## Assignment 8/MATH 247/Winter, 2010

### Part 1. Notes on systems of differential equations

(*First installment*: only the text part is here now. Hopefully, later today, you'll have the problems too.)

I give a *recipe* that describes the general solution of a system of differential equations the kind we saw in class, this time without any talk of the matrix exponential or Jordan's S/N decomposition. Of course, the recipe is justified by the theory, learned in class, involving those ideas just mentioned. And, of course, *it is much better* to remember the solution via the steps through those two ideas than memorizing the recipe.

Given

$$\dot{X} = A \cdot X \quad , \qquad \textbf{(1)}$$

a <u>s</u>ystem of <u>*h*</u>*omogeneous* <u>*l*</u>*inear* <u>*d*</u>*ifferential* <u>*e*</u>*quations with* <u>*c*</u>*onstant* <u>*c*</u>*oefficients* (SHLDECC) , a complete system of linearly independent solutions are formed by the columns of the matrix

$$\mathbf{X} \;=\; P \cdot \,\mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}) \;\cdot\; (I + t\cdot\hat{N} + \frac{1}{2}t^2\cdot\hat{N}^2 + \dots + \frac{1}{(r-1)!}t^{r-1}\cdot\hat{N}^{r-1}) \quad \textbf{(2)}$$

(in other words: all solutions of (1) are of the form $X = \mathbf{X}\cdot C$, for some (uniquely determined) value of the constant $(n\times 1)$-vector $C$ )

*where:*

- the numbers

$$\lambda_1 = \dots = \lambda_{m_1} = \mu_1,\ \lambda_{m_1+1} = \dots = \lambda_{m_1+m_2} = \mu_2,\ \dots,\ \lambda_{m_1+m_2\dots+m_{k-1}+1} = \dots = \lambda_{m_1+m_2\dots+m_{k-1}+m_k = n} = \mu_k$$

form the *complete system of eigenvalues* of the matrix $A$, with $\mu_1,\ \mu_2,\ \dots,\ \mu_k$ the *distinct* eigenvalues of $A$ , with respective algebraic multiplicities $m_1,\ m_2,\ \dots,\ m_k$ ;

- $(P_1, \dots, P_{m_1})$ is ***any*** basis of the subspace $U_1 = \mathrm{nullsp}((A-\mu_1 I)^{m_1})$ of $\mathbb{C}^n$ (complex numbers may be involved!);

$(P_{m_1+1}, \ldots, P_{m_1+m_2})$ is ***any*** basis of the subspace $U_2 = \text{nullsp}((A-\mu_2 I)^{m_2})$;

$\vdots$

$(P_{m_1+m_2\ldots+m_{k-1}+1}, \ldots, P_{m_1+m_2\ldots+m_{k-1}+m_k=n})$ is ***any*** basis of the subspace

$$U_k = \text{nullsp}((A-\mu_k I)^{m_k}) \ .$$

$(U_1, U_2, \ldots, U_k$ are the *generalized eigenspaces* of $A$; their non-zero elements are the *generalized eigenvectors* of $A$).

- $P = [P_1, \ldots, P_{m_1}, P_{m_1+1}, \ldots, P_{m_1+m_2}, \ldots, P_{m_1+m_2\ldots+m_{k-1}+1}, \ldots, P_{m_1+m_2\ldots+m_{k-1}+m_k=n}]$: "stack the bases of the spaces $U_1, U_2, \ldots, U_k$ into matrix $P$";

- $I$ is the $n \times n$ identity matrix;

- $\hat{N} = P^{-1} \cdot A \cdot P - \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$;

- $r$ is the minimal number $r$ such that $\text{nullsp}((A-\mu_i I)^r) = m_i$ for all $i = 1, 2, \ldots, k$; of course, any number greater than this $r$ also works just as well, and thus $r = \min(m_1, m_2, \ldots m_k)$, and then, of course, $r = n$ are also values that make the formula correct ($r$ taken larger than the minimum value just adds zero terms to the third factor of $\mathbf{X}$).

The theorem on the *Jordan normal form* says that the matrix $P$ can be chosen in such a way that the matrix $J = P^{-1} \cdot A \cdot P$ is in **Jordan normal form**, meaning that

1) all entries of $J$ outside the main diagonal and the diagonal immediately above the main diagonal are equal to zero;
2) the main diagonal of $J$ is (the same as that of) $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$;
3) all entries in $J$ in the diagonal immediately above the main diagonal are either equal to 0, or equal to 1; and
4) the entry immediately above the *first* element of each of the $k-1$ uniform blocks, other than the first one, of eigenvalues in the main diagonal, is equal to zero.

***Note again:*** the above prescription works for ***any*** choice of the matrix $P$ *as described*, not only those that transform the matrix $A$ to a Jordan form.

In the non-trivial cases, that is, when the matrix $A$ is non-diagonalizable, the transforming matrix $P$ is "highly" non-unique, and hence, the resulting form of the general solution is "highly" non-unique. This is the same phenomenon as the one in the fact that the basis of a (sub)space of dimension $\ge 2$ is "highly" non-unique.

