
February 6, 2014
Notes

Contents:

1. Face-structure categories ( = frames, for short)  \[ 1 \]

2. L-equivalence ( = span equivalence) \[ 6 \]

3. Augmented structures and classes of such \[ 9 \]

4. Quantification \[ 14 \]

5. Initial embeddings of "frames" \[ 20 \]

6. The category of augmented frames (signatures) \[ 24 \]

7. Reducts and coreducts \[ 27 \]

8. Generalized FOLDS \[ 29 \]
1. **Face-structure categories** (= "frames") for short

Let: $\Lambda$ : small category

$K, K', \ldots$ : objects of $\Lambda$ ("kinds")

proper arrow: any non-identity arrow,

$\Rightarrow \quad \dim(K) \overset{\text{def}}{=} \text{largest } n \in \mathbb{N} \text{ such that}$

there exists a composable string of length $n$

of proper arrows ending in $K$:

$K = K_0 \xleftarrow{\neq id} K_1 \xleftarrow{\ldots} K_{n-1} \xleftarrow{\neq id} K_n$

if such largest $n$ exists;

$\dim(K) = \infty$ otherwise.

$\Rightarrow \quad \text{A-functor } X : \Lambda^{op} \to \text{Set is finite if}$

$el(X)$, the category of elements of $X$ is

a finite category (having finitely many arrows (and objects)).

(\text{NB: } el(X): \text{ objects: } (K, x) :: K \in \text{Ob}(\Lambda) \quad x \in X(K)$

arrows: $(K, x) \xrightarrow{f} (K', x')$

\[ K' \xrightarrow{f} K \quad X(K) \xrightarrow{X(f)} X(K') \quad \text{(covariance)} \]

$\quad x \mapsto x' = X(f)(x)$
we have:
\[
\begin{align*}
el(X) & \quad (K, x) \\
\downarrow & \quad \downarrow \\
\Lambda^{op} & \quad K
\end{align*}
\]

thus, for every \( K \in \text{Ob}(\Lambda) \),

1) \( \dim(K) \) is finite;

and 2) \( \Lambda = \text{hom}_\Lambda (-, K) : \Lambda^{op} \to \text{Set} \) is a finite functor.

\( \Lambda \): always denotes a frame.

Properties of a frame \( \Lambda \):

- \( \text{End}(K) = \text{hom}_\Lambda (K, K) = \{1_K\} \).

- \( \text{Ob}(\Lambda) = \bigcup \Lambda_n \), where \( n \in \mathbb{N} \)

  \( K \in \Lambda_n \iff \dim(K) = n. \)

- \( K' \xrightarrow{f} K \) implies that \( \dim(K') \leq \dim(K) \).
and $K' \xrightarrow{p} K$ proper implies $\dim(K') < \dim(K)$.

- $K \in \mathcal{L}_0 \iff$ there is no proper $p : K' \rightarrow K$.

- for $n > 0$, $K \in \mathcal{L}_n \iff$
  
  for all proper $p : K' \rightarrow K$, $K' \in \mathcal{L}_{n-1} \subseteq \bigcup_{k \leq n-1} \mathcal{L}_k$

  and there is $p : K' \rightarrow K$ with $K' \in \mathcal{L}_{n-1}$.

- hom$_{\mathcal{L}}(K', K)$ is finite.

- Therefore, $\chi : \mathcal{L}^{op} \rightarrow \text{Set}$ is a finite functor iff

  $\bigvee_{K \in \mathcal{L}^{op}} X(K) \subseteq \text{Ob}(\text{el}(X))$

  in a finite set.
Denote by $\hat{\mathcal{L}}$ the presheaf category 
\[ \hat{\mathcal{L}} = \text{Set} \mathcal{L}^{\text{op}}, \]
and by
\[ \text{Context}[\mathcal{L}], \text{ or } C[\mathcal{L}], \]
the full subcategory of $\hat{\mathcal{L}}$ on the objects the finite functors.
An object of $C[\mathcal{L}]$ is called a context (or; "context of variables"...).

