ ]$$

We have the relation

Fact 2: ` $$[T(w)]_{\mathcal{U}} = A \cdot [w]_{\mathcal{U}}, \qquad (1)$$

where $A = [T]_{\mathcal{U}}$, w any vector in V.

Proof: Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [w]_{\mathcal{U}}$. This means $w = \sum_{i=1}^{n} x_i \cdot u_i$. Therefore, since T is linear,

$T(w) = \sum_{i=1}^{n} x_i \cdot T(u_i)$. Using Fact 1 (“coordinates are linear”), we infer

$[T(w)]_{\mathcal{U}} = \sum_{i=1}^{n} x_i \cdot [T(u_i)]_{\mathcal{U}} = \sum_{i=1}^{n} [T(u_i)]_{\mathcal{U}} \cdot x_i = [\ [T(u_1)]_{\mathcal{U}}, \ldots, [T(u_n)]_{\mathcal{U}}\ ] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$

$A \cdot X = A \cdot [w]_{\mathcal{U}}$.

Fact 2': The equality (1) determines the matrix A as $A=[T]_{\mathcal{U}}$.

Proof: Apply (1) to the vector $w=u_i$. We have $[u_i]_{\mathcal{U}}=e_i$, the standard $n\times 1$ column vector with all but the i th component equal zero, and with the i th component equal 1. For any $n\times n$ matrix A, $A\cdot e_i=A_i$, the i th column of A. Therefore, (1) now says that $[T(u_i)]_{\mathcal{U}}=A_i$. This is precisely to say that $A=[T]_{\mathcal{U}}$.

3. Change of basis

Suppose $\mathcal{U}=(u_1,...,u_n)$ and $\mathcal{V}=(v_1,...,v_n)$ are both bases of the space V; $\dim(V)=n$. The *change-of –basis matrix from* $\mathcal{U}$ *to* $\mathcal{V}$ is the matrix P, denoted sometimes by $[\mathcal{U}\to\mathcal{V}]$, and defined as

$$P = [\mathcal{U}\to\mathcal{V}] = [\ [v_1]_{\mathcal{U}}\ ,...,\ [v_n]_{\mathcal{U}}\ ]\ .$$

Fact 3: For any vector w in V, we have

$$[w]_{\mathcal{U}} = P\cdot[w]_{\mathcal{V}}$$

(and *not* the other way around!);

and, equivalently,

$$[w]_{\mathcal{V}} = P^{-1}\cdot[w]_{\mathcal{U}}\ . \qquad (2)$$

Proof: Let us write $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ for $X = [w]_{\mathcal{U}}$, and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [w]_{\mathcal{V}}$. We want $X = P\cdot Y$.

With $P = (p_{ij})^{n\times n}$, we have that the column vector $[v_j]_{\mathcal{U}}$ is the same as

$[v_j]_{\mathcal{U}} = \begin{bmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{bmatrix}$. We have

$$w \overset{1}{=} \sum_{j=1}^{n} y_j \cdot v_j \overset{2}{=} \sum_{j=1}^{n} y_j \cdot (\sum_{i=1}^{n} p_{ij} \cdot u_i) \overset{3}{=} \sum_{i,j=1}^{n} y_j \cdot p_{ij} \cdot u_i \overset{4}{=} \sum_{i=1}^{n} (\sum_{j=1}^{n} y_j \cdot p_{ij}) \cdot u_i$$

[*explanations*: $\overset{1}{=}$ by the definition $[w]_{\mathcal{V}}$; $\overset{2}{=}$ by the definitions of $[v_j]_{\mathcal{U}}$ and P; $\overset{3}{=}$ expresses the fact that the sum on the left can be written, in any order, as the sum of the n^2 terms $y_j \cdot p_{ij} \cdot u_i$, one for each pair (i, j) of indices i, j from 1 to n. On the right of $\overset{4}{=}$, the sum is regrouped by collecting the terms that contain the same u_i.]

and

$$w = \sum_{i=1}^{n} x_i \cdot u_i .$$

Therefore: $x_i = \sum_{j=1}^{n} y_j \cdot p_{ij} = \sum_{j=1}^{n} p_{ij} \cdot y_j$, which is the same as $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = P \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, that is, $X = P \cdot Y$.

Fact 3': Equality (2) determines the matrix P as $[\mathcal{U} \to \mathcal{V}]$.

Proof: Apply (2) to the vector $w = u_i$.

