Midterm examination/MATH 247/Winter, 2010
March 11
Corrected copy: April 6
Numbers in ***BOLD ITALIC***: part marks

1. Details are all-important, both in exposition and calculation. Results without justification, and numerical answers that seem to be guessed, are severely down-scored.

2. Calculators are allowed, but they are not really necessary. Do not calculate irrational quantities such as $\sqrt{2}$ in decimals; leave them as they are. In general, it is not important to calculate everything completely. An answer in the form $\sqrt{23^2+34^2}$ may be considered final.

3. All five questions [1], [2], [3], [4], [5] are worth the same number of marks.

[1] Let $Z_1=\begin{pmatrix}1\\2\\-1\\1\end{pmatrix}$, $Z_2=\begin{pmatrix}0\\1\\3\\-1\end{pmatrix}$, $Z_3=\begin{pmatrix}2\\5\\1\\1\end{pmatrix}$, $Z_4=\begin{pmatrix}1\\1\\-4\\2\end{pmatrix}$, vectors in $\mathbb{R}^4$.

Let $U = \text{span}(Z_1,Z_2,Z_3,Z_4)$, a subspace of $\mathbb{R}^4$.

7 **1) Determine** a basis and the dimension of each of the subspaces U and $U^\perp$ of $\mathbb{R}^4$.

6 **2**) Let $T:\mathbb{R}^4\to\mathbb{R}^4$ be the operator for which $T(X) = \text{proj}_U(X)$, the orthogonal projection of the vector X onto the subspace U. **Prove** that T is linear.

7 **3) Determine** the matrix $[T]_{\mathcal{E}}$ of T given in 2) relative to the standard basis $\mathcal{E}$ of $\mathbb{R}^4$ in the form of a product of numerically given matrices and, possibly, inverses of such. There is no need to calculate the matrix completely. Justify your formula.

[2] Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

$U = \mathbf{span}(X_1, X_2)$ and $W = \mathbf{span}(Y)$ are subspaces of $\mathbb{R}^4$.

10 **1) Determine** the orthogonal projections $\text{proj}_{(U+W)}(g - f)$ and $\text{proj}_{(U+W)^{\perp}}(g - f)$. The result may be left in the form of an expression of matrix operations applied to numerically given items.

10 **2)** Let $F = U + f$ and $G = W + g$, flats in $\mathbb{R}^4$ (a plane and a line in $\mathbb{R}^4$). **Determine** points (vectors) $f_0 \in F$, $g_0 \in G$ such that $\|g_0 - f_0\|$ is minimal among all $\|\hat{g} - \hat{f}\|$ for $\hat{f} \in F$, $\hat{g} \in G$. Also, determine the value $\|g_0 - f_0\|$, the *distance* between F and G.

[3] Let the four points P_1, P_2, P_3, P_4 in the x, y-plane be given as follows:

$$P_1(0\ ,\ 1.1)\ ,\quad P_2(1\ ,\ -0.1)\ ,\quad P_3(2\ ,\ 1.1)\ ,\quad P_4(3\ ,\ 3.8)\ .$$

10 **1) Determine** the values of the parameters a, b, c that make the parabola

$$y = f(x) = ax^2 + bx + c$$

the best fit, in the sense of least squares, to the four given points. The calculations should be carried out to the point where only expressions containing numerical determinants are left to be calculated; the final numerical values are not important.

2 **2) Write** down precisely the quantity that is being minimized by the calculation in 1).

6 **3) Verify** that for the four points

$$P_1^*(0\ ,\ 1)\ ,\quad P_2^*(1\ ,\ 0)\ ,\quad P_3^*(2\ ,\ 1)\ ,\quad P_4^*(3\ ,\ 4)\ ,$$

there is a parabola $y = f^*(x)$ that fits precisely the points $P_1^*,\ P_2^*,\ P_3^*,\ P_4^*$ **.**
Calculate $f^*(2.5)$.

2 **4)** **Write** down the value in algebraic form, without actually calculating it, of the error that we make in estimating the value $f^*(2.5)$ when we use our parabola $y = f(x)$ obtained in 1) in place of $y = f^*(x)$.

[4] Let A be the symmetric matrix $A = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & 0 & 0 & 2 \end{pmatrix}$; here, a is a real number, *assumed to be non-zero*. Let $q(X)$ be the quadratic form $q(X) = X^{tr}AX$ $\left(X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right)$.

12 **1)** **Determine** an orthonormal basis $\mathcal{P} = (P_1, P_2, P_3, P_4)$ of $\mathbb{R}^4$ such that if

$$[X]_{\mathcal{P}} = Y = \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}, \text{ then } q(X) = \lambda_1 s^2 + \lambda_2 t^2 + \lambda_3 u^2 + \lambda_4 v^2$$ with suitable constants

$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$; **determine** the constants $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ as well. The vectors P_1, P_2, P_3, P_4 will be algebraic expressions of a . The expressions need not be simplified.

4 **2)** **Determine** the values $M = \max \dfrac{q(X)}{\|X\|^2}$ and $m = \min \dfrac{q(X)}{\|X\|^2}$ and vectors $X_1,\ X_2$ such that $\dfrac{q(X_1)}{\|X_1\|^2} = M$, $\dfrac{q(X_2)}{\|X_2\|^2} = m$.

4 **3)** **Determine** the values of a for which $q(X)$ is positive/negative definite, positive/negative semidefinite but not definite, indefinite.

[5] Let $F[x,y]$ be the vector space of all functions $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ having partial derivatives of all orders. ***Reminder:*** we have the linear operators

$$I,\ D_x,\ D_y,\ M_x,\ M_y \qquad (1)$$

on $F[x,y]$, defined by

$$I(f)=f\ ,\quad D_x(f)=\frac{\partial f}{\partial x},\quad D_y(f)=\frac{\partial f}{\partial y},\quad (M_x(f))(x,y)\ =\ x\cdot f(x,y)$$

and similarly for M_y.

10 **1) Prove** the following identities:

$M_xM_y=M_yM_x\,,\qquad D_xM_y=M_yD_x\,,$

$D_xM_x=I+M_xD_x\ ,\qquad IT=TI=T$ for any linear operator on $F[x,y]$.

10 **2)** Consider the linear operator $T=(D_x+D_y)^2M_xM_y$. **Write** T in the form of a sum each of whose terms is a product (possibly of one factor only) of items in the list (1) in such a way that in the product all M-operators precede all D-operators. (For instance, M_xD_y is allowed as a term in the sum, but D_yM_x is not.) To produce the desired expression, use general laws of operator algebra, the identities in 1), the obvious analogs of the latter obtained by interchanging x and y, *plus* the basic commutation law $D_yD_x=D_xD_y$.

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