By condition 2) of the definition of "frame" the Yoneda functor $\mathcal{L} \to \hat{\mathcal{L}}$ factors through $C[\mathcal{L}]$; we have $y : \mathcal{L} \to C[\mathcal{L}]$

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{Y} & \hat{\mathcal{L}} \\
\downarrow{y} & \nearrow{\text{inclusion}} & \\
& C[\mathcal{L}] & \\
\end{array}
\]

L will denote $\mathcal{L}^{\text{op}}$; L is a signature

\[
\begin{cases}
L = \mathcal{L}^{\text{op}} \\
\mathcal{L} = L^{\text{op}}
\end{cases}
\]
$X, Y, Z, \ldots$ denote $(\mathcal{L}-)$ contexts;

finite functors $\mathcal{L}^{\text{op}} \rightarrow \text{Set}$;

$M, N, P, \ldots$ arbitrary functors $\mathcal{L}^{\text{op}} \rightarrow \text{Set}$.

A functor $\{M : L \rightarrow \text{Set}
\begin{align*}
\begin{array}{c}
M : \mathcal{L}^{\text{op}} \rightarrow \text{Set} \\
\end{array}
\end{align*}$

is also called an $L$-structure, and also: $L$-set.

Particular contexts:

$\hat{K} = \text{hom}_L(-, K)$, the $K$-ball;

$\hat{K}$, the $K$-sphere:

Subfunctor of $\hat{K}$:

$\hat{K}(U) = \begin{cases} 
\hat{K}(U) & \text{when } U \neq K \\
\emptyset & \text{when } U = K
\end{cases}$

$(U \in \text{Ob}(\mathcal{L}))$

$\hat{K} \rightarrow \hat{K}$, inclusion
2. \( L \)-equivalence

\[ = \text{span equivalence} \]

We are in the category \( \hat{\mathcal{A}} = \text{Set}^{\mathcal{A}^{op}} \) of \( \mathcal{A} \)-sets; \( M, N, P, \{ X, Y, Z \} \) objects of \( \hat{\mathcal{A}} \); \( X, Y, Z \) usually finite functors.

\[ \rightarrow \text{A map } P \xrightarrow{m} M \text{ (natural transformation) is called fiberwise surjective (f.s.) (or: a trivial fibration...)} \text{ if it has the right lifting property with respect to the sphere inclusions:} \]

for all \( K \in \text{Ob}(\mathcal{A}) \):

Whenever

\[
\begin{array}{ccc}
K & \xrightarrow{\beta} & P \\
\downarrow^{i_K} & \# & \downarrow^m \\
\hat{\mathcal{A}} & \xrightarrow{\cong} & M
\end{array}
\]

there exists \( \xi \) such that

\[
\begin{array}{ccc}
K & \xrightarrow{\beta} & P \\
\downarrow^{i_K} & \# & \downarrow^m \\
\hat{\mathcal{A}} & \xrightarrow{\cong} & M
\end{array}
\]

\[ \# : \text{Commutes} \]
If \( P \rightarrow M \) is f.s., it has the right lifting property w.r.t. to all monomorphisms \( X \rightarrow Y \) (\( X, Y \) not necessarily finite functors) — because every monomorphism is a transfinite composite of pushouts of sphere inclusions.

For \( L \)-structures (\( = L \)-sets) \( M \) and \( N \), a span equivalence of \( M \) and \( N \) is a span \((P, m, n)\),

\[
\begin{array}{c}
\text{P} \\
\downarrow m \quad \downarrow n \\
M & \rightarrow & N
\end{array}
\]

where both \( m \) and \( n \) are f.s.

We write
\[
(P, m, n) : M \sim N.
\]

Writing "\( M \sim N \)" (or "\( M \sim L N \)"")
means that there is \((P, m, n) : M \sim N\);
we say "M and N are span-equivalent".

Indeed, \(M \sim N\) is an equivalence relation.

