Assignment 2/MATH 247/Winter 2010
Due: Thursday, January 21

The problems to solve are numbered [1] to [5] below.

First, some explanatory notes.

§ 1. *Finding a basis of the column-space of a matrix – and proving that the* ***column rank*** (dimension of the column space) *is the same as the* ***row rank*** (dimension of the row space) of any matrix. ***Rank*** of A = row rank of A = column rank of A.

Let A be an $m\times n$ matrix. The column space of A, $\mathbf{colsp}(A)$, is the subspace of $\mathbb{R}^m$ (m is the length of columns of A) spanned by the columns of A: $\mathbf{colsp}(A)=\mathbf{span}(A_1,...,A_n)$, where $A_1,...,A_n$ are the n columns of A.

One way of finding a basis for $\mathbf{colsp}(A)$ is do column reduction on A; that is, row reduction on A^{tr}. Another way is more interesting, however.

We *row*-reduce the matrix A, $A\xrightarrow{row}B$, B row-reduced. Denote the columns of B by $B_1,...,B_n$; that is, $B=[B_1,...,B_n]$. Select those columns of B that contain a pivot; say those are $B_{i_1},...,B_{i_r}$, where $1\le i_1<...<i_r\le n$. (Of course, then $r=$ row rank of A). The desired basis of $\mathbf{colsp}(A)$ is $(A_{i_1},...,A_{i_r})$. In words: a basis of $\mathbf{colsp}(A)$ is obtained by selecting those columns of the *original matrix* A that correspond to the pivoted columns of the row-reduced version of A.

(Note that this shows that dimcolsp(A) = r =dimrowsp(B)=dimrowsp(A) as promised.)

Why is this true? The first point to observe is that $(B_1,...,B_n)$ is a basis of $\mathbf{colsp}(B)$. This is easily seen, similarly to the way one sees how B gives rise to a basis of $\mathbf{nullsp}(A)$.

Second point: as we know, for any $X\in\mathbb{R}^n$,

$$A\cdot X=0 \text{ if and only if } B\cdot X=0 \qquad (1)$$

(this is the basis fact of the equality $\mathbf{nullsp}(A)=\mathbf{nullsp}(B)$). Therefore,

$$x_{i_1}\cdot A_{i_1}+...+x_{i_r}\cdot A_{i_r}=0 \quad \text{if and only if} \quad x_{i_1}\cdot B_{i_1}+...+x_{i_r}\cdot B_{i_r}=0 \ : (2)$$

in (1), just take $X = [0,...,0, x_{i_1}, 0,......, 0, x_{i_r}, 0,...0]^{tr}$, meaning that each entry x_i in the vector X for which i is *not* one of the "pivot-indices" $i_1,...,i_r$ is zero: $x_i = 0$.
Since $B_{i_1},...,B_{i_r}$ are linearly independent, $x_{i_1} \cdot B_{i_1} + ... + x_{i_r} \cdot B_{i_r} = 0$ holds only if $x_{i_1} = ... = x_{i_r} = 0$; but then, by (2), the same holds for $A_{i_1},...,A_{i_r}$ in place of $B_{i_1},...,B_{i_r}$.

We just proved that $A_{i_1},...,A_{i_r}$ are *linearly independent*. We can also see that $(A_{i_1},...,A_{i_r})$ is a *spanning set* for $\mathbf{colsp}(A)$ (thereby: a *basis* of $\mathbf{colsp}(A)$) ; in fact, in a way that is useful for calculations. Briefly, an arbitrary column A_i of A is expressed as a linear combination of $A_{i_1},...,A_{i_r}$ *in the same way* (with the same coefficients) as B_i is expressed in terms of $B_{i_1},...,B_{i_r}$. Since B is row-reduced, to express B_i as a linear combination of $B_{i_1},...,B_{i_r}$ is easy: it is similar to writing the general solution of $B \cdot X = 0$. We are saying that, then, A_i is expressed as a linear combination of $A_{i_1},...,A_{i_r}$ *in the same way*.