4. Change of basis and matrix representation

Suppose $\mathcal{U} = (u_1, ..., u_n)$ and $\mathcal{V} = (v_1, ..., v_n)$ are both bases of the space V; $\dim(V) = n$. Suppose $T : V \to V$ be a linear operator on V. We have the three matrices

$$A = [T]_{\mathcal{U}}, \quad B = [T]_{\mathcal{V}}, \quad P = [\mathcal{U} \to \mathcal{V}].$$

We have the following connection between them

Fact 4: $$A = P \cdot B \cdot P^{-1}$$

or equivalently,

$$B = P^{-1} \cdot A \cdot P$$

Proof : By Fact 2', it suffices to show that $(P \cdot B \cdot P^{-1}) \cdot [w]_{\mathcal{U}} = A \cdot [w]_{\mathcal{U}} = [T(w)]_{\mathcal{U}}$. But

$$P^{-1} \cdot [w]_{\mathcal{U}} = [w]_{\mathcal{V}} \qquad \text{(see Fact 3)}$$
$$B \cdot [w]_{\mathcal{V}} = [T(w)]_{\mathcal{V}} \qquad \text{(meaning of } B = [T]_{\mathcal{V}} \text{)}$$

and

$$P \cdot [T(w)]_{\mathcal{V}} = [T(w)]_{\mathcal{U}} .$$

Therefore,

$$P \cdot B \cdot P^{-1} \cdot [w]_{\mathcal{U}} = P \cdot B \cdot [w]_{\mathcal{V}} = P \cdot [T(w)]_{\mathcal{V}} = [T(w)]_{\mathcal{U}}$$

as desired.

Part 2: The questions

[1] This problem is an exercise in comprehension of Part 1. Answer the questions economically, for each question using previously obtained items when possible. Pretend that the questions concerned vectors and matrices of large dimensions for which economy of calculation would be of importance.

Let : $V = \mathbb{R}^2$, $\mathcal{E} = (e_1, e_2)$ the standard basis of $\mathbb{R}^2$;

$\mathcal{U} = (u_1, u_2)$ a basis of $\mathbb{R}^2$, where $u_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $u_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$;

$\mathcal{V} = (v_1, v_2)$ another basis of $\mathbb{R}^2$, where $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$,

$w_1 = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$, $w_2 = \begin{pmatrix} 6 \\ -10 \end{pmatrix}$, $w = 5w_1 + 6w_2$.

1) **Determine** the following entities:

$$P = [\mathcal{E} \to \mathcal{U}], \quad Q = [\mathcal{U} \to \mathcal{E}], \quad X_1 = [w_1]_{\mathcal{E}}, \quad X_2 = [w_2]_{\mathcal{E}}, \quad Y_1 = [w_1]_{\mathcal{U}}, \quad Y_2 = [w_2]_{\mathcal{U}},$$

$$X = [w]_{\mathcal{E}}, \quad Y = [w]_{\mathcal{U}} \ .$$

Write down the equalities that are true for these entities on general grounds.

2) Let $R = [\mathcal{U} \to \mathcal{V}]$, $S = [\mathcal{V} \to \mathcal{U}]$. **Write down** and **prove** equalities that express R and S in terms of $P = [\mathcal{E} \to \mathcal{U}]$ and $M = [\mathcal{E} \to \mathcal{V}]$; the equalities should hold on general grounds (**hints:** use Fact 3', and argue similarly to the proof of Fact 4)).

Determine M, R and S.

3) Let the linear operator $T : V \to V$ be defined by:

$$T(u_1) = -2u_1 + 4u_2, \qquad T(u_2) = -3u_1 - 9u_2 \ .$$

Use ,when reasonable, previously obtained matrices, and **determine** the matrices $[T]_{\mathcal{U}}$, $[T]_{\mathcal{E}}$, $[T]_{\mathcal{V}}$.

4) For w, w_1 and w_2 given above, **determine** the vectors $T(w_1)$, $T(w_2)$, $T(w)$.

5) Suppose $v = a \cdot v_1 - (a+1) \cdot v_2$; v_1, v_2 are as given before. **Determine** $T(v)$.

[2] We define the following linear operators:

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection in the line l whose equation is $2x - 5y = 0$.

$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the line l (l as before);
that is, $T_2(X) = \mathrm{proj}_l(X)$.

$T_3 : \mathbb{R}^4 \to \mathbb{R}^4$ is the reflection of $\mathbb{R}^4$ (!) in the plane $U = \text{span}(u_1, u_2)$, where $u_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ -1 \end{pmatrix}$. This means that $T(X) = \text{proj}_U(X) - \text{proj}_{U^\perp}(X)$; remember that always $X = \text{proj}_U(X) + \text{proj}_{U^\perp}(X)$.

$T_4 : \mathbb{R}^4 \to \mathbb{R}^4$ is the orthogonal projection onto the plane U (U is as before).

For each of the four cases $T : V \to V$,

a) determine a basis $\mathcal{U}$ of V for which $[T]_{\mathcal{U}}$ is easily obtained;

b) determine $[T]_{\mathcal{E}}$, with $\mathcal{E}$ the standard basis (of $V = \mathbb{R}^n$ for $n = 2$ or 4);

c) calculate the values $T(w)$ where $w = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$ for the first two operators, and $w = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}$ for the last two.

[3] Let $u_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$, and let $U = \text{span}(u_1, u_2)$, a subspace (plane) of $\mathbb{R}^3$.