**Transitivity:** suppose
\[
(P, m, n) : M \sim N \\
(R, n', d) : N \sim D
\]

Construct pullback \(S\):

```
      S
     / \  \\
    P   R
   / \   /   \
  m  n n' d \\
 M  N  D
```

One proves the lemma that a pullback of
an f.s. map is f.s.; and the composite of
two f.s. maps is f.s. — from which

\((S, m, p, d*): M \sim D\)
3. Augmented structures and classes of such

Let: \( L: \) signature

\[ L = L^\text{op}, \quad \Lambda: \text{frame} \]

\( X: L\)-context

(\text{finite functor } \Lambda^\text{op} \to \text{Set})

\( \Rightarrow \) The pair \((L, X)\) is an \underline{augmented signature}.

\( \Rightarrow \) An \underline{augmented structure} over \((L, X)\) is a pair \((M, \alpha)\), where

\[ \Lambda \in \Lambda \quad (= \text{category of } L\text{-structures}) \]

and

\[ \alpha: X \to M \]

\[ \Lambda^\text{op} \xrightarrow{X} \text{Set} \]

\[ M \]

\( \Rightarrow \) A \underline{class} over \((L, X)\) is any class of augmented structures over \((L, X)\).

We also say: \((L, X)\)-class.
A span equivalence of augmented structures \((M, \alpha), (N, \beta)\) and the same augmented signature \((L, X)\) in a quadruple \((P, m, n, \gamma)\) such that

\[(P, m, n) : M \sim N\]

\[\gamma : X \to P\]

and \(m \gamma = \alpha, \ n \gamma = \beta:\]

We write: \((P, m, n, \gamma) : (M, \alpha) \sim (N, \beta)\);

and if such \((P, m, n, \gamma)\) exists, we say that \((M, \alpha)\) and \((N, \beta)\) are span equivalent and write \((M, \alpha) \sim (N, \beta)\).
Span equivalence of augmented structures is an equivalence relation:

Transitivity: Assume: \((P, m, n, \gamma) : (M, \alpha) \sim (N, \beta)\)

\((R, n', d, \rho) : (N, \beta) \sim (D, \delta)\)

Construct \(S, p, \pi, \tau\) as before; the following commutes:

Define \(X \xrightarrow{\sigma} S\) by the universal property of \(S\) such that \(\gamma = p \sigma\), \(\rho = r \sigma\):
this is possible since $\gamma = \sigma \gamma' = \rho$.

We have

$$(S, m_p, d_r, \sigma) : (M, \lambda) \sim (D, \delta) :$$

$\rightarrow$ A $(L, X)$-class $\mathcal{C}$ is cloned under span-equivalence if
\[ M \in \Phi \land M \sim N \text{ imply } (\mathbf{\Phi}, \mathbf{\alpha}) \text{ that } N \in \mathbf{\Phi}. \]

An \((L, X)\)-class \(\Phi\) that is closed under span-equivalence may be called a generalized \((L, X)\)-formula.

\text{\textit{NB:}} \quad \Phi \text{ is closed under span-equivalence iff the following holds:}

Every time
\[ X \xrightarrow{g} P \xrightarrow{m} M \]
and \(m\) is f.c., then
\[ (M, m \gamma) \in \Phi \]

\[ \iff (P, \gamma) \in \Phi. \]
Let $X, Y$ be arbitrary functors $\mathcal{A}^\circ \to \text{Set}$ (possibly "infinite contexts", although we are interested mainly in the case when $X, Y$ are contexts; finite functors), and let

$$\delta^1 : X \to Y,$$

any natural transformation (eventually: a monomorphism).

We make three definitions. As usual, $L = \mathcal{A}^\circ$.

$\rightarrow 1.$ For a class $\mathcal{Y}$ over $(L, X)$,

the class $\delta^1 \times \mathcal{Y}$ over $(L, Y)$ is defined thus:

$$(M, \mu) \in \delta^1 \times \mathcal{Y} \iff (M, \mu\delta) \in \mathcal{Y}$$

$\mu : Y \to M$

$X \overset{\delta^1}{\to} Y \overset{\mu}{\to} M$

$\mu\delta$

$\rightarrow 2.$ For a class $\mathcal{\Omega}$ over $(L, Y)$,

the class $\forall \delta \mathcal{\Omega}$ over $(L, X)$ is defined thus:
\[(M, \alpha) \in \forall \bar{\phi} \bar{s}\]
\[\alpha : X \to M\]

\[
\iff \\
\text{def}
\]

for all \(\mu : Y \to M\),
\[\mu \bar{s} = \alpha \text{ implies } (M, \alpha) \in \bar{\phi}\].