§2. *Reminder: row rule and column rule for matrix multiplication.*

The *column rule* of matrix multiplication says: for matrices

$$A = [A_1,...,A_n] : \quad A \text{ is } m \times n$$
$$B = [B_1,...,B_p] : \quad B \text{ is } n \times p$$

(where we have displayed the *columns* of both matrices), we have

$$A \cdot B = A \cdot [B_1,...,B_p] = [A \cdot B_1,...,A \cdot B_p];$$

and each $A \cdot X = \sum_{j=1}^{n} x_j \cdot A_j$ for any *column* $X = [x_1,...,x_n]$. In words,

when forming the product $A \cdot B$, we take a linear combination of the *columns* of A, using the entries of a *column* of B as coefficients, to get a column of $A \cdot B$.

There is a similar *row-rule:*

to get a *row* of $A \cdot B$, we take a linear combination of the *rows* of B, using the entries of a *row* of A as coefficients.

(Note the interchange of A and B in the two rules.)

§3 *Writing a subspace U in the form $U = \mathbf{nullsp}(C)$*, with C a suitable matrix.

Suppose a subspace U of $\mathbb{R}^n$ is given in the form $U = \mathbf{span}(A_1,\ldots,A_m)$. Let us write A for the matrix $A = [A_1,\ldots,A_m]$. A is an $n\times m$ (!) matrix. This makes $U = \mathbf{span}(A_1,\ldots,A_m) = \mathbf{colsp}(A)$. Consider the matrix A^{tr}; A^{tr} is an $m\times n$ matrix. Calculate a basis of $\mathbf{nullsp}(A^{tr})$ in the usual way; denote this basis as $(C_1,\ldots,C_k)$. Let C be the matrix whose columns are the basis vectors just found: $C = [C_1,\ldots,C_k]$. Then C^{tr}, the transpose of C, is the desired matrix: $U = \mathbf{span}(A_1,\ldots,A_m) = \mathbf{nullsp}(C^{tr})$.

Can you prove this?

§4 *Sum and intersection of subspaces*

For the *intersection* $U\cap W$ of two subspaces (of the same space $\mathbb{R}^n$), see p. 123 in Lipschutz. For the *sum* $U+W$ of two subspaces, see p. 134. We have the important formula

$$\dim(U+W) = \dim U + \dim W - \dim(U\cap W)$$

stated as Theorem 4.20, p. 134, and proved in solved problem 4.58 on p. 157.

General instructions: Use “advanced” matrix methods always. Your methods should be efficient for problems involving large matrices. For smaller matrices, such as the ones in the problem set here, one could often use direct methods, which are less efficient, but would work in the example at hand. It is not bad to try a direct method just to see what is going on; but for the answer submitted as part of the assignment, you should use the (better) matrix method(s).

Many of the problems come with the instructions to use a specific method, or something previously done. Sometimes, other, probably less efficient, ways would also work for the problem at hand, especially since the problem is, usually, made numerically easy. For full marks (this applies to the examinations too!), you should follow the instructions, and not go the way of possible other methods. As a general remark: the course and the assignments are not geared towards solving “toy” problems , but rather, to *learning certain specific methods of solving possibly large problems*, ones whose complete handling requires using a computer.

Having said all the above, I have to say this: you are free and even encouraged to point out (for discussion with me or in class) if and when you find that some method proposed here (or any other assignment) is not optimal. I am not pretending that I in fact know the last word about everything in linear algebra.

[1] Suppose that $X_1, X_2, X_3, X_4 \in \mathbb{R}^5$. We let

$$Z_1 = X_1 - X_3 \ ,$$
$$Z_2 = X_3 + 2X_4 \ ,$$
$$Z_3 = X_3 + X_4 \ ,$$
$$Z_4 = X_2 + X_3 + X_4 \ ,$$
$$Z_5 = X_1 + 2X_3 \ .$$

1) Use the method of the proof of the "Main Lemma" in class and **find** scalars y_i for $i = 1,2,3,4,5$ such that **a)** *not all* y_i *are equal to* 0, and **b)** $\sum_{i=1}^{5} y_i Z_i = 0$.

