FOLDS, Lindström's theorem, and back-and-forth properties in infinitary logic. Part 1.

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Notes

Contents:

Feb 05/014

1

1. Face-structure categories (= "frames", for short)

Let: Λ : small category;

$K, K', \ldots$: objects of Λ ("kinds");

proper arrow: any non-identity arrow.

→ $\dim(K) \overset{\text{def}}{=}$ largest $n \in \mathbb{N}$ such that

there exists a composable string of length n

of proper arrows ending in K:

$$K = K_0 \xleftarrow[\neq id]{p_0} K_1 \leftarrow \cdots \leftarrow K_{n-1} \xleftarrow[\neq id]{p_{n-1}} K_n$$

if such largest n exists;

$\dim(K) = \infty$ otherwise.

→ A functor $X : \Lambda^{op} \longrightarrow \mathrm{Set}$ is finite if

$el(X)$, the category of elements of X is

a finite category (having finitely many arrows (and objects)).

NB: $el(X)$: objects: (K, x) :: $K \in Ob(L)$
$x \in X(K)$

arrows: $(K, x) \xrightarrow{f} (K', x')$

$$\overline{\quad K' \xrightarrow{f} K \qquad X(K) \xrightarrow{X(f)} X(K') \quad}$$ (contravariance)

$$x \longmapsto x' = X(f)(x)$$

we have:

$$\begin{tikzcd} el(X) \arrow[d, "\pi"] & (K,x) \arrow[d, mapsto] \\ \Lambda^{op} & K \end{tikzcd}$$

→

Λ is face-structure category; or frame

if, for every $K \in Ob(\Lambda)$,

1) $\dim(K)$ is finite;

and 2) $\hat{K} \underset{def}{=} \hom_{\Lambda}(-,K) : \Lambda^{op} \to Set$ is a finite functor.

Λ: always denotes a frame.

Properties of a frame Λ:

- $End(K) = \hom_{\Lambda}(K,K) = \{1_K\}$.
- $Ob(\Lambda) = \dot{\bigcup}_{n \in \mathbb{N}} \Lambda_n$, where

  $K \in \Lambda_n \iff \dim(K) = n$.
- $K' \xrightarrow{f} K$ implies that $\dim(K') \leq \dim(K)$

and $K' \xrightarrow{p} K$ proper implies

$$\dim(K') < \dim(K).$$

- $K \in \Lambda_0 \iff$ there is no proper $p : K' \to K$.

- for $n > 0$, $K \in \Lambda_n \iff$

  for all proper $p : K' \to K$, $K' \in \Lambda_{\leq n-1} = \bigcup_{k \leq n-1} \Lambda_k$

  and there is $p : K' \to K$ with $K' \in \Lambda_{n-1}$.

- $\hom_\Lambda(K', K)$ is finite.

- therefore, $X : \Lambda^{op} \longrightarrow Set$ is a finite functor iff

  $$\coprod_{K \in Ob(\Lambda)} X(K) \quad (= Ob(el(X))$$

  is a finite set.

Denote by $\hat{\Lambda}$ the presheaf category

$$\hat{\Lambda} = \mathrm{Set}^{\Lambda^{op}},$$

and by

$$\mathrm{Context}[\Lambda] \quad , \text{ or } \quad C[\Lambda],$$

the full subcategory of $\hat{\Lambda}$ on the objects the finite functors.
An object of $C[\Lambda]$ is called a context (or: "context of variables"...).