However, all issues concerning the question what form the general solution should take disappears in the serious applications, when one asks for a solution with additional, "initial", conditions. The most important case is when the solution $X$ of $\dot{X} = A \cdot X$ is asked for that satisfies the *initial condition* $X(t_0) = X_0$: here, $t_0$ is any fixed value of the (real) variable $t$, and $X_0$ is any fixed (numerically given) vector.

(The equation $\dot{X} = A \cdot X$ describes the interacting motions of $n$ particles, each moving in a one-dimensional space, the position at time $t$ of the $i$th particle being $x_i(t)$. The initial condition represents the knowledge that the $i$th particle at time $t_0$ is at the given position $x_{0,i}$; here

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad X_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ \vdots \\ x_{0,n} \end{pmatrix}. )$$

It is an important **fact** that any initial condition $X(t_0) = X_0$ determines the solution $X$ *uniquely*.

## Example

Let $A = \begin{pmatrix} -3 & 5 & 1 & -1 \\ -3 & 4 & 2 & -1 \\ -2 & 2 & 2 & 0 \\ -1 & 1 & 0 & 2 \end{pmatrix}$. Then $n = 4$,

$$\text{char}_A(\lambda) = \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = (\lambda - 1)^3(\lambda - 2) \;;$$

$\lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = 1$ , $m_1 = 3$ ; $\lambda_4 = \mu_2 = 2$ ; $m_2 = 1$ .

$$U_1 = \text{nullsp}((A-\mu_1 I)^{m_1}) = \text{nullsp}((A-I)^3) = \text{span}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\\1\end{pmatrix}\right)$$

The subspaces $\text{nullsp}(A-I)$, $\text{nullsp}((A-I)^2)$ have respective dimensions 1 and 2. Therefore, the minimal value for the integer parameter $r$ in the formula will be 3.

$$U_2 = \text{nullsp}((A-\mu_2 I)^{m_2}) = \text{nullsp}(A-2I) = \text{span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right)$$

(Of course, these bases are not unique; the basis for $U_1$ is “highly” non-unique; the basis vector of $U_2$ is determined up to a constant factor.)

$$P = \begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&2&1\\0&0&1&1\end{pmatrix};$$

$$\hat{N} = P^{-1}\cdot A\cdot P - \text{diag}(\lambda_1,\lambda_2,\dots,\lambda_n) = = \begin{pmatrix}-4&5&1&0\\-3&3&3&0\\-1&1&1&0\\0&0&0&0\end{pmatrix};$$

$r = 3$ ; $r-1 = 2$ ;

$$\mathbf{X} = P\cdot \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t},\dots,e^{\lambda_n t}) \cdot (I + t\cdot\hat{N} + \frac{1}{2}t^2\cdot\hat{N}^2 + \dots + \frac{1}{(r-1)!}t^{r-1}\cdot\hat{N}^{r-1}) =$$

$$\mathbf{X} \;=\; \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \operatorname{diag}(e^t, e^t, e^t, e^{2t}) \;\cdot\; (I + t \cdot \begin{pmatrix} -4 & 5 & 1 & 0 \\ -3 & 3 & 3 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} t^2 \cdot \begin{pmatrix} -4 & 5 & 1 & 0 \\ -3 & 3 & 3 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 )$$

which is equal to

$$\mathbf{X} \;=\; \begin{pmatrix} \mathrm{e}^t(1-4t) & \mathrm{e}^t(5t-2t^2) & \mathrm{e}^t(t+6t^2) & \mathrm{e}^{2t} \\ -3\mathrm{e}^t t & \mathrm{e}^t(1+3t-\frac{3}{2}t^2) & \mathrm{e}^t(3t+\frac{9}{2}t^2) & \mathrm{e}^{2t} \\ -2\mathrm{e}^t t & 2\mathrm{e}^t(t-\frac{1}{2}t^2) & 2\mathrm{e}^t(1+t+\frac{3}{2}t^2) & \mathrm{e}^{2t} \\ -\mathrm{e}^t t & \mathrm{e}^t(t-\frac{1}{2}t^2) & \mathrm{e}^t(1+t+\frac{3}{2}t^2) & \mathrm{e}^{2t} \end{pmatrix}$$

Let us solve the initial value problem that prescribes $X(t_0 = 0) \;=\; X_0 \;=\; \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

We have $\mathbf{X}(t_0 = 0) \;=\; \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. **(3)**

Therefore, for

$$X(t) \;=\; \mathbf{X}(t) \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

we have $X(0) = \mathbf{X}(0)\cdot\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

which says that

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{-1} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ -3 \end{pmatrix}.$$

