

FOLDS, Lindström's theorem,
and back-and-forth properties
in infinitary logic. Part 1.

February 6, 2014
Notes

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1. face-structure categories (= "frames") for short

Let: $\mathcal{L}$: small category;

$K, K', \dots$: objects of $\mathcal{L}$ ("kinds");

proper arrow: any non-identity arrow.

→ $\dim(K) \stackrel{\text{def}}{=} \underline{\text{largest}} \ n \in \mathbb{N} \text{ such that}$

there exists a composable string of length n of proper arrows ending in K :

$$K = K_0 \xleftarrow[\neq \text{id}]{p_0} K_1 \leftarrow \dots \leftarrow K_{n-1} \xleftarrow[\neq \text{id}]{p_{n-1}} K_n$$

if such largest n exists;

$\dim(K) = \infty$ otherwise.

→ A-functor $X: \mathcal{L}^{\text{op}} \rightarrow \text{Set}$ is finite if

$\text{el}(X)$, the category of elements of X is a finite category (having finitely many arrows (and objects)).

(NB: $\text{el}(X)$: objects: $(K, x) :: K \in \text{Ob}(\mathcal{L})$
 $x \in X(K)$)

arrows: $(K, x) \xrightarrow{f} (K', x')$

$$\begin{array}{ccc} K' & \xrightarrow{f} & K \\ X(K') & \xrightarrow{X(f)} & X(K) \end{array} \quad (\text{contravariance})$$

$x \mapsto x' = X(f)(x)$

we have:

$$\begin{array}{ccc} \text{el}(X) & (K, x) \\ \downarrow \pi & \downarrow \\ \mathcal{L}^{\text{op}} & K \end{array}$$

→

$\mathcal{L}$ is face-structure category; or frame.

if, for every $K \in \text{Ob}(\mathcal{L})$,

1) $\dim(K)$ is finite;

and 2) $\hat{K} \stackrel{\text{def}}{=} \text{hom}_{\mathcal{L}}(-, K) : \mathcal{L}^{\text{op}} \rightarrow \text{Set}$ is a finite functor.

$\mathcal{L}_i$: always denotes a frame.

Properties of a frame $\mathcal{L}$:

- $\text{End}(K) = \text{hom}_{\mathcal{L}}(K, K) = \{1_K\}.$

- $\text{Ob}(\mathcal{L}) = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n$, where

$$K \in \mathcal{L}_n \Leftrightarrow \dim(K) = n.$$

- $K' \xrightarrow{f} K$ implies that $\dim(K') \leq \dim(K)$

and $K' \xrightarrow{P} K$ proper implies
 $\dim(K') \leq \dim(K)$.

• $K \in \mathcal{L}_0 \iff$ there is no proper
 $p: K' \rightarrow K$.

• for $n > 0$, $K \in \mathcal{L}_n \iff$

for all proper $p: K' \rightarrow K$,

$$K' \in \mathcal{L}_{\leq n-1} = \bigcup_{k \leq n-1} \mathcal{L}_k$$

and there is $p: K' \rightarrow K$
 with $K' \in \mathcal{L}_{n-1}$.

• $\text{hom}_{\mathcal{L}}(K', K)$ is finite.

• therefore, $X: \mathcal{L}^{\text{op}} \rightarrow \text{Set}$ is a
 finite functor iff

$$\coprod_{K \in \text{ob}(\mathcal{L})} X(K) (= \text{ob}(\text{el}(X)))$$

is a finite set.

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Denote by $\hat{\mathcal{L}}$ the presheaf category

$$\hat{\mathcal{L}} = \text{Set}^{\mathcal{L}^{\text{op}}},$$

and by

$$\text{Context}[\mathcal{L}], \text{ or } C[\mathcal{L}],$$

the full subcategory of $\hat{\mathcal{L}}$ on the objects the finite functors.

An object of $C[\mathcal{L}]$ is called a context (or: "context of variables" ...).

By condition 2) of the definition of "frame",

the Yoneda functor $\mathcal{L} \xrightarrow{Y} \hat{\mathcal{L}}$ factors

through $C[\mathcal{L}]$: we have $y: \mathcal{L} \longrightarrow C[\mathcal{L}]$

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{Y} & \hat{\mathcal{L}} \\ & \searrow y & \nearrow \text{inclusion} \\ & C[\mathcal{L}] & \end{array}$$

#

$$\left\{ \begin{array}{l} L \text{ will denote } \mathcal{L}^{\text{op}}; \text{ } L \text{ is a } \text{signature} \\ L = \mathcal{L}^{\text{op}} \\ \mathcal{L} = L^{\text{op}} \end{array} \right.$$

called

$X, Y, Z, \dots$: denote ^(usually) $(\mathcal{L}-)$ contexts;
 (finite functors $\mathcal{L}^{\text{op}} \rightarrow \text{Set}$);

$M, N, P, \dots$: arbitrary functors

$$\mathcal{L}^{\text{op}} \rightarrow \text{Set}.$$

A functor $\begin{cases} M : \mathcal{L} \rightarrow \text{Set} \\ M : \mathcal{L}^{\text{op}} \rightarrow \text{Set} \end{cases}$

is also called an $\mathcal{L}$ -structure, and
 also: $\mathcal{L}$ -set.

Particular contexts:

$$\hat{K} = \text{hom}_{\mathcal{L}}(-, K), \text{ the } K\text{-ball};$$

$\overset{\circ}{K}$, the K -sphere:

subfunctor of $\hat{K}$:

$$\overset{\circ}{K}(U) = \begin{cases} \hat{K}(U) & \text{when } U \neq K \\ \emptyset & \text{when } U = K \end{cases}$$

$(U \in \text{ob}(\mathcal{L}))$

$$\overset{\circ}{K} \xrightarrow{i_K} \hat{K} \quad \text{inclusion}$$

2. $\mathcal{L}$ -equivalence

(= span equivalence)

We are in the category $\hat{\mathcal{L}} = \text{Set}^{\mathcal{L}^{\text{op}}}$
 of $\mathcal{L}$ -sets; M, N, P, X, Y, Z objects
 of $\hat{\mathcal{L}}$; X, Y, Z usually finite functors.

