Assignment 9/MATH 247/Winter 2010
Due: Wednesday, April 14

Let $\mathbf{E}$ be the subspace spanned by the Fourier functions $E_m = \mathbf{e}^{imx}$ ($m = 0, -1, 1, -2, 2, \ldots; m \in \mathbb{Z}$) of the $\mathbb{C}$-vector space of all functions $f(x)$, $f : \mathbb{R} \to \mathbb{C}$. In symbols: $\mathbf{E}$=span(E_m: $m \in \mathbb{Z}$). The functions in $\mathbf{E}$ are the (finite) linear combinations $\sum_{k=1}^{n} a_k \cdot E_{m_k}$, with complex coefficients $a_k \in \mathbb{C}$. Thus, $\{E_m : m \in \mathbb{Z}\}$ is an *orthogonal* basis for $\mathbf{E}$, with respect to the Fourier inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \cdot \bar{g}(x) \cdot dx .$$

Remember that $\mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\}$ and $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

[1] Prove that $\{\cos(nx) : n \in \mathbb{N}\} \cup \{\sin(nx) : n \in \mathbb{N}\}$ is also an orthogonal basis of $\mathbf{E}$, with respect to the same inner product. (**Hint**s: instead of trying to calculate the integrals involved directly, write the functions $\cos(nx)$, $\sin(nx)$ as appropriate linear combinations of the functions E_m, and use the orthogonality properties of the E_m.)

The *Chebyshev polynomials* $T_n(x)$ for $n = 0,1,2,3,\ldots$, the first three of them being

$$T_0(x) = 1 , \; T_1(x) = x , \; T_2(x) = 2x^2 - 1 ,$$

can be defined in the following simple way.

Using the addition formulas

$$\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$$

we can show that $\cos(n\alpha)$ can be expressed as a polynomial with integer coefficients of $\cos(\alpha)$; that is, there is a polynomial $T_n(x)$ such that

$$\cos(n\alpha) = T_n(\cos(\alpha)) .$$

Indeed,

$$\cos(0\cdot\alpha) \;=\; \cos(0) \;=\; 1 \;=\; T_0(\cos(\alpha)) \;;$$

$$\cos(1\cdot\alpha) \;=\; \cos(\alpha) \;=\; T_1(\cos(\alpha))\,;$$

$$\cos(2\cdot\alpha) \;=\; \cos(\alpha)\cdot\cos(\alpha)-\sin(\alpha)\cdot\sin(\alpha) \;=\; \cos^2(\alpha)-\sin^2(\alpha)$$
$$=\; \cos^2(\alpha)-(1-\cos^2(\alpha)) \;=\; 2\cos^2(\alpha)-1 \;=\; T_2(\cos(\alpha)) \;\;;$$

and the general $T_n(x)$ can be figured out recursively. The $T_n(x)$ are the *Chebyshev polynomials (of the first kind),* after the nineteenth century Russian mathematician P. Chebyshev.

The Chebyshev polynomials have many interesting properties. One of them is the fact that

*they form an orthogonal system relative to the inner product (*Chebyshev inner product)

$$\left\langle f,g \right\rangle \;=\; \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, f(x)\cdot g(x)\cdot dx \;\; .$$

Note the use of the weight-function $w(x) \;=\; \dfrac{1}{\sqrt{1-x^2}}$. Since

$\lim\limits_{x\to -1+} \dfrac{1}{\sqrt{1-x^2}} \;=\; \lim\limits_{x\to 1-} \dfrac{1}{\sqrt{1-x^2}} \;=\; +\infty$, the integral in the definition of $\left\langle f,g \right\rangle$ is an *improper integral* ; it is well-defined as the limit $\lim\limits_{a\to 1-} \displaystyle\int_{-a}^{a} \frac{1}{\sqrt{1-x^2}}\, f(x)\cdot g(x)\cdot dx$.

[2] **Prove** the fact stated in the italicized sentence and **calculate** $\left\|T_n\right\|^2 \;=\; \left\langle T_n,T_n \right\rangle$. (**Hint:** in calculating the required integrals, apply the substitution $x=\cos(u)$. Note carefully the exact range of the variable u corresponding to the variable x ranging from -1 to 1. Then use what you learned in problem [1] on certain trigonometric integrals, instead of trying to calculate the integrals directly.)

Let $w(x)$ be any weight function defined on the interval $(-1,1)$ (continuous, non-negative, not identically zero). (The Chebyshev weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ is an example; but there are several other interesting choices for $w(x)$.)