There are two rotations of the plane U in itself, through 30° about the origin. (It is no use now to say which one is clockwise, which one is counterclockwise, since such a characterization depends on from which side of the plane we are viewing the plane itself.) For each of these rotations, say T, and for a vector X in U, $T(X)$ is a vector in U such that $\|T(X)\| = \|X\|$ and $\measuredangle(X, T(X)) = 30^\circ$.

Determine the matrix $[T]_{\mathcal{U}}$ for both rotations as T as follows:

1) Determine an orthogonal basis $\mathcal{V} = (v_1, v_2)$ of U by the following formulas: $v_1 = u_1$, $v_2 = u_2 - \dfrac{< u_2, v_1 >}{< v_1, v_1 >} \cdot v_1$ (this is called the Gram-Schmidt process, applied to two vectors; soon we will learn more about it).

2) Normalize the basis $\mathcal{V}$: form $\hat{\mathcal{V}} = (\hat{v}_1, \hat{v}_2)$, where $\hat{v}_i = \frac{1}{\|v_i\|} \cdot v_i$.

$\hat{\mathcal{V}} = (\hat{v}_1, \hat{v}_2)$ is an *orthonormal* basis of U: $\hat{v}_1 \perp \hat{v}_2$, and $\|v_1\| = \|v_2\| = 1$. (Do not use decimal approximations. Leave quantities such as $\sqrt{2}$ as they are, without calculating them.)

3) Write down the matrix $[T]_{\hat{\mathcal{V}}}$: this is just an ordinary rotation matrix, $[T]_{\hat{\mathcal{V}}} = [T']_{\mathcal{E}}$, for either of the two rotations $T' : \mathbb{R}^2 \to \mathbb{R}^2$ of $\mathbb{R}^2$ through 30°.

4) Apply change-of-basis to **calculate** $[T]_{\mathcal{U}}$.

As an application,

5) determine the value of $T(2u_1 + 3u_2)$ numerically as a 3×1 vector in $\mathbb{R}^3$, for both rotations as T. (Leave expressions “algebraically’; do not calculate square roots.)

Rotations of $\mathbb{R}^3$ (preliminaries to problem [4])

Let l be a line passing through the origin, and let α be an angle. Let U be the plane through the origin that is perpendicular to l: in our general terminology, $U = l^\perp$, the orthogonal complement of l.

A rotation in $\mathbb{R}^3$ about the axis l through the angle α is a linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which

a) $T(v) = v$ for the direction vector v of the line l; and

b) whenever $u \in U$, we also have $T(u) \in U$ (for which we say that U is invariant under T), and the operator $T \restriction U : U \to U$ on U, defined by $(T \restriction U)(u) = T(u)$ ($u \in U$), is a rotation of the plane U through angle α. In short, T acts in the plane U as a rotation of a plane

Since, unless $\alpha = 0$ or $\alpha = 180^\circ$, there are two possibilities for the rotation $T \restriction U : U \to U$ as in [3], there are two rotations T about the axis l through angle α.

[4] This is a continuation of problem [3]. We let U be the plane defined in [3]. We let $\alpha = 30^\circ$ as in [3]. We let l be the line perpendicular to U through the origin: $l = U^\perp$ (and thus $U = l^\perp$).

The task is to **calculate** $[T]_{\mathcal{E}}$ with the *standard* basis $\mathcal{E}$ of $\mathbb{R}^3$, for the two rotations $T:\mathbb{R}^3 \to \mathbb{R}^3$ about the axis l through the angle $\alpha = 30^\circ$.

To do this, use the work in [3]. *Recipe:* Calculate a unit vector $\hat{v}_3$ that lies in the line l. (there are two possibilities; it is enough to take one, either one) . The three vectors $\hat{v}_1, \hat{v}_2, \hat{v}_3$ form an orthonormal basis $\tilde{\mathcal{V}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ of $\mathbb{R}^3$. Write down the matrix $[T]_{\tilde{\mathcal{V}}}$ directly, using the work in [3]. Apply change-of basis to obtain $[T]_{\mathcal{E}}$. As an application ("tasting"), calculate the two rotated vectors $T(\begin{pmatrix}1\\1\\1\end{pmatrix})$, for both rotations $T:\mathbb{R}^3 \to \mathbb{R}^3$.

(Leave expressions "algebraically'; do not calculate square roots.)

Remark: When $\hat{\mathcal{V}} = (\hat{v}_1,\ldots,\hat{v}_n)$ is an *orthonormal* system of $n\times 1$ column vectors, then the $n\times n$ matrix $P = [\hat{v}_1,\ldots,\hat{v}_n]$ is called an *orthogonal* matrix. An orthogonal matrix P has the property that $P^{tr}\cdot P = I_n$ (as it is easily seen), and thus $P^{-1} = P^{tr}$: the inverse is easy to calculate!