→ A map $P \xrightarrow{m} M$ (natural transformation)
 is called fiberwise surjective (f.s.) (or a "trivial
 fibration"...) if it has the right lifting
 property with respect to the sphere inclusions:
 for all $K \in \text{Ob}(\mathcal{L})$,

whenever

$$\begin{array}{ccc} K^0 & \xrightarrow{\beta} & P \\ i_K \downarrow & \# & \downarrow m \\ \hat{K} & \xrightarrow{\hat{\alpha}} & M \end{array}$$

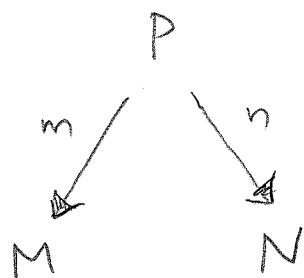
: 'commutes'

there exist b such that

$$\begin{array}{ccc} K^0 & \xrightarrow{\beta} & P \\ i_K \downarrow & \# & \downarrow m \\ \hat{K} & \xrightarrow{\hat{\alpha}} & M \end{array} \quad \begin{array}{c} \nearrow b \\ \searrow \end{array}$$

If $P \xrightarrow{m} M$ is f.s., it has the right lifting property v.r. to all monomorphisms $X \xrightarrow{f} Y$ (X, Y not necessarily finite functors) — Because every monomorphism is a transfinite composite of pushouts of sphere inclusions.

→ For L -structures (= L -sets) M and N , a span equivalence of M and N is a span (P, m, n) ,



where both m and n are f.s.

We write

$$(P, m, n) : M \sim N.$$

Writing " $M \sim N$ " (or " $M \sim_L N$ ")

means that there is $(P, m, n): M \sim N$;

we say "M and N are span-equivalent".

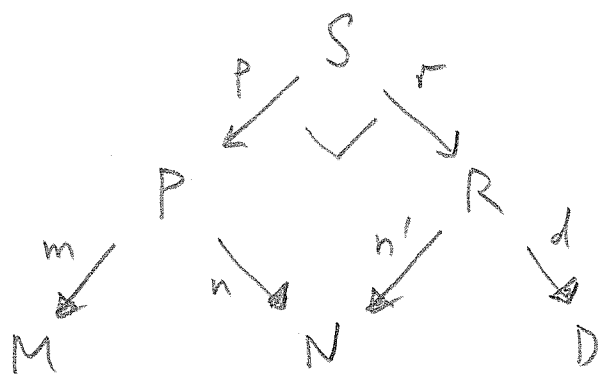
Indeed: $M \sim N$ is an equivalence relation.

Transitivity: suppose

$$(P, m, n): M \sim N$$

$$(R, n', d): N \sim D$$

Construct pullback S:



One proves the lemmas that a pullback of an f.s. map is f.s., and the composite of two f.s. maps is f.s. — from which

$$(S, m_P, d_r): M \sim D$$

3. Augmented structures and classes of such

Let: L : signature

$$(L = \mathcal{L}^{\text{op}}, \quad \mathcal{L}: \text{frame})$$

X : L -context

$$(\text{finite functor } \mathcal{L}^{\text{op}} \rightarrow \text{Set})$$

→ The pair (L, X) is an augmented signature.

→ An augmented structure over (L, X) is a pair (M, α) , where

$$M \in \hat{\mathcal{L}} \quad (= \text{category of } L\text{-structures})$$

and

$$\alpha: X \rightarrow M$$

$$\left(\mathcal{L}^{\text{op}} \begin{array}{c} \xrightarrow{X} \\ \downarrow \alpha \\ M \end{array} \rightarrow \text{Set} \right)$$

→ A class over (L, X) is any class of augmented structures over (L, X) .

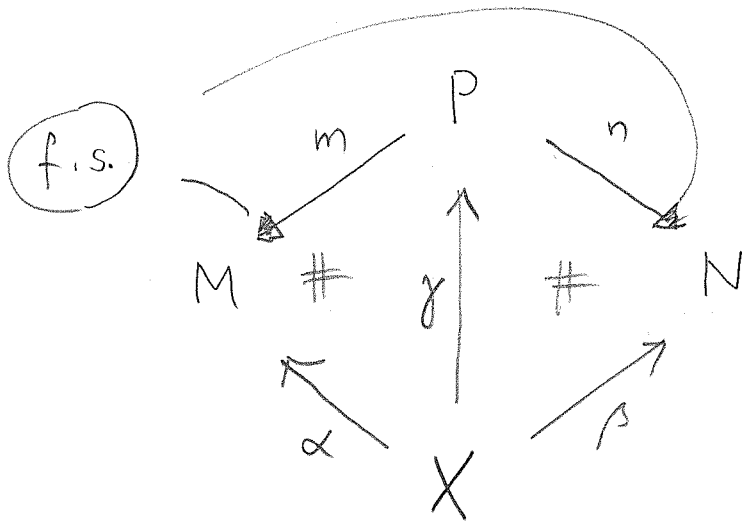
We also say: (L, X) -class.

→ A span equivalence of augmented structures (M, α) , (N, β) over the same augmented signature (L, X) is a quadruple (P, m, n, γ) such that