By condition 2) of the definition of "frame", the Yoneda functor $\Lambda \xrightarrow{Y} \hat{\Lambda}$ factors through $C[\Lambda]$: we have $y: \Lambda \longrightarrow C[\Lambda]$

$$\begin{tikzcd}
\Lambda \arrow[rr, "Y"] \arrow[dr, "y"'] & \arrow[d, phantom, "\#"] & \hat{\Lambda} \\
& C[\Lambda] \arrow[ur, "\text{inclusion}"'] &
\end{tikzcd}$$

$$\begin{cases} L \text{ will denote } \Lambda^{op}; \ L \text{ is called a signature} \\ L = \Lambda^{op} \\ \Lambda = L^{op} \end{cases}$$

$X, Y, Z, \dots$: denote (usually) ($\mathcal{L}$-) contexts;

(finite functors $\mathcal{L}^{op} \longrightarrow \mathrm{Set}$);

$M, N, P, \dots$: arbitrary functors

$$\mathcal{L}^{op} \longrightarrow \mathrm{Set}.$$

A functor $\begin{cases} M : L \to \mathrm{Set} \\ M : \mathcal{L}^{op} \to \mathrm{Set} \end{cases}$

is also called an L- <u>structure</u>, and also: $\mathcal{L}$-<u>set</u>.

<u>Particular</u> contexts:

$\hat{K} = \hom_{\mathcal{L}}(-, K)$, the K-<u>ball</u>;

$\overset{\circ}{K}$, the K-<u>sphere</u>:

subfunctor of $\hat{K}$:

$$\overset{\circ}{K}(U) = \begin{cases} \hat{K}(U) & \text{when } U \neq K \\ \emptyset & \text{when } U = K \end{cases}$$

$(U \in \mathrm{Ob}(\mathcal{L}))$

$$\overset{\circ}{K} \xrightarrow[i_K]{} \hat{K} \qquad \text{inclusion}$$

2. L-equivalence (= span equivalence)

We are in the category $\hat{\Lambda} = \mathrm{Set}^{\Lambda^{op}}$ of Λ-sets; M, N, P, X, Y, Z objects of $\hat{\Lambda}$; X, Y, Z usually finite functors.

→ A map $P \xrightarrow{m} M$ (natural transformation) is called *fiberwise* *surjective* (f.s.) (or: a "trivial fibration" ...) if it has the right lifting property with respect to the sphere inclusions: for all $K \in \mathrm{Ob}(\Lambda)$,

whenever

$$\begin{tikzcd} \mathring{K} \arrow[r, "\beta"] \arrow[d, "i_K"'] \arrow[dr, phantom, "\#"] & P \arrow[d, "m"] \\ \hat{K} \arrow[r, "\hat{a}"'] & M \end{tikzcd}$$

$\#$: 'commutes'

there exists b such that

$$\begin{tikzcd} \mathring{K} \arrow[r, "\beta"] \arrow[d, "i_K"'] \arrow[dr, phantom, "\#" near start] & P \arrow[d, "m"] \\ \hat{K} \arrow[r, "\hat{a}"'] \arrow[ur, dashed, "b"'] \arrow[ur, phantom, ""] & M \arrow[ul, phantom, "\#" near start] \end{tikzcd}$$

.

If $P \xrightarrow{m} M$ is f.s., it has the right lifting property w.r. to all monomorphisms $X \xrightarrow{f} Y$ (X, Y not necessarily finite functors) — because every monomorphism is a transfinite composite of pushouts of sphere inclusions.

→ For L-structures ($= \Lambda$-sets) M and N, a _span equivalence_ of M and N is a span (P, m, n),

$$\begin{tikzcd} & P \arrow[dl, "m"'] \arrow[dr, "n"] & \\ M & & N \end{tikzcd}$$

where both m and n are f.s.

We write

$$(P, m, n) : M \sim N.$$

Writing "$M \sim N$" (or "$M \underset{L}{\sim} N$")

means that there is $(P, m, n) : M \sim N$;

we say "M and N are span-equivalent".

Indeed: $M \sim N$ is an equivalence relation.