2) Assume, in addition to what we had so far, that the vectors X_1, X_2, X_3, X_4 are linearly independent. **Prove** that, with the Z_i's defined as before, if we have two solutions $Y = (y_i)^{5\times 1}$ and $Y' = (y'_i)^{5\times 1}$ of 1) [that is, there is i such that $y_i \neq 0$; there is i' such that $y'_{i'} \neq 0$; $\sum_{i=1}^{5} y_i Z_i = 0$; and $\sum_{i=1}^{5} y'_i Z_i = 0$], then $Y' = a \cdot Y$ for a unique scalar a. [Warning: this is not a general fact; it holds in this special numerical case.]

[2] Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 3 & -1 & 4 & 1 \\ 1 & 0 & -1 & 1 & -2 \end{pmatrix} .$$

1) Find a basis of $\mathrm{colsp}(A)$, and write each of the five columns A_1, A_2, A_3, A_4, A_5 of A as a linear combination of the basis vectors. (**Hint:** consult §1 above.)

2) Use the work for 1), and write the matrix A in the form $A = B \cdot C$, B a suitable $4\times k$ matrix, C a $k\times 5$ matrix**,** with the number k chosen the least possible value. (**Hint:** consult §2.)

3) Use the work for 2) to find a basis of $\text{rowsp}(A)$, and write each row of A as a linear combination of the basis (row-)vectors found. (**Hint:** consult §2.)

[3] Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $X_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{pmatrix}$, vectors in $\mathbb{R}^5$. Let $U = \text{span}(X_1, X_2, X_3)$, a subspace of $\mathbb{R}^5$.

1) **Find** a $5 \times k$ matrix C, with the smallest possible value for k, such that $U = \text{nullsp}(C)$. (**Hint:** consult §3.)

2) Using the work for 1), **find** values for the parameters a and b such that the vector $Y = [a, b, a, b, 1]^{tr}$ belongs to the subspace U.

[4] **1)** **Find** a basis and the dimension of $U = \textbf{span}(X_1, X_2, X_3, X_4, X_5)$, subspace of $\mathbb{R}^5$, where

$$X_1 = \begin{pmatrix} 1 \\ a \\ 2 \\ 3 \\ a \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ a \\ 2 \\ -a \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ a \\ a \end{pmatrix}, \quad X_4 = \begin{pmatrix} 1 \\ a+1 \\ a+3 \\ a+5 \\ a \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 \\ 1 \\ a+2 \\ 2a+2 \\ a \end{pmatrix}.$$

(a is any given number; its value is unspecified).

2) **Find** a basis and the dimension of the subspaces V, W and $V+W$ of $\mathbb{R}^5$, where $V = \textbf{span}(X_1, X_4, X_5)$, $W = \textbf{span}(X_2, X_3, X_4)$, and the X_i ($i = 1, 2, 3, 4, 5$) are as before.

3) Without further calculations, **determine** a basis for $V \cap W$. Justify your answer. (**Hint:** consult §4.)

[5] We let

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 \\ 7 \\ 3 \\ -2 \\ 3 \\ 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 3 \\ 6 \\ 1 \\ -3 \\ 3 \\ -1 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

vectors in $\mathbb{R}^6$. Let $U = \text{span}(X_1, X_2, X_3)$ and $W = \text{span}(Y_1, Y_2, Y_3, Y_4)$, subspaces of $\mathbb{R}^6$.

Determine a basis and the dimension of the subspace $U \cap W$ of $\mathbb{R}^6$.

(**Hints**: Find matrices C and D such that $U = \text{nullsp}(C)$ and $W = \text{nullsp}(D)$. Let $A = \begin{bmatrix} C \\ D \end{bmatrix}$, the matrix whose rows are those of C followed by those of D. Then $U \cap W = \text{nullsp}(A)$ (why is this true?).)