$$(P, m, n) : M \sim N$$

$$\gamma : X \rightarrow P$$

$$\text{and } m\gamma = \alpha, \quad n\gamma = \beta:$$



We write: $(P, m, n, \gamma) : (M, \alpha) \sim (N, \beta);$

and if such (P, m, n, γ) exists, we say

that (M, α) and (N, β) are span equivalent,

and write $(M, \alpha) \sim (N, \beta).$

Span equivalence of augmented

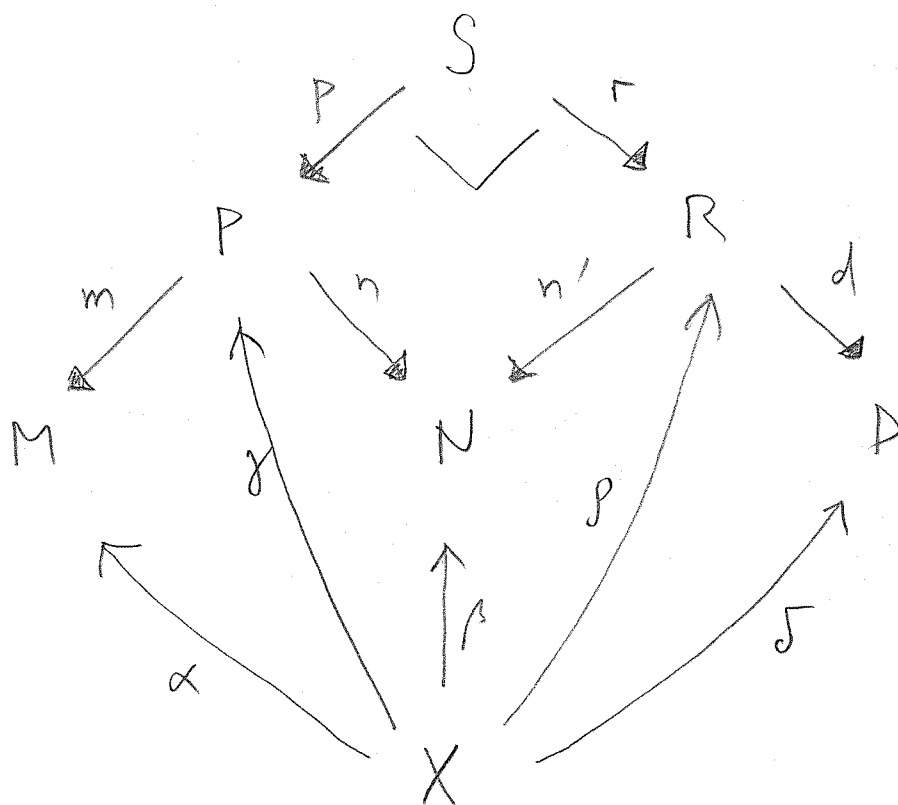
structures is an equivalence relation:

Transitivity:

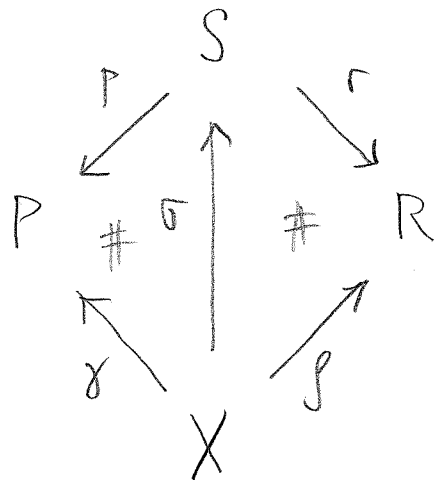
assume: $(P, m, n, \gamma) : (M, \alpha) \sim (N, \beta)$

$(R, n', d, \delta) : (N, \beta) \sim (D, \delta)$

Construct S, p, r as before; the following commutes:



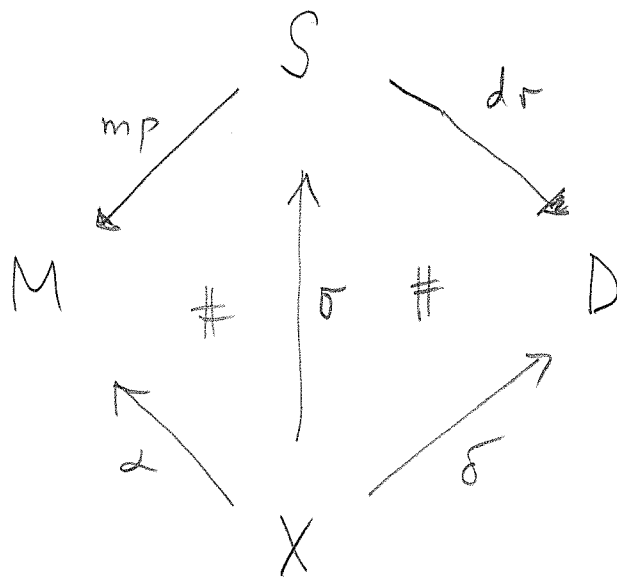
Define $X \xrightarrow{\sigma} S$ by the universal property of S
such that $\gamma = p\sigma$, $\beta = r\sigma$:



this is possible since $\eta\gamma = \rho\eta' = \rho$.

We have

$$(S, m_p, d_r, \sigma) : (M, \alpha) \sim (D, \delta) :$$



→ An (L, X) -class Φ is closed
under span-equivalence if

$\underline{M} \in \underline{\Phi}$ & $\underline{M} \sim \underline{N}$ imply
 $\left(\begin{smallmatrix} \text{"} \\ (M, \alpha) \end{smallmatrix} \right)$ that $\underline{N} \in \underline{\Phi}$.

An (L, X) -class $\underline{\Phi}$ that is
 closed under span-equivalence may be
 called a generalized (L, X) -formula.

(NB: $\underline{\Phi}$ is closed under span-equivalence
 iff the following holds:

every time

$$X \xrightarrow{\gamma} P \xrightarrow{m} M$$

and m is f.s., then

$$(M, m\gamma) \in \underline{\Phi}$$

$$\Leftrightarrow (P, \gamma) \in \underline{\Phi}.$$

4. Quantification

Let X, Y be arbitrary functors $\mathcal{L}^{\text{op}} \rightarrow \text{Set}$ (possibly "infinite contexts", although we are interested mainly in the case when X, Y are contexts; finite functors), and let

$$\xi: X \rightarrow Y,$$

any natural transformation (eventually: a monomorphism).

We make three definitions. As usual, $\boxed{L = \mathcal{L}^{\text{op}}}$.