Let $\langle f, g \rangle$ denote the corresponding inner product on $P_\infty(x)$, the space of real polynomials $f(t)$:

$$\langle f, g \rangle = \int_{-1}^{1} w(x) \cdot f(x) \cdot g(x) \cdot dx \ .$$

In what follows, concepts such as orthogonality refer to this inner product.

Using the Gram-Schmidt algorithm with the starting *infinite*, standard, basis $\{1,\ x,\ x^2,\ x^3,\ ...\}$ of $P_\infty(x)$, one obtains an infinite series of polynomials $p_0(x),\ p_1(x),\ p_2(x),\ p_3(x),\ ...$ such that

1) the degree of $p_n(x)$ is exactly n,

2) $p_n \perp p_m$ when $n \neq m$,

and

the p_n span the whole space $P_\infty(x)$; *in fact,*

3) for each $n = 0,1,2,...$, the polynomials $p_0,\ ...\ p_n$ span the subspace $P_n(x)$ of all polynomials of degree at most n. (Note that $\dim(P_n(x)) = n+1$.).

It is easy to see that the properties 1), 2) and 3) determine the polynomials p_n uniquely up to a non-zero constant factor. It is convenient to require that p_n have leading coefficient equal to 1:

$$p_n(x) = x^n + a_{n-1}^{(n)} \cdot x^{n-1} + \ ...\ + a_k^{(n)} \cdot x^k + \ ...\ + a_0^{(n)} \ ; \quad (*)$$

this makes the polynomials p_n with the description 1), 2) and 3) completely unique, depending only on the inner product chosen.

When we use the inner product with the Chebyshev weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$, we obtain as p_n, up to a non-zero constant factor, the Chebyshev polynomials $T_n(x)$. In fact, it turns out that if we choose p_n to be of the form above, then $p_0 = T_0 = 1$, and $p_n = \frac{1}{2^{n-1}} \cdot T_n(x)$ for $n \geq 1$.

[3] 1) Prove the equalities

$$\int_{-1}^{1} \frac{x^{2k+1}}{\sqrt{1-x^2}} \cdot dx = 0 \qquad (k = 0,1,2,\dots)$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot dx = \pi$$

$$\int_{-1}^{1} \frac{x^{2k}}{\sqrt{1-x^2}} \cdot dx = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2k)} \cdot \pi \qquad (k = 1,2,\dots)$$

Instructions: The first equality follows from the fact that the function $f(x) = \frac{x^{2k+1}}{\sqrt{1-x^2}}$ is an *odd function*: $f(-x) = -f(x)$.

For the second and third equalities (the second being the special case of the third for $k = 0$), use the substitution mentioned in problem [2], and show that the left-hand side equals $\int_{0}^{\pi} \cos^{2k}(x) \cdot dx$ ($k = 0,1,2,\dots$).

Next, deduce the second equality $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot dx = \pi$ (the case $k = 0$).

Next, compute

$$\int_{0}^{\pi} \cos^{2k+2}(x) \cdot dx \qquad (*)$$

from $\int_{0}^{\pi} \cos^{2k}(x) \cdot dx$ in the following manner. Apply integration by parts to (*) with $\cos^{2k+2}(x) = \cos(x) \cdot \cos^{2k+1}(x)$, with $u' = \cos(x)$, $v = \cos^{2k+1}(x)$. In the expression so obtained, use the identity $\sin^2(x) = 1 - \cos^2(x)$. The integral (*) becomes expressed by

$\int_0^\pi \cos^{2k}(x)\cdot dx$ *and* $\int_0^\pi \cos^{2k+2}(x)\cdot dx$ itself (!). Therefore, by a simple algebraic step, (*) can be expressed by $\int_0^\pi \cos^{2k}(x)\cdot dx$ alone: one gets that

$$\int_0^\pi \cos^{2k+2}(x)\cdot dx \quad = \quad \frac{2k+1}{2k+2}\cdot \int_0^\pi \cos^{2k}(x)\cdot dx \quad .$$

The desired formula then follows inductively.

2) Using the Gram-Schmidt algorithm, **calculate** six of the Chebyshev polynomials T_n :

$$T_0(x),\ T_1(x),\ T_2(x),\ T_3(x),\ T_4(x),\ T_5(x)\ . \qquad\qquad (**)$$

(**Hint:** Using the results of part 1) makes the calculation easy.)

3) Determine the polynomial $f(x)$ of degree at most 5 for which the value of the integral $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}(x^6 - f(x))^2 \cdot dx$ is minimal. (**Hints:** The "general Fourier formula" should be used: note that (**) is an orthogonal basis of $P_5(x)$. Besides the previous parts of this problem, also results of problem [2] can be used.)

