MATH 247/Winter, 2007
A calculation of a Fourier series

The function is:

$$f(x) = \begin{cases} x & 0<x<\pi \\ -\pi & -\pi<x<0 \end{cases} \quad ; \quad f \text{ periodic with period } 2\pi .$$

$$f(x) = \sum_{-\infty}^{\infty} c_m e^{imx} \; ; \quad c_m = \frac{\langle f, e^{imx}\rangle}{\langle e^{imx}, e^{imx}\rangle} = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(x)e^{-imx}dx .$$

$$2\pi c_0 = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} -\pi\cdot dx + \int_0^{\pi} x dx = [-\pi x]_{-\pi}^{0} + [\frac{x^2}{2}]_0^{\pi} = -\pi^2 + \frac{\pi^2}{2} = -\frac{\pi^2}{2} ;$$

$$c_0 = \frac{\pi}{4}$$

For $m\neq 0$:

$$2\pi c_m = -\pi \int_{-\pi}^{0} e^{-imx}dx + \int_0^{\pi} x e^{-imx}dx = -\pi I_1 + I_2 .$$

Indefinite integral: $\int e^{-imx}dx = -\frac{1}{im}\cdot e^{-imx} = \frac{i}{m}\cdot e^{-imx}$;

$$I_1 = \frac{i}{m}\cdot[e^{-imx}]_{-\pi}^{0} = i\cdot\frac{1}{m}\cdot(1-(-1)^m) \quad .$$

$$I_2 = \int_0^{\pi} x e^{-imx} dx \quad = \frac{i}{m}\cdot[x e^{-imx}]_0^{\pi} - \frac{i}{m}\cdot\int_0^{\pi} e^{-imx} dx = \frac{i}{m}\cdot(\pi(-1)^m - [\frac{i}{m}\cdot e^{-imx}]_0^{\pi})$$

$u=x$

$v'=e^{-imx}$

$u'=1$

$v=\frac{i}{m}\cdot e^{-imx}$

$$= \frac{i}{m}\cdot(\pi(-1)^m - \frac{i}{m}\cdot((-1)^m-1)) = \frac{1}{m^2}\cdot((-1)^m-1) + i\cdot\frac{\pi}{m}\cdot(-1)^m$$

$$2\pi c_m = -\pi I_1 + I_2 = -i\cdot\frac{\pi}{m}\cdot(1-(-1)^m) + \frac{1}{m^2}\cdot((-1)^m-1) + i\cdot\frac{\pi}{m}\cdot(-1)^m =$$

$$= \frac{1}{m^2} \cdot ((-1)^m - 1) + i \cdot \frac{\pi}{m} \cdot (2 \cdot (-1)^m - 1)$$

$$c_m = \begin{cases} i \cdot \frac{1}{2m} & \text{if } m \text{ is even} \\ -\frac{1}{\pi m^2} - i \cdot \frac{3}{2m} & \text{if } m \text{ is odd} \end{cases}$$

"Real" expression: for f a real-valued function, we have $c_{-n} = \overline{c_n}$.

$$f(x) = \sum_{-\infty}^{\infty} c_m e^{imx} = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$$

$$= c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + \overline{c_n e^{inx}}) = c_0 + \sum_{n=1}^{\infty} 2\mathrm{Re}(c_n e^{inx})$$

For n even:

$$\mathrm{Re}(c_n e^{inx}) = \mathrm{Re}(i \cdot \frac{1}{2n} \cdot (\cos(nx) + i \cdot \sin(nx))) = -\frac{1}{2n} \cdot \sin(nx) .$$

For n odd:

$$\mathrm{Re}(c_n e^{inx}) = \mathrm{Re}((-\frac{1}{\pi n^2} - i \cdot \frac{3}{2n}) \cdot (\cos(nx) + i \cdot \sin(nx)) =$$

$$= -\frac{1}{\pi n^2} \cdot \cos(nx) + \frac{3}{2n} \cdot \sin(nx) .$$

The Fourier expansion of our f :

$$f(x) = -\frac{\pi}{4} + \sum_{\substack{n \text{ even} \\ n>1}} -\frac{1}{n} \cdot \sin(nx) + \sum_{\substack{n \text{ odd} \\ n\geq 1}} (-\frac{2}{\pi n^2} \cdot \cos(nx) + \frac{3}{n} \cdot \sin(nx))$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} \cdot \cos(x) + 3 \cdot \sin(x) - \frac{1}{2} \cdot \sin(2x) - \frac{2}{9\pi} \cdot \cos(3x) + \sin(3x) + \ldots .$$

For $x=0$: $f(0) = \frac{f(0+)+f(0-)}{2} = \frac{0-\pi}{2} = -\frac{\pi}{2}$;

the series gives: $-\frac{\pi}{4} + \sum_{\substack{n \text{ odd} \\ n\geq 1}} -\frac{2}{\pi n^2}$; from the equality of the two:

$$\sum_{\substack{n \text{ odd} \\ n\geq 1}} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2k+1)^2} + \ldots = \frac{\pi^2}{8} .$$