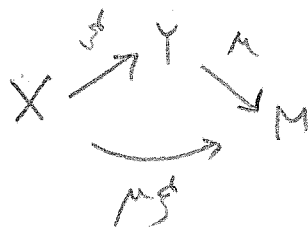
→ 1. For a class Ψ over (L, X) ,

the class $\xi^* \Psi$ over (L, Y) is

defined thus:

$$(M, \mu) \in \xi^* \Psi \iff_{\text{def}} (M, \mu \xi) \in \Psi$$

$\mu: Y \rightarrow M$



→ 2. For a class Φ over (L, Y) ,

the class $\forall_{\xi} \Phi$ over (L, X) is

defined thus:

$$(M, \alpha) \in \bigvee_{\mathcal{F}} \overline{\Phi}$$

$$\alpha: X \rightarrow M$$

$$\stackrel{\text{def}}{(\Leftrightarrow)}$$

for all $\mu: Y \rightarrow M$,

$\mu \upharpoonright_{\mathcal{F}} = \alpha$ implies $(M, \mu) \in \overline{\Phi}$.

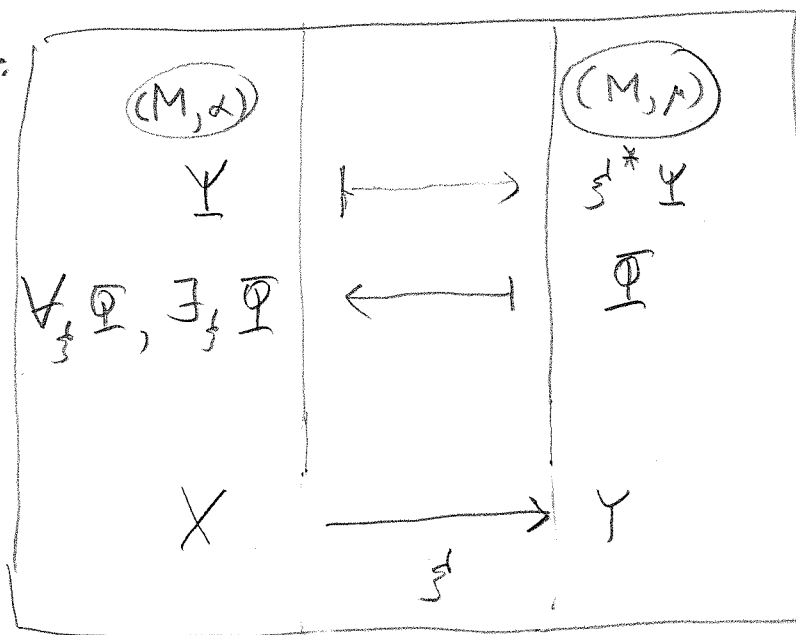
→ 3. For a class $\overline{\Phi}$ over (L, Y)

$(\exists_{\mathcal{F}} \overline{\Phi})$ is over (L, X) , and

$(M, \alpha) \in \bigvee_{\mathcal{F}} \overline{\Phi} \stackrel{\text{def}}{(\Leftrightarrow)} \underline{\text{there is } \mu: Y \rightarrow M}$

such that $\mu \upharpoonright_{\mathcal{F}} = \alpha$ and $(M, \mu) \in \overline{\Phi}$

Picture:



It is easy to see that we have the following adjunctions:

$$\frac{\Psi \subseteq \bigvee_{\mathcal{S}} \Phi}{\mathcal{S}^* \Psi \subseteq \Phi} \quad (\text{if and only if})$$

and

$$\frac{\bigexists_{\mathcal{S}} \Phi \subseteq \Psi}{\Phi \subseteq \mathcal{S}^* \Psi}$$

For brevity, let us say that a class is closed if it is closed under span equivalence; see pp [12], [13].

Continuing the above notation, we have:

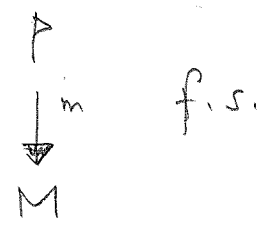
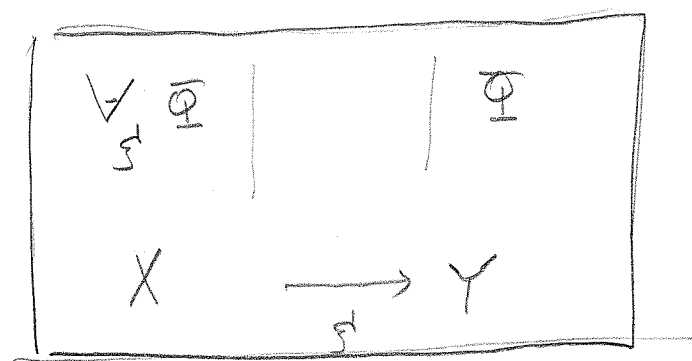
Proposition (i) Ψ is closed $\Rightarrow \mathcal{S}^* \Psi$ is closed.

(ii) If $X \xrightarrow{f} Y$ is a monomorphism, then

Φ is closed $\Rightarrow \bigvee_{\mathcal{S}} \Phi, \bigexists_{\mathcal{S}} \Phi$ are closed.

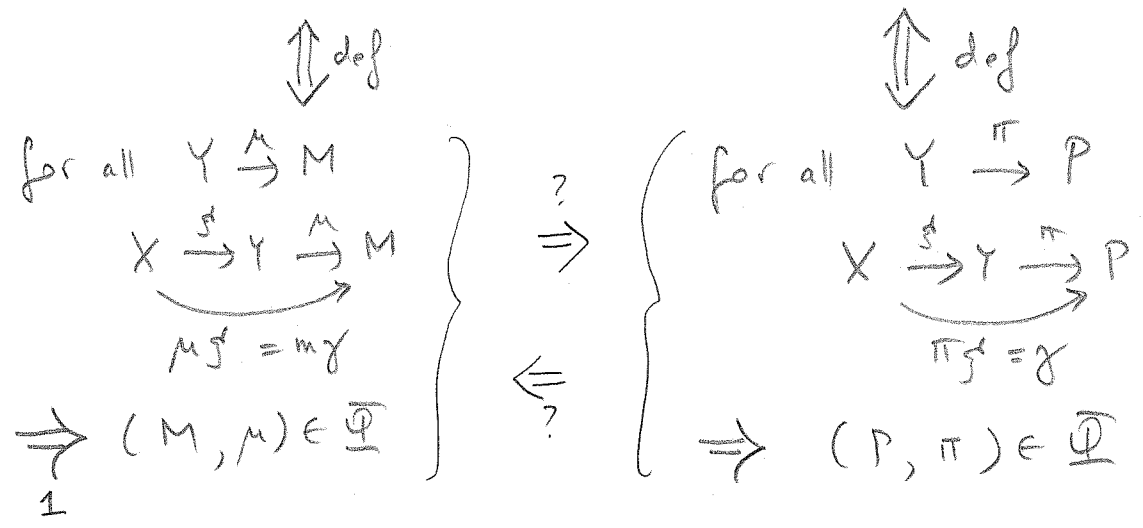
Proof of Proposition: assume: $f^!$ is mono, Φ is closed.