\(\rightarrow 3.\) For a class \(\bar{\phi}\) over \((L, Y)\)

\[\exists_y \bar{\phi}\]
is over \((L, X)\), and

\[(M, \alpha) \in \forall \bar{\phi} \bar{s} \iff \text{there is } \mu : Y \to M\]
such that \(\mu \bar{s} = \alpha\) and \((M, \mu) \in \bar{\phi}\).

Picture:

\[
\begin{array}{c|c|c}
(M, \alpha) & Y & (M, \alpha) \\
\forall \bar{\phi}, \exists_y \bar{\phi} & X \to Y & \exists_y \bar{\phi} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 & Y & \bar{\phi} \\
\downarrow & \downarrow & \downarrow \\
X & \rightarrow & Y \\
\end{array}
\]
It is easy to see that we have the following adjunctions:

\[ \psi \subseteq \mathcal{V} \quad \text{if and only if} \quad \mathcal{S} \ast \psi \subseteq \mathcal{V} \]

and

\[ \exists \mathcal{V} \subseteq \psi \quad \text{if and only if} \quad \mathcal{V} \subseteq \mathcal{S} \ast \psi \]

For brevity, let us say that a class is \underline{closed} if it is closed under span equivalence; see pp \[12], [13].

Continuing the above notation, we have:

**Proposition (i)** \[ \psi \text{ is closed } \Rightarrow \mathcal{S} \ast \psi \text{ is closed.} \]

(ii) \[ \psi \text{ is closed } \Rightarrow \forall \mathcal{V}, \exists \mathcal{S} \ast \psi \text{ are closed.} \]
Proof of Proposition: Assume $\gamma$ is monic, $\bar{\mathcal{C}}$ is closed.

Let: $X \xrightarrow{\delta} P$.

Want: $(M, m') \in \forall_{\delta} \mathcal{Q} \iff (P, \gamma) \in \forall_{\delta} \mathcal{Q}$

\[ \begin{align*}
\text{for all } Y & \xrightarrow{\delta} M \quad \text{for all } Y \xrightarrow{\pi} P \\
\Rightarrow (M, m') & \in \mathcal{Q} \quad \Rightarrow (P, \gamma) \in \mathcal{Q} \\
\text{(the trivial direction)} & \Rightarrow \quad \text{assume left-hand side.}
\end{align*} \]

Let $Y \xrightarrow{\pi} P$ s.t. $m'^{-1} = \gamma$, to show $(P, \pi) \in \mathcal{Q}$

Define: $m \overset{\text{def}}{=} m \pi$; we have $m'^{-1} = \gamma$.

$\Rightarrow m \pi = m'$. \[ \square \]
Therefore, by assumption (see \( \Rightarrow \))

\((M, \mu) \in \overline{\Phi} \). We have

\[
\begin{array}{ccc}
Y & \xrightarrow{\gamma} & P \\
\downarrow \Phi & \sc{\mu} & \downarrow m \\
Y & \rightarrow & M
\end{array}
\]

\(\mu = m \circ \Phi\)

Since \( \Phi \) is done, \((M, m \circ \Phi) \in \overline{\Phi} \) \(\Rightarrow\) \((P, \pi) \in \overline{\Phi} \).

2) \(\Leftarrow\) : assume right-hand side.

Let \(Y \xrightarrow{\gamma} M \) s.t. \(\mu \circ \gamma = m Y\):

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma} & P \\
\downarrow \delta & \sc{\#} & \downarrow \mu \\
Y & \rightarrow & M
\end{array}
\]

\(\delta \) mono: there is diagonal \(\pi\):

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma} & P \\
\downarrow \delta & \sc{\#} & \downarrow \mu \\
Y & \rightarrow & M
\end{array}
\]

\(\mu \circ \pi = \gamma\)

By definition of \( (M, \mu) \) is f.s., \(\delta\) mono: there is diagonal \(\pi\):

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma} & P \\
\downarrow \delta & \sc{\#} & \downarrow \mu \\
Y & \rightarrow & M
\end{array}
\]

\(\mu \circ \pi = \gamma\)
By assumption \( \pi f = \gamma \), \((P, \pi) \in \Omega \).

\[
Y \overset{\pi}{\longrightarrow} P \overset{m}{\longrightarrow} M
\]

\( \mu = m \pi \)

Since \( \Omega \) is closed, \((P, \pi) \in \Omega \) \( \Rightarrow \) \((M, \mu) \in \Omega \).