Transitivity: suppose

$$(P, m, n) : M \sim N$$

$$(R, n', d) : N \sim D$$

Construct pullback S:

$$\begin{tikzcd} & & S \arrow[dl, "p"'] \arrow[dr, "r"] \arrow[dd, phantom, "\vee" near start] & & \\ & P \arrow[dl, "m"'] \arrow[dr, "n"] & & R \arrow[dl, "n'"'] \arrow[dr, "d"] & \\ M & & N & & D \end{tikzcd}$$

One proves the lemmas that a pullback of an f.s. map is f.s., and the composite of two f.s. maps is f.s. — from which

$$(S, mp, dr) : M \sim D$$

3. Augmented structures and classes of such

Let : L : signature

$(L = \Lambda^{op}, \ \Lambda$: frame$)$

X : L-context

(finite functor $\Lambda^{op} \longrightarrow Set$)

→ The pair (L, X) is an *augmented signature*.

→ An *augmented structure* over (L, X) is a pair (M, α), where

$M \in \hat{\Lambda}$ (= category of L-structures)

and $\alpha : X \longrightarrow M$

$$\begin{tikzcd} (\ \Lambda^{op} \arrow[r, shift left=2, "X"] \arrow[r, shift right=2, "M"'] \arrow[r, phantom, "\downarrow\alpha"] & Set \ ) \end{tikzcd}$$

→ A $\boxed{\text{class}}$ over (L, X) is *any* class of augmented structures over (L, X).

We also say : (L, X)-class.

→ A <u>span equivalence</u> of augmented structures (M, α), (N, β) over the same augmented signature (L, X) is a quadruple (P, m, n, γ) such that

$$(P, m, n) : M \sim N$$

$$\gamma : X \longrightarrow P$$

and $m\gamma = \alpha$, $n\gamma = \beta$:

$$\begin{tikzcd}
\boxed{\text{f.s.}} & P \arrow[dl, "m"'] \arrow[dr, "n"] & \\
M \arrow[r, phantom, "\#"] & {} \arrow[r, phantom, "\#"] & N \\
& X \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, "\gamma"] &
\end{tikzcd}$$

We write: $(P, m, n, \gamma) : (M, \alpha) \sim (N, \beta)$;

and if such (P, m, n, γ) exists, we say that (M, α) and (N, β) are span equivalent, and write $(M, \alpha) \sim (N, \beta)$.

Span equivalence of augmented structures is an equivalence relation:

Transitivity:

assume: $(P, m, n, \gamma) : (M, \alpha) \sim (N, \beta)$

$(R, n', d, \rho) : (N, \beta) \sim (D, \delta)$

Construct S, p, r as before; the following commutes:

$$\begin{tikzcd}
 & & S \arrow[dl, "p"'] \arrow[dr, "r"] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & P \arrow[dl, "m"'] \arrow[dr, "n"] & & R \arrow[dl, "n'"'] \arrow[dr, "d"] & \\
M & & N & & D \\
 & & X \arrow[llu, "\alpha"] \arrow[luu, "\gamma"] \arrow[u, "\beta"'] \arrow[ruu, "\rho"'] \arrow[rru, "\delta"'] & &
\end{tikzcd}$$

Define $X \xrightarrow{\sigma} S$ by the universal property of S such that $\gamma = p\sigma$, $\rho = r\sigma$:

$$\begin{tikzcd}
 & & S \arrow[dll, "p"'] \arrow[drr, "r"] & & \\
P & \# & & \# & R \\
 & & X \arrow[ull, "\gamma"] \arrow[urr, "\rho"'] \arrow[uu, "\sigma"] & &
\end{tikzcd}$$

this is possible since $n\gamma = \rho n' = \rho$.

We have

$(S, mp, dr, \sigma) : (M, \alpha) \sim (D, \delta)$:

$$\begin{tikzcd}
 & & S \arrow[dll, "mp"'] \arrow[drr, "dr"] & & \\
M & \# & & \# & D \\
 & & X \arrow[ull, "\alpha"] \arrow[urr, "\delta"'] \arrow[uu, "\sigma"'] & &
\end{tikzcd}$$

— · —

→ An (L, X)-class Φ is *closed under span-equivalence* if

$\underline{M} \in \underline{\Phi}$ & $\underline{M} \sim \underline{N}$ imply

$\left(\overset{\shortparallel}{(M, \alpha)} \right)$ that $\underline{N} \in \underline{\Phi}$.