(proof of cii) for $\forall$



let: $X \xrightarrow{f} P$.

want: $(M, m_Y) \in \forall_{f^!} \Phi \iff (P, \gamma) \in \forall_{f^!} \Phi$



1) (the trivial direction) $\Rightarrow$: assume left-hand side.

let $Y \xrightarrow{\pi} P$ s.t. $\pi_{f^!} = \gamma$, to show $(P, \pi) \in \Phi$

define: $\mu \stackrel{\text{def}}{=} m \pi$; we have $\mu_{f^!} \stackrel{?}{=} m \gamma$
 $\parallel \checkmark \parallel$
 $m \pi_{f^!}$

Therefore, by assumption (see $\Rightarrow_1$)

$(M, \mu) \in \Phi$. We have

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & P \xrightarrow{m} M \\ & \searrow \mu = m\pi & \end{array}$$

Since Φ is closed, $(M, m\pi) \in \Phi \Rightarrow (P, \pi) \in \Phi$.

2) $\Leftarrow$: assume right-hand side.

let: $Y \xrightarrow{\mu} M$ s.t. $\mu \circ \xi = m \circ \gamma$:

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & P \\ \xi \downarrow & \# & \downarrow m \\ Y & \xrightarrow{\mu} & M \end{array}$$

, to show $(M, \mu) \in \Phi$

By: definition of ' m is f.s.', ξ mono: there is diagonal π :

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & P \\ \xi \downarrow & \nearrow \pi & \downarrow m \\ Y & \xrightarrow{\mu} & M \end{array}$$

$$\pi \circ \xi = \gamma$$

&

$$m \circ \pi = \mu$$

By assumption & $\pi f' = \gamma : (P, \pi) \in \bar{\Phi}$.

$$\begin{array}{ccccc} Y & \xrightarrow{\pi} & P & \xrightarrow{m} & M \\ & & \searrow & \nearrow & \\ & & \mu = m\pi & & \end{array}$$

Since $\bar{\Phi}$ is closed, $(P, \pi) \in \bar{\Phi} \Rightarrow (M, \mu) \in \bar{\Phi}$.

proof of (ii) for $\forall$: $\square$

proof of (ii) for $\exists$: dual

proof of (i): assume Ψ is closed.

let: $Y \xrightarrow{\pi} P$

$$\begin{array}{ccccc} X & \xrightarrow{f'} & Y & \xrightarrow{\pi} & P & \xrightarrow{m} & M \\ (M, m\pi) \in f'^* \Psi & \stackrel{?}{\Leftrightarrow} & (P, \pi) \in f'^* \Psi \end{array}$$

$$(M, m\pi f') \in \Psi \Leftrightarrow (P, \pi f') \in \Psi$$

↑
by Ψ being closed

(i) $\square$

5. Initial embeddings of frames

Let $Q, \mathcal{A}$ be frames. We say that Q is an initial segment of $\mathcal{A}$, $Q \leq \mathcal{A}$, if Q is a full subcategory of $\mathcal{A}$, and whenever $U \in \text{Ob}(Q)$, $K \in \text{Ob}(\mathcal{A})$ and there is an arrow $K \rightarrow U$ (in $\mathcal{A}$) then $K \in \text{Ob}(Q)$.

More generally: the functor $F: Q \rightarrow \mathcal{A}$ is an initial embedding, or simply initial, if F is one-to-one (injective) on objects, full and faithful, and whenever $U \in \text{Ob}(Q)$, $K \in \text{Ob}(\mathcal{A})$ and there is an arrow $K \rightarrow F(U)$ (in $\mathcal{A}$), then there is (a necessarily unique) $V \in \text{Ob}(Q)$ such that $K = F(V)$.

If $F: Q \rightarrow \mathcal{A}$ is initial, then F can be uniquely factored as

$$\begin{array}{ccc}
 Q & \xrightarrow{F} & \mathcal{A} \\
 \downarrow I & \nearrow F' & \uparrow \# \\
 & Q' &
 \end{array}$$

where I is an isomorphism of categories, 21
 and F' is the inclusion of an initial
 segment Q' of $\mathcal{L}$ to $\mathcal{L}$.

Suppose $F: Q \rightarrow \mathcal{L}$ is an initial embedding,
 $P, M \in \hat{\mathcal{L}}$, $m: P \rightarrow M$. By
 restriction, we have

$$mF: PF \longrightarrow MF$$

$$\begin{array}{ccc} Q^{op} & \xrightarrow{F^{op}} & \mathcal{L}^{op} \\ & & \downarrow m \\ & & M \end{array} \quad \begin{array}{c} P \\ \xrightarrow{\quad} \\ \text{Set} \end{array}$$

$$\begin{array}{ccc} Q^{op} & \xrightarrow{PF} & \text{Set} \\ & \downarrow mF & \\ & MF & \end{array}$$

Claim: If m is f.s., then mF is f.s.

This is essentially obvious — but here are some
 details: First of all, it suffices to consider
 the case when F is the inclusion of an
 initial segment, since for an isomorphism of
 categories, the assertion is obviously true.