Therefore, our desired solution is:

$$X = \begin{pmatrix} \mathrm{e}^t(1-4t) & \mathrm{e}^t(5t-2t^2) & \mathrm{e}^t(t+6t^2) & \mathrm{e}^{2t} \\ -3\mathrm{e}^t t & \mathrm{e}^t(1+3t-\frac{3}{2}t^2) & \mathrm{e}^t(3t+\frac{9}{2}t^2) & \mathrm{e}^{2t} \\ -2\mathrm{e}^t t & 2\mathrm{e}^t(t-\frac{1}{2}t^2) & 2\mathrm{e}^t(1+t+\frac{3}{2}t^2) & \mathrm{e}^{2t} \\ -\mathrm{e}^t t & \mathrm{e}^t(t-\frac{1}{2}t^2) & \mathrm{e}^t(1+t+\frac{3}{2}t^2) & \mathrm{e}^{2t} \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 4 \\ 2 \\ -3 \end{pmatrix}. =$$

$$X = \begin{pmatrix} \mathrm{e}^t(4+6t+4t^2)-3\mathrm{e}^{2t} \\ \mathrm{e}^t(4+6t+3t^2)-3\mathrm{e}^{2t} \\ \mathrm{e}^t(4+4t+2t^2)-3\mathrm{e}^{2t} \\ \mathrm{e}^t(2+2t+t^2)-3\mathrm{e}^{2t} \end{pmatrix}.$$

*Is it an accident that* $\mathbf{X}(t_0 = 0) = P$ ? No: if we substitute $t = 0$ in the general formula (2), we obtain exactly this!

As a comparison, let us note that the Jordan normal form of our matrix $A$, as given by MAPLE 9, is

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with transforming matrix

$$R = \begin{pmatrix} -1 & 8 & -\frac{2}{3} & \frac{5}{3} \\ -1 & 6 & 1 & 1 \\ -1 & 4 & \frac{2}{3} & 2 \\ -1 & 2 & \frac{1}{3} & 1 \end{pmatrix}$$

This means that $J = R^{-1} \cdot A \cdot R$ . $R$ is an example for what $P$ is above; with $R$ as $P$, then

$$\tilde{N} = R^{-1} \cdot A \cdot R - \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

works as $\hat{N}$ above did. Therefore, *another* formula for the general solution is $X = \mathbf{X} \cdot C$ , where

$$\mathbf{X} = R \cdot \text{diag}(e^t, e^t, e^t, e^{2t}) \cdot (I + t \cdot \tilde{N} + \frac{1}{2} t^2 \cdot \overline{N}^2) =$$

$$\mathbf{X} = \begin{pmatrix} -e^{2t} & 8e^t & e^t(-\frac{2}{3} + 8t) & e^t(\frac{5}{3} - \frac{2}{3}t + 4t^2) \\ -e^{2t} & e^t & e^t(1 + 6t) & e^t(1 + t + 3t^2) \\ -e^{2t} & e^t & e^t(\frac{2}{3} + 4t) & e^t(2 + \frac{2}{3}t + 2t^2) \\ -e^{2t} & e^t & e^t(\frac{1}{3} + 2t) & e^t(1 + \frac{1}{3}t + t^2) \end{pmatrix} .$$

For the initial value problem , with $X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, we get

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1 & 8 & -\frac{2}{3} & \frac{5}{3} \\ -1 & 6 & 1 & 1 \\ -1 & 4 & \frac{2}{3} & 2 \\ -1 & 2 & \frac{1}{3} & 1 \end{pmatrix}^{-1} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -2 \\ \frac{1}{12} & \frac{11}{72} & -\frac{1}{18} & -\frac{13}{72} \\ -\frac{1}{2} & \frac{7}{12} & \frac{1}{3} & -\frac{5}{12} \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \mathrm{C}$$

$$C = \begin{pmatrix} 3 \\ \frac{13}{36} \\ \frac{5}{6} \\ 1 \end{pmatrix}.$$ And finally,

$$X = \begin{pmatrix} -e^{2t} & 8e^{t} & e^{t}(-\frac{2}{3}+8t) & e^{t}(\frac{5}{3}-\frac{2}{3}t+4t^2) \\ -e^{2t} & e^{t} & e^{t}(1+6t) & e^{t}(1+t+3t^2) \\ -e^{2t} & e^{t} & e^{t}(\frac{2}{3}+4t) & e^{t}(2+\frac{2}{3}t+2t^2) \\ -e^{2t} & e^{t} & e^{t}(\frac{1}{3}+2t) & e^{t}(1+\frac{1}{3}t+t^2) \end{pmatrix} \bullet \begin{pmatrix} 3 \\ \frac{13}{36} \\ \frac{5}{6} \\ 1 \end{pmatrix} = \begin{pmatrix} e^{t}(4+6t+4t^2)-3e^{2t} \\ e^{t}(4+6t+3t^2)-3e^{2t} \\ e^{t}(4+4t+2t^2)-3e^{2t} \\ e^{t}(2+2t+t^2)-3e^{2t} \end{pmatrix}$$

*same as above!*

As a matter of fact, the “standard” solution, via the Jordan normal form, have used more fractions than the more direct method!

Note, again, that the Jordan normal form solution is in fact just a special case, a particular choice of parameters, in our general solution scheme. The point is that there is

***no need for the extra care of choosing the transforming matrix that provides the Jordan normal form***.