An (L, X)-class $\underline{\Phi}$ that is closed under span-equivalence may be called a <u>generalized</u> (L, X)-<u>formula</u>.

NB: $\underline{\Phi}$ is closed under span-equivalence iff the following holds:

every time

$$X \xrightarrow[\gamma]{} P \xrightarrow{m} M$$

and m is f.s., then

$$(M, m\gamma) \in \underline{\Phi}$$

$$\iff (P, \gamma) \in \underline{\Phi}.$$

4. Quantification

Let X, Y be arbitrary functors $\Lambda^{op} \to Set$ (possibly "infinite contexts", although we are interested mainly in the case when X, Y are <u>contexts</u>; finite functors), and let

$$\xi : X \longrightarrow Y,$$

any natural transformation (eventually: a monomorphism). We make three definitions. As usual, $\boxed{L = \Lambda^{op}}$.

→ 1. For a class Ψ over (L, X), the class $\xi^{*}\Psi$ over (L, Y) is defined thus:

$$(M, \mu) \in \xi^{*}\Psi \underset{def}{\Longleftrightarrow} (M, \mu\xi) \in \Psi$$

$\mu : Y \to M$

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[rr, bend right, "\mu\xi"'] & Y \arrow[r, "\mu"] & M \end{tikzcd}$$

→ 2. For a class Φ over (L, Y), the class $\forall_{\xi}\Phi$ over (L, X) is defined thus:

$(M, \alpha) \in \forall_{\xi} \underline{\Phi}$

$\alpha : X \to M$

$$\underset{\text{def}}{\Longleftrightarrow}$$

<u>for all</u> $\mu : Y \to M$,

$\mu\xi = \alpha$ implies $(M, \mu) \in \underline{\Phi}$.

→ 3. For a class $\underline{\Phi}$ over (L, Y)

$\boxed{\exists_{\xi} \underline{\Phi}}$ is over (L, X), and

$(M, \alpha) \in \forall_{\xi} \underline{\Phi} \overset{\text{def}}{\Longleftrightarrow}$ <u>there is</u> $\mu : Y \to M$

such that $\mu\xi = \alpha$ and $(M, \mu) \in \underline{\Phi}$

<u>Picture</u>:

$$\begin{tikzcd}
(M, \alpha) & & (M, \mu) \\
\underline{\Psi} \arrow[rr, mapsto] & & \xi^{*}\underline{\Psi} \\
\forall_{\xi} \underline{\Phi}, \exists_{\xi} \underline{\Phi} & & \underline{\Phi} \arrow[ll, mapsto] \\
X \arrow[rr, "\xi"'] & & Y
\end{tikzcd}$$

It is easy to see that we have the following adjunctions:

$$\frac{\underline{\Psi} \quad \subseteq \quad \forall_{\xi}\ \underline{\Phi}}{\xi^{*}\underline{\Psi} \quad \subseteq \quad \underline{\Phi}}$$ (if and only if)

and

$$\frac{\exists_{\xi}\ \underline{\Phi} \quad \subseteq \quad \underline{\Psi}}{\underline{\Phi} \quad \subseteq \quad \xi^{*}\underline{\Psi}}$$

For brevity, let us say that a class is <u>closed</u> if it is closed under span equivalence; see pp $\boxed{12}$, $\boxed{13}$.

Continuing the above notation, we have:

<u>Proposition</u> (i) $\underline{\Psi}$ is closed $\Rightarrow \xi^{*}\underline{\Psi}$ is closed.