Suppose $Q \leq \mathcal{L}$, $F: Q \rightarrow \mathcal{L}$ the inclusion. Let $U \in \text{Ob}(Q)$. I write

$$\hat{U}^{(Q)} = \text{hom}_Q(-, U): Q^{\text{op}} \rightarrow \text{Set}$$

$$\hat{U}^{(\mathcal{L})} = \text{hom}_{\mathcal{L}}(-, U): \mathcal{L}^{\text{op}} \rightarrow \text{Set}$$

$$U^{(Q)} \doteq \text{the subfunctor } U^{\circ} \text{ of } \hat{U}^{(Q)}$$

$$U^{(\mathcal{L})} \doteq \text{the subfunctor } U^{\circ} \text{ of } \hat{U}^{(\mathcal{L})}$$

Then we have:

$$\hat{U}^{(\mathcal{L})}(K) = \begin{cases} \hat{U}^{(Q)}(K) & \text{if } K \in Q \\ \emptyset & \text{if } K \in \mathcal{L} - Q \end{cases}$$

(note that if $K \in \mathcal{L} - Q$ and $U \in Q$,
then $\text{hom}_{\mathcal{L}}(K, U) = \emptyset$)

and

$$U^{(\mathcal{L})}(K) = \begin{cases} U^{(Q)}(K) & \text{if } K \in Q \\ \emptyset & \text{if } K \in \mathcal{L} - Q \end{cases}$$

It follows that, for $U \in \text{Ob}(Q)$,

the diagrams

$$\begin{array}{ccc}
 \mathcal{U}^{\circ}(Q) & \xrightarrow{\beta} & PF \\
 i_U^{(Q)} \downarrow & \# & \downarrow m_F \\
 \hat{\mathcal{U}}^{(Q)} & \xrightarrow{\hat{\alpha}} & MF
 \end{array}
 \quad \text{in } \hat{Q}$$

are in a bijective correspondence with the diagrams

$$\begin{array}{ccc}
 \mathcal{U}^{\circ}(\Lambda) & \xrightarrow{\beta^*} & P \\
 i_U^{(\Lambda)} \downarrow & \# & \downarrow m \\
 \hat{\mathcal{U}}^{(\Lambda)} & \xrightarrow{\hat{\alpha}^*} & M
 \end{array}
 \quad \text{in } \hat{\Lambda},$$

the bijection effected by restriction from the second to the first, and likewise,

the diagrams

$$\begin{array}{ccc}
 \mathcal{U}^{\circ}(Q) & \xrightarrow{\beta} & PF \\
 i \downarrow & \# \nearrow & \downarrow m_F \\
 \hat{\mathcal{U}}^{(Q)} & \xrightarrow{\hat{\alpha}} & MF
 \end{array}$$

are bijectively related to the diagrams

$$\begin{array}{ccc}
 \mathcal{U}(\mathcal{A}) & \xrightarrow{\beta^*} & P \\
 i \downarrow & \nearrow \# & \downarrow m \\
 \hat{\mathcal{U}}(\mathcal{A}) & \xrightarrow{\hat{\alpha}^*} & M
 \end{array}$$

The Claim follows.

6. The category of augmented frames (signatures)

Let $\underline{\mathcal{A}}_1 = (\mathcal{A}_1, X_1)$, $\underline{\mathcal{A}}_2 = (\mathcal{A}_2, X_2)$ be augmented frames: $\mathcal{A}_1$ is a frame (face-structure category), $X_1: \mathcal{A}_1^{\text{op}} \rightarrow \text{Set}$ is a finite functor; similarly for $\underline{\mathcal{A}}_2$.

A morphism $\underline{F}: \underline{\mathcal{A}}_1 \rightarrow \underline{\mathcal{A}}_2$ is $\underline{F} = (F, \xi)$, where $F: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is an initial embedding, and $\xi: X_1 \rightarrow X_2 F:$

$$\begin{array}{ccc}
 \mathcal{A}_1^{\text{op}} & \xrightarrow{X_1} & \text{Set} \\
 F^{\text{op}} \downarrow & \Downarrow \xi & \uparrow X_2 \\
 \mathcal{A}_2^{\text{op}} & \xrightarrow{X_2} &
 \end{array}$$

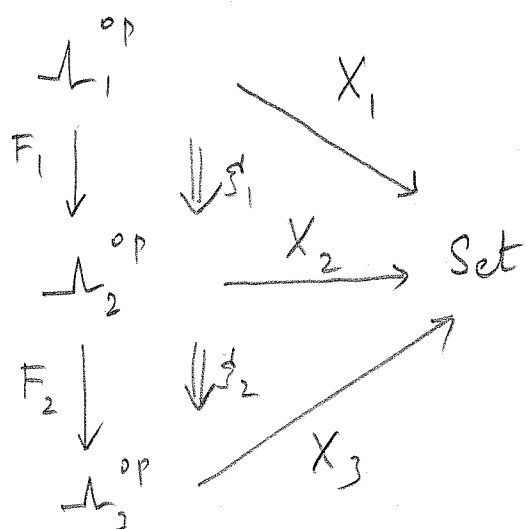
We can compose these morphisms:

$$(\mathcal{A}_1, X_1) \xrightarrow{(F_1, f_1)} (\mathcal{A}_2, X_2) \xrightarrow{(F_2, f_2')} (\mathcal{A}_3, X_3)$$

gives rise to

$$(\mathcal{A}_1, X_1) \xrightarrow{(F_2 F_1, \underbrace{(f_2' F_1) f_1}_{f_3})} (\mathcal{A}_3, X_3)$$

(compare:

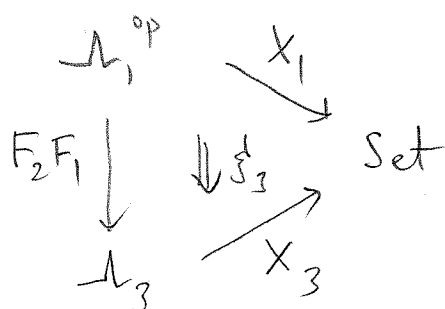


$$f_1: X_1 \rightarrow X_2 F_1$$

$$f_2: X_2 \rightarrow X_3 F_2$$

$$(f_2 F_1: X_2 F_1 \rightarrow X_3 F_1 F_2)$$

$$X_1 \xrightarrow{f_1} X_2 F_1 \xrightarrow{f_2 F_1} X_3 F_1 F_2$$



This composition is associative, and

has identities $(\mathcal{L}, X) \xrightarrow{(Id_{\mathcal{L}}, id_X)} (\mathcal{L}, X)$.