---

Proof of (ii) for \( \exists \): dual

Proof of (i): assume \( \Psi \) is closed.

Let: \( Y \overset{\pi}{\longrightarrow} P \)

\[
X \overset{\delta}{\longrightarrow} Y \overset{\pi}{\longrightarrow} P \overset{m}{\longrightarrow} M
\]

\((M, m \pi f) \in \delta \Psi \) \( \iff \) \((P, \pi) \in \delta \Psi \)

\((M, m \pi f) \in \Psi \) \( \iff \) \((P, \pi f) \in \Psi \)

by \( \Psi \) being closed

(i) \( \square \)
Let \( Q, \lambda \) be frames. We say that \( Q \) is an initial segment of \( \lambda \), \( Q \subseteq \lambda \), if \( Q \) is a full subcategory of \( \lambda \), and whenever \( U \in \text{Ob}(Q) \), \( K \in \text{Ob}(\lambda) \) and there is an arrow \( K \to U \) (in \( \lambda \)) then \( K \in \text{Ob}(Q) \).

More generally: the functor \( F: Q \to \lambda \) is an initial embedding, or simply initial, if \( F \) is one-to-one (injective) on objects, full and faithful, and whenever \( U \in \text{Ob}(Q) \), \( K \in \text{Ob}(\lambda) \) and there is an arrow \( K \to F(U) \) (in \( \lambda \)), then there is (a necessarily unique) \( V \in \text{Ob}(Q) \) such that \( K = F(V) \).

If \( F: Q \to \lambda \) is initial, then \( F \) can be uniquely factored as

\[
\begin{array}{ccc}
Q & \xrightarrow{F} & \lambda \\
I & \xrightarrow{\#} & Q'
\end{array}
\]
where $I$ is an isomorphism of categories, and $F'$ is the inclusion of an initial segment $\mathcal{Q}'$ of $\Lambda$ to $\Lambda$.

Suppose $F : Q \to \Lambda$ is an initial embedding, $P, M \in \Lambda$, $m : P \to M$. By restriction, we have

$$mF : PF \longrightarrow MF$$

$$\begin{array}{ccc}
\mathcal{Q} & \xrightarrow{\text{op}} & \Lambda^{\text{op}} \\
\xrightarrow{F^{\text{op}}} & & \xrightarrow{\downarrow m} \\
\xrightarrow{\text{Set}} & & \text{Set} \\
\end{array}$$

$$\begin{array}{ccc}
\mathcal{Q}^{\text{op}} & \xrightarrow{\text{op}} & \Lambda^{\text{op}} \\
\xrightarrow{PF} & & \xrightarrow{\downarrow mF} \\
\xrightarrow{\text{Set}} & & \text{Set} \\
\end{array}$$

**Claim:** If $m$ is f.s., then $mF$ is f.s.

This is essentially obvious — but here are some details: First of all, it suffices to consider the case when $F$ is the inclusion of an initial segment, since for an isomorphism of categories, the assertion is obviously true.
Suppose \( Q \subseteq \Lambda \), \( F : Q \rightarrow \Lambda \) the inclusion. Let \( U \in \text{ob}(Q) \), I write

\[
\hat{U}(Q) = \text{hom}_Q (-, U) : Q^{op} \rightarrow \text{Set}
\]

\[
\hat{U}(\Lambda) = \text{hom}_\Lambda (-, U) : \Lambda^{op} \rightarrow \text{Set}
\]

\( \hat{U}(Q) \) is the subfunctor \( \hat{U}^\circ \) of \( \hat{U}(Q) \)