Parseval's formula says that if the Fourier series of the 2π-periodic function f is $\sum_{m\in\mathbb{Z}} c_m \cdot E_m$, then

$$\|f\|^2 \ = \ \sum_{m\in\mathbb{Z}} |c_m|^2 \cdot \|E_m\|^2 \quad ;$$

here, of course, $\|-\|$ is the norm with respect to the Fourier inner product (mentioned at the beginning of this assignment).

[4] **1) Calculate** the (real) Fourier series of the 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ for which $f(x) = \mathrm{e}^x$ when $-\pi < x < \pi$.

2) Compare the value of the function f and the sum of the series at $x = 0$ to **get** the value of the sum of the infinite series $\sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^2}$ as an expression of e and π. Use your calculator to double-check your formula, at least approximately, by calculating the partial sum $\sum_{n=0}^{10} \frac{(-1)^n}{1+n^2}$ and noting that the sum of the series is approximated by this partial sum with an error less than $\frac{1}{122}$ (why?).

3) **Obtain** similar summation formulas for two other infinite series using values at $x = \pi$ and $x = \frac{\pi}{2}$.

4) Use Parseval's theorem to the Fourier series obtained to **get** a formula for the sum $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$.

Given any bilinear form $\langle u, v \rangle$ on a real vector space V of dimension n, and given any basis $\mathcal{U} = (u_1, u_2, \ldots u_n)$ of V, we have a matrix A such that

$$\langle u, v \rangle \;=\; ([u]_{\mathcal{U}})^{tr} \cdot A \cdot [v]_{\mathcal{U}}$$

for any $u, v \in V$. We have $[u_i]_{\mathcal{U}} = e_i$, the standard basis vector of $\mathbb{R}^n$ with 1 as the i the entry, 0 as the j the entry for all $j \neq i$. Thus $\langle u_i, u_j \rangle \;=\; ([u_i]_{\mathcal{U}})^{tr} \cdot A \cdot [u_j]_{\mathcal{U}} \;=$ $(e_i)^{tr} \cdot A \cdot e_j \;=\; a_{ij}$, the (i, j)-entry of A. In other words, $A = \left(\langle u_i, u_j \rangle\right)^{n \times n}$. We call the matrix A the matrix of the bilinear form $\langle u, v \rangle$ relative to the basis $\mathcal{U} = (u_1, u_2, \ldots u_n)$.

Suppose we also have another basis $\mathcal{V} = (v_1, v_2, \ldots v_n)$, and suppose that the change of basis matrix $[\mathcal{U} \to \mathcal{V}]$ is P. Then $[u]_{\mathcal{U}} = P \cdot [u]_{\mathcal{V}}$; thus

$$\langle u, v \rangle \;=\; ([u]_{\mathcal{U}})^{tr} \cdot A \cdot [v]_{\mathcal{U}} = \; (P \cdot [u]_{\mathcal{V}})^{tr} \cdot A \cdot (P \cdot [v]_{\mathcal{V}}) = \; ([u]_{\mathcal{V}})^{tr} \cdot P^{tr} A P \cdot [v]_{\mathcal{V}}.$$

This says that if the matrix of $\langle u, v \rangle$ relative to $\mathcal{U}$ is A, and $[\mathcal{U} \to \mathcal{V}]$ is P, then the matrix relative to $\mathcal{V}$ is $P^{tr} A P$.

Note that every inner product is a bilinear form in particular; therefore, we can talk about the matrix of an inner product relative to a basis.

[5] **1) Determine** the matrix of the Chebyshev inner product on the space $P_2(x)$ relative to the standard basis $(1, x, x^2)$.

2) Suppose $\langle u,v\rangle$ is an inner product on the real vector space V of dimension n, and $\mathcal{U} = (u_1, u_2, \dots u_n)$ is a basis of V. Let A be the matrix of $\langle u,v\rangle$ relative to $\mathcal{U}$. **Prove** that $\mathcal{U}$ is an orthogonal basis of V if and only if A is diagonal; and $\mathcal{U}$ is an orthonormal basis of V if and only if A is the identity matrix diag(1,1,…,1).

3) Define the symmetric bilinear form $\langle X,Y\rangle$ on $\mathbb{R}^3$ by $\langle X,Y\rangle = X^{tr}\cdot A\cdot Y$, where A is the matrix $A = \frac{1}{9}\cdot\begin{pmatrix} 14 & -2 & -4 \\ -2 & 17 & -2 \\ -4 & -2 & 14 \end{pmatrix}$.

3.1) Prove that $\langle X,Y\rangle$ is an inner product on $\mathbb{R}^3$.

3.2) Determine non-zero vectors $u_1,\ u_2,\ u_3$ that form an orthogonal system relative to both the dot product and the inner product of part 3.1.