We have the category $\underline{AugFr}$ of augmented frames.

The morphism $(F, f): (\mathcal{L}_1, X_1) \longrightarrow (\mathcal{L}_2, X_2)$ can be factored as the composite of "simpler" morphisms:

$$\begin{array}{ccc}
 (\mathcal{L}_1, X_1) & \xrightarrow{(F, f)} & (\mathcal{L}_2, X_2) \\
 & \searrow (Id_{\mathcal{L}_1}, f) & \nearrow (F, id_{X_2} F_1) \\
 & (\mathcal{L}_1, X_2 F_1) &
 \end{array}$$

We will work with the full subcategory $\underline{AugFr}_{\leq \aleph_0}$ of $\underline{AugFr}$ whose objects $(\mathcal{L}, X)$ have

$\mathcal{L}$ a countable frame: $Ob(\mathcal{L})$ is a countable (possibly finite) set.

7. Reducts and coreducts

Let $\underline{Q} = (Q, \gamma)$, $\underline{\Delta} = (\Delta, \chi)$

be augmented frames, $\underline{F} = (F, \mathfrak{f}) : \underline{Q} \rightarrow \underline{\Delta}$

a morphism (see above). For an

augmented structure $\underline{M} = (M, \alpha)$ over $\underline{\Delta}$,

→ the reduct $\underline{M} \restriction \underline{F}$ is the a.s. over $\underline{Q}$

defined by

$$\underline{M} \restriction \underline{F} = (MF, (\alpha F)\mathfrak{f})$$

(compare:

$$Q^{op} \xrightarrow{F} \Delta^{op} \xrightarrow[\alpha]{\chi} Set$$

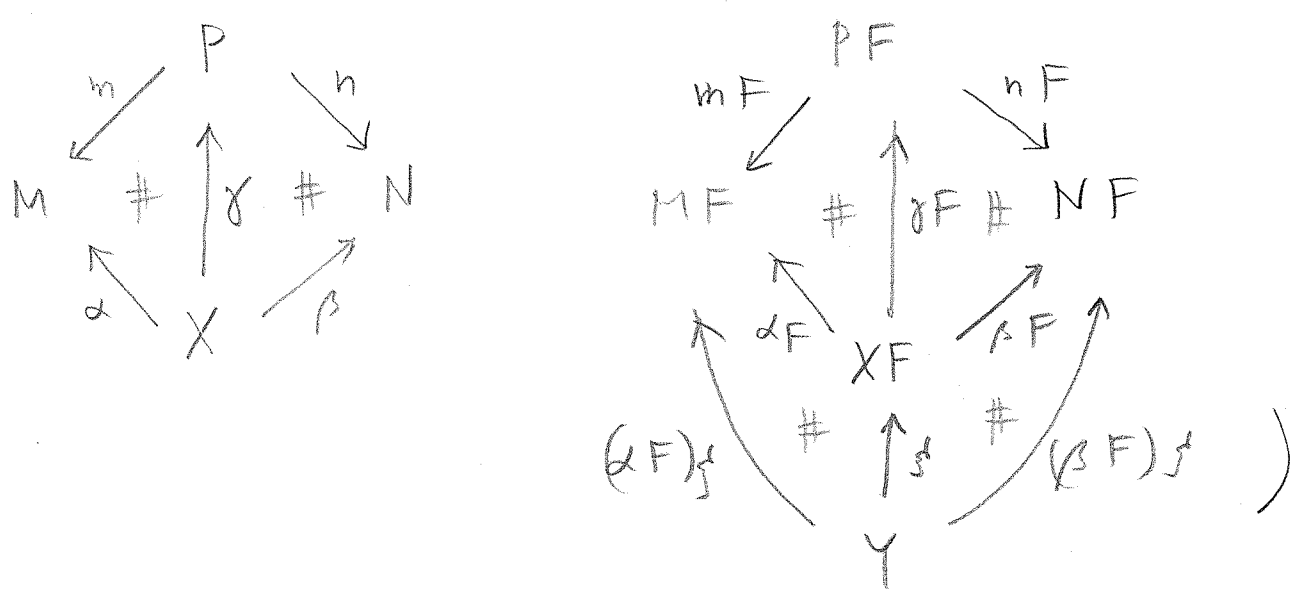
$$Q^{op} \xrightarrow[\alpha F]{\begin{array}{c} \gamma \\ XF \downarrow \mathfrak{f} \end{array}} Set$$

Note: if $(P, m, n, \gamma) : \underline{M} \sim \underline{N}$

then

$$(P, mF, nF, (\gamma F)\mathfrak{f}) : \underline{M} \restriction \underline{F} \sim \underline{N} \restriction \underline{F}$$

(compare:



remember that m, n are f.s.

$\Rightarrow mF, nF$ are f.s.

Hence:

$$\underline{M} \underset{\underline{\Lambda}}{\sim} \underline{N} \quad \Rightarrow \quad \underline{M} \underset{\underline{Q}}{\uparrow} \underline{F} \underset{\underline{Q}}{\sim} \underline{N} \underset{\underline{Q}}{\uparrow} \underline{F}$$

With $\underline{F} : \underline{Q} \rightarrow \underline{\Lambda}$ as before, let

$\underline{\Phi}$ be a class over $\underline{Q}$. The wreath of $\underline{Q}$ along $\underline{F}$, $\underline{\Phi} \uparrow \underline{F}$, is the class over $\underline{\Lambda}$ for which

$$\underline{M} \in \underline{\Phi} \uparrow \underline{F} \Leftrightarrow_{\text{def}} \underline{M} \underset{\underline{Q}}{\uparrow} \underline{F} \in \underline{\Phi}$$

It immediately follows that

$$\underline{\Phi} \text{ is closed} \Rightarrow \underline{\Phi} \mid \underline{E} \text{ is closed}$$

Def. Generalized FOLDS

A generalized first-order logic with dependent sorts, a GFOLDS for short, is a collection of classes of augmented structures over countable augmented signatures, satisfying conditions given below. With $\mathcal{L}$ the given GFOLDS, for any countable augmented signature $\underline{L}$, I denote by $\mathcal{L}_{\underline{L}}$ the collection of augmented structures over $\underline{L}$ in $\mathcal{L}$. Thus, $\mathcal{L}$ is the disjoint union of all $\mathcal{L}_{\underline{L}}$, with $\underline{L}$ ranging over countable augmented signatures — the empty class is counted as many times as there are $\underline{L}$'s. The conditions are (1) (1)-(4) given below.