\( \hat{U}(\Lambda) \) is the subfunctor \( \hat{U}^\circ \) of \( \hat{U}(\Lambda) \)

Then we have:

\[
\hat{U}(\Lambda)(K) = \begin{cases} 
\hat{U}(Q)(K) & \text{if } K \in Q \\
\emptyset & \text{if } K \in \Lambda - Q 
\end{cases}
\]

(note that if \( K \in \Lambda - Q \) and \( U \in Q \), then \( \text{hom}_\Lambda (K, U) = \emptyset \)

and

\[
\hat{U}(\Lambda)(K) = \begin{cases} 
\hat{U}(Q)(K) & \text{if } K \in Q \\
\emptyset & \text{if } K \in \Lambda - Q 
\end{cases}
\]

It follows that, for \( U \in \text{ob}(Q) \), the diagrams
\[ \hat{U}(a) \xrightarrow{\hat{\alpha}} MF \]
\[ \hat{U}(a) \xrightarrow{\hat{\alpha}} MF \]

are in a bijective correspondence with the diagrams
\[ U(a) \xrightarrow{p^*} P \]
\[ U(a) \xrightarrow{\hat{\alpha}^*} M \]

the bijection is effected by restriction from the second to the first, and likewise, the diagrams
\[ \hat{U}(a) \xrightarrow{\hat{\alpha}} MF \]
\[ \hat{U}(a) \xrightarrow{\hat{\alpha}} MF \]
are bijectively related to the diagrams

\[\begin{align*}
\text{diag:} & \quad 
\begin{array}{c}
\Delta^x & \rightarrow & P \\
i & \rightarrow & \# \\
\downarrow & & \downarrow m \\
\hat{\Delta}^x & \rightarrow & M \\
\end{array}
\end{align*}\]

The Claim follows.

6. The category of augmented frames (signature)

Let \( \Lambda_1 = (\Lambda_1, X_1) \), \( \Lambda_2 = (\Lambda_2, X_2) \) be augmented frames: \( \Lambda_1 \) is a frame (force-structure category), \( X_1: \Lambda_1^{\text{op}} \rightarrow \text{Set} \) is a finite functor, similarly for \( \Lambda_2 \).

A morphism \( F: \Lambda_1 \rightarrow \Lambda_2 \) is \( F = (F, \xi) \), where \( F: \Lambda_1^{\text{op}} \rightarrow \Lambda_2^{\text{op}} \) is an initial embedding, and \( \xi: X_1 \rightarrow X_2^F \):
We can compose these morphisms:

\[(A_1, X_1) \xrightarrow{(F_1, s_1)} (A_2, X_2) \xrightarrow{(F_2, s_2)} (A_3, X_3)\]

gives rise to

\[(A_1, X_1) \xrightarrow{(F_2 F_1, (s_2 F_1) s_1)} (A_3, X_3)\]

(Compare:

\[
\begin{align*}
\Lambda_1^{op} & \xrightarrow{F_1} \Lambda_2^{op} \\
\x_1 & \xrightarrow{s_1} \x_2 \\
F_2 & \downarrow s_2 \\
\Lambda_2^{op} & \xrightarrow{s_2} \Lambda_3^{op} \\
\x_2 & \xrightarrow{s_3} \x_3 \\
\end{align*}
\]

\[
\begin{align*}
\x_1 & \xrightarrow{s_1} \x_2 F_1 \\
\x_2 & \xrightarrow{s_2} \x_3 F_2 \\
(\x_2 F_1 & : \x_2 F_1 \longrightarrow \x_3 F_1 F_2) \\
\x_1 & \xrightarrow{s_1} \x_2 F_1 \\
\x_2 & \xrightarrow{s_2} \x_3 F_2 \\
\x_3 & \xrightarrow{s_3} \x_3 \\
\end{align*}
\]
This composition is associative, and has identities $(\Lambda, X) \longrightarrow (\Lambda, X)$:

$$(1_{\Lambda}, \text{id}_X)$$

We have the category $\text{Aug Fr}$ of augmented frames.

The morphism $(F, f) : (\Lambda_1, X_1) \longrightarrow (\Lambda_2, X_2)$ can be factored as the composite of "simpler" morphisms:

$$(\Lambda_1, X_1) \xrightarrow{(F, f')} \Lambda_2, X_2 \xrightarrow{(F, \text{id}_{X_2})}$$

$$(\text{id}_{\Lambda_1}, f) \quad \Lambda_1, X_2 \quad (\Lambda_1, X_2 \xrightarrow{f}$$

We will work with the full subcategory $\text{Aug Fr}_{\leq \aleph_0}$ of $\text{Aug Fr}$ whose objects $(\Lambda, X)$ have a countable frame: $\text{Ob} (\Lambda)$ is a countable (possibly finite) set.
7. Reducts and Coreducts

Let \( Q = (Q, \gamma), \; \Lambda = (\Lambda, \chi) \) be augmented frames, \( \mathcal{F} = (F, \xi) : Q \to \Lambda \) a morphism (see above). For an augmented structure \( M = (M, \alpha) \) over \( \Lambda \), the reduct \( M \mathcal{F} \) is the a.f. over \( Q \) defined by

\[
M \mathcal{F} = (MF, (\alpha F) \xi) \]

(Compare:

\[
\begin{array}{ccc}
Q^\text{op} & \xrightarrow{F} & \Lambda^\text{op} \\
\downarrow{X} & \searrow{\text{la}} & \downarrow{\text{Set}} \\
M & \xrightarrow{MF} & \text{Set}
\end{array}
\]

\[
\begin{array}{ccc}
Q^\text{op} & \xrightarrow{Y} & \text{Set} \\
\downarrow{XF} & \searrow{\text{laF}} & \downarrow{\text{Set}} \\
MF & \xrightarrow{MF} & \text{Set}
\end{array}
\]

Note: if \( (P, m, n, \gamma) : M \sim N \), then

\[
(P, mF, nF, (\gamma F)\xi) : M \mathcal{F} \sim N \mathcal{F}
\]
(Compare:

\[ \begin{array}{c}
M \xrightarrow{\alpha} X \xrightarrow{\beta} N \\
\uparrow \quad \uparrow \\
\# \quad \# \\
\downarrow \quad \downarrow \\
\# \quad \# \\
\end{array} \]

Remember that \( m, n \) are f.s.

\[ \Rightarrow mF, nF \text{ are f.s.} \]

Hence:

\[ M \sim N \quad \Rightarrow \quad M \downarrow F \sim N \downarrow F \]

With \( F : \mathcal{Q} \rightarrow \Lambda \) as before, let

\( \mathcal{Q} \) be a class over \( \mathcal{Q} \). The \text{\underline{co}}\text{\underline{re}}\text{\underline{duct}}

of \( \mathcal{Q} \) along \( F \), \( \mathcal{Q} \downarrow F \), is the class over

\( \Lambda \) for which

\[ M \in \mathcal{Q} \downarrow F \quad \Leftrightarrow \quad M \downarrow F \in \mathcal{Q} \]
It immediately follows that

\[ \overline{\phi} \text{ is closed } \Rightarrow \overline{\phi} F \text{ is closed} \]

8. Generalized GFOLDS

A *generalized* first-order logic with **dependent sorts**, a GFOLDS for short, is a collection of classes of augmented structures over countable augmented signatures, satisfying conditions given below. With \( \mathcal{L} \) the given GFOLDS, for any countable augmented signature \( \mathcal{L} \), I denote by \( \mathcal{L}_L \) the collection of augmented structures over \( \mathcal{L} \) in \( \mathcal{L} \). Thus, \( \mathcal{L} \) is the disjoint union of all \( \mathcal{L}_L \), where \( L \) ranging over countable augmented signatures, the empty class is counted as many times as there are \( L \)’s. The conditions are (1) (11) (4),
given below.

[For an augmented frame \((A, X)\), \( L(A, X) \) is the same as \( L(A^{op}, X) \).]
(1) Every class in \( L \) is closed under span equivalence. For any \( L \) (as above), and \( \Omega \in L \), \( M \in \Omega \) & \( M \equiv N \) imply that \( N \in \Omega \).