(ii) If $X \xrightarrow{\xi} Y$ is a monomorphism, then

$\underline{\Phi}$ is closed $\Rightarrow \forall_{\xi}\underline{\Phi}$, $\exists_{\xi}\underline{\Phi}$ are closed.

proof of (ii) for $\forall$

Proof of Proposition: **assume**: ξ is mono, $\overline{\Phi}$ is closed.

$$\boxed{\begin{tikzcd} \forall_{\xi}\overline{\Phi} & & \overline{\Phi} \\ X \arrow[rr, "\xi"'] & & Y \end{tikzcd}}$$

$$\begin{tikzcd} P \arrow[d, "m"] \\ M \end{tikzcd} \quad \text{f.s.}$$

let: $X \xrightarrow{\gamma} P$.

want: $(M, m\gamma) \in \forall_{\xi} \overline{\Phi} \overset{?}{\Longleftrightarrow} (P, \gamma) \in \forall_{\xi} \overline{\Phi}$

$$\Big\Updownarrow \text{def} \qquad\qquad\qquad \Big\Updownarrow \text{def}$$

$$\left.\begin{array}{l} \text{for all } Y \xrightarrow{\mu} M \\ \begin{tikzcd} X \arrow[r, "\xi"] \arrow[rr, bend right] & Y \arrow[r, "\mu"] & M \end{tikzcd} \\ \mu\xi = m\gamma \\ \underset{1}{\Rightarrow} (M, \mu) \in \overline{\Phi} \end{array}\right\} \begin{array}{c} \overset{?}{\Rightarrow} \\ \\ \underset{?}{\Leftarrow} \end{array} \left\{\begin{array}{l} \text{for all } Y \xrightarrow{\pi} P \\ \begin{tikzcd} X \arrow[r, "\xi"] \arrow[rr, bend right] & Y \arrow[r, "\pi"] & P \end{tikzcd} \\ \pi\xi = \gamma \\ \Rightarrow (P, \pi) \in \overline{\Phi} \end{array}\right.$$

1) (the trivial direction) $\overset{?}{\Rightarrow}$: assume left-hand side.

let $Y \xrightarrow{\pi} P$ s.t. $\pi\xi = \gamma$, to show $(P, \pi) \overset{?}{\in} \overline{\Phi}$

define: $\mu \underset{\text{def}}{=} m\pi$; we have $\mu\xi \overset{?}{=} m\gamma$

$$\underset{m\pi\xi}{\overset{\mu\xi}{\|}} \quad \checkmark \; /\!/$$

Therefore, by assumption (see $\underset{1}{\Rightarrow}$)

$(M, \mu) \in \Phi$. We have

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[rr, bend right, "\mu = m\pi"'] & P \arrow[r, two heads, "m"] & M \end{tikzcd}$$

Since Φ is closed, $(M, m\pi) \in \Phi$ $\overset{\checkmark}{\Rightarrow}$ $(P, \pi) \underset{\checkmark}{\in} \Phi$.

2) $\underset{?}{\Leftarrow}$: assume right-hand side.

let: $Y \xrightarrow{\mu} M$ s.t. $\mu\xi = m\gamma$:

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\#"] & P \arrow[d, "m"] \\ Y \arrow[r, "\mu"'] & M \end{tikzcd}$$

, to show $(M, \mu) \overset{?}{\in} \Phi$

By definition of 'm is f.s.', ξ mono: there is diagonal π:

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\xi"'] & P \arrow[d, "m"] \\ Y \arrow[r, "\mu"'] \arrow[ur, dashed, "\pi"' near end, "\#" near start, "\#"' near end] & M \end{tikzcd}$$

: $\boxed{\pi\xi = \gamma}$ & $\boxed{m\pi = \mu}$

By assumption & $\pi\xi = \gamma$: $(P, \pi) \in \underline{\Phi}$.

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[rr, bend right, "\mu = m\pi"'] & P \arrow[r, "m"] & M \end{tikzcd}$$

Since $\underline{\Phi}$ is closed, $(P, \pi) \underset{\checkmark}{\in \underline{\Phi}