[For an augmented frame $(\mathcal{A}, X)$, $\mathcal{L}_{(\mathcal{A}, X)}$ is the same as $\mathcal{L}_{(\mathcal{A}^{op}, X)}$].

(1) Every class in $\mathcal{L}$ is closed under span equivalence. For any $\underline{L}$ (as above), and $\underline{\Phi} \in \mathcal{L}_{\underline{L}}$, $\underline{M} \in \underline{\Phi}$ & $\underline{M} \sim_{\underline{L}} \underline{N}$ imply that $\underline{N} \in \underline{\Phi}$.

(2) $\mathcal{L}_{\underline{L}}$ (any $\underline{L}$) is a Boolean subalgebra of the collection of all classes over $\underline{L}$. That is, $\underline{T}_{\underline{L}}$, the class of all augmented structures over $\underline{L}$, belongs to $\mathcal{L}_{\underline{L}}$;

whenever $\underline{\Phi}, \underline{\Psi} \in \mathcal{L}_{\underline{L}}$, we have that $\underline{\Phi} \cap \underline{\Psi}$, $\underline{\Phi} \cup \underline{\Psi}$, $\underline{T}_{\underline{L}} - \underline{\Phi}$ (complement) belong to $\mathcal{L}_{\underline{L}}$ as well

(3) $\mathcal{L}$ is closed under first-order FOLDS quantification. For any countable signature L , X, Y finite L -contexts (finite functors $L \rightarrow \text{Set}$) and a monomorphism (!) $f: X \rightarrow Y$,

$$\underline{\Phi} \in \mathcal{L}_{(L, Y)} \Rightarrow \forall_f \underline{\Phi} \in \mathcal{L}_{(L, X)}$$

$$(\text{and } \exists_f \underline{\Phi} \in \mathcal{L}_{(L, X)})$$

(4) $\mathcal{L}$ is closed under coreduction.

For every morphism $\underline{F} : \underline{Q} \rightarrow \underline{\Lambda}$ of countable augmented frames,

$$\underline{\Phi} \in \mathcal{L}_{\underline{Q}} \Rightarrow \underline{\Phi} \upharpoonright \underline{F} \in \mathcal{L}_{\underline{\Lambda}}.$$

end of def 'GFOLDS'

The classical model-theoretical conditions on $\mathcal{L}$ (not necessarily holding for an arbitrary GFOLDS $\mathcal{L}$):

CC: Countable compactness: Given countable augmented signature $\underline{L}$, and classes $\underline{\Phi}_n \in \mathcal{L}_{\underline{L}}$, $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} \underline{\Phi}_n = \emptyset$ implies that there is $m \in \mathbb{N}$ such that $\bigcap_{n \leq m} \underline{\Phi}_n = \emptyset$.

CdLS: Countable downward Löwenheim-Skolem property:

For $\underline{L}$ and $\underline{\Phi}_n \in \mathcal{L}_{\underline{L}}$, $n \in \mathbb{N}$ as above $\bigcap_{n \in \mathbb{N}} \underline{\Phi}_n \neq \emptyset$ implies that there is a countable $\underline{M} \in \bigcap_{n \in \mathbb{N}} \underline{\Phi}_n$. ($\underline{M} = (M, \alpha)$ is

countable means that $M: L \rightarrow \text{Set}$ is a countable functor: $\coprod_{K \in \text{Ob}(L)} M(K)$ is a countable set).

Thm Lindström for FOLDS.

For a GFOLDS $\mathcal{L}$,

$\mathcal{L}$ satisfies $\underline{C_c}$ and $\underline{\underline{C_{dLS}}}$

if and only if $\mathcal{L}$ is the minimal

GFOLDS: for any GFOLDS $\mathcal{L}'$, $\mathcal{L} \subseteq \mathcal{L}'$.

NB: For any collection $\mathcal{L}_0$ of classes

such that every Φ in $\mathcal{L}_0$ is closed under

span-equivalence, there is a least GFOLDS

$\langle \mathcal{L}_0 \rangle$ generated by $\mathcal{L}_0$. The conditions 2), 3) and

4) are closure conditions; $\langle \mathcal{L}_0 \rangle$ is the

closure of $\mathcal{L}_0$ under these. $\langle \mathcal{L}_0 \rangle$ also satisfies

the main FOLDS condition 1), since

each of the closure steps 2), 3), 4)

generate closed classes from closed classes. For 2), this is trivial.

For 3), this is proved in §4., "Quantification";

for 4), in §7. "Reducts and coreducts".

The minimal GFOLDS is the collection

$\mathcal{L}_{\text{ww}} \stackrel{\text{def}}{=} \langle \emptyset \rangle$ generated by the empty collection $\mathcal{L}_0 = \emptyset$.

The statement as well as the proof (to be given in Part 2 of these notes) of the Lindström theorem for FOLDS are given without any use of traditional syntax (formulas and the like), and even without categorical-logical formulations of syntax versus semantics. We can make the connections to traditions easily, however.

The classes in $\mathcal{L}_{\text{ww}}$, the minimal GFOLDS, are exactly the classes

$$\text{Mod}_L [X: \varphi]$$

Where:

- L is a countable FOLDS signature;
- X is an L -context, construed as a finite context of typed variables, with the dependent typing rules given by L ;
- φ is a (finite) FOLDS formula over L such that the free variables of φ are all in X ;
- and $\text{Mod}_L [X: \varphi]$ is the class of all augmented structures (M, α) , $M: L \rightarrow \text{Set}$, $\alpha: X \rightarrow M$ for which

$$M \models \varphi[\alpha]$$
 "M satisfies φ at α ".