(2) \( L \) (any \( L \)) is a Boolean subalgebra of the collection of all classes over \( L \). That is, the class of all augmented structures over \( L \), belongs to \( L \); whenever \( \Omega, \psi \in L \), we have that \( \Omega \cap \psi, \Omega \cup \psi, \Omega^c - \Omega \) (complement) belong to \( L \) as well.

(3) \( L \) is closed under first-order FOLDS quantification. For any countable signature \( L \), \( X, Y \) finite \( L \)-contexts (finite functions \( L \rightarrow \text{Set} \)) and a monomorphism (1) \( \exists^L : X \rightarrow Y \),

\[
\Omega \in L(L, Y) \implies \forall^L \Omega \in L(L, X)
\]

(and \( \exists^L \Omega \in L(L, X) \))
(4) $L$ is closed under reduction. 

For every morphism $E : L \rightarrow \Lambda$ of countable augmented frames,

$$\varphi \in L \rightarrow \varphi 1E \in L \Lambda.$$ 

The classical model-theoretical conditions on $L$ (not necessarily holding for an arbitrary G-Forss $L$):

$C_c$: Countable compactness: Given countable augmented signature $\Sigma$ and classes $\varphi_n \in L_\Sigma$, $n \in \mathbb{N}$, then $\bigcap \varphi_n = \emptyset$ implies that there is $m \in \mathbb{N}$ such that $\bigcap \varphi_n = \emptyset$.

$C_{dls}$: Countable downward Löwenheim–Skolem property:

For $L$ and $\varphi_n \in L_\Sigma$, $n \in \mathbb{N}$ as above,

$\bigcap \varphi_n \neq \emptyset$ implies that there is a $M \in \bigcap \varphi_n$, $(M, \mathbb{A}) \in$ countable.
countable means that \( M : L \to \text{Set} \) is a countable functor: \( \prod_{K \in \mathcal{B}(L)} M(K) \) is a countable set.

**Thm.** Lindström for FOLDS.

For a GFOLDS \( L \),

\( L \) satisfies \( C_0 \) and \( C_{dL} \)

if and only if \( L \) is the minimal GFOLDS: for any GFOLDS \( L' \), \( L \subseteq L' \).

**NB:** For any collection \( L_0 \) of classes such that every \( E \) in \( L_0 \) is closed under span-equivalence, there is at least GFOLDS \( \langle L_0 \rangle \) generated by \( L_0 \). The conditions 2), 3) and 4) are closure conditions; \( \langle L_0 \rangle \) is the closure of \( L_0 \) under these. \( \langle L_0 \rangle \) also satisfies the main FOLDS condition 1), since each of the closure steps 2), 3), 4)
generate closed classes from closed classes. For 2), this is trivial.
For 3), this is proved in §4, "Quantification";
for 4), in §7, "Reducts and Conduits".
The minimal GFOLDS in the collection
\[
L_{\emptyset} \overset{\text{def}}{=} \langle \emptyset \rangle \text{ generated by the empty collection } L_0 = \emptyset.
\]
The statement as well as the proof (to be given in Part 2 of these notes) of
the Lindström theorem for FOLDS are
given without any use of traditional
Syntax (formulas and the like), and
even without categorical-logical formulations
of syntax versus semantics. We can make
the connections to traditional easily, however.
The classes in \(L_{\emptyset}\), the minimal GFOLDS,
are exactly the classes
\[
\text{Mod}_L [X : \emptyset]
\]
Where:

- \( L \) is a countable \textit{FOLDS} signature;
- \( X \) is an \( L \)-context, construed as a finite context of typed variables, with the dependent typing rules given by \( L \); [limiting]
- \( \varphi \) is a \textit{FOLDS} formula over \( L \) such that the free variables of \( \varphi \) are all in \( X \);
- \( \text{Mod}_L [X : \varphi] \) is the class of all augmented structures \((M, \alpha)\), \( M : L \to \text{Set}, \alpha : X \to M \) for which \( M = \varphi [\alpha] \) "\( M \) satisfies \( \varphi \) at \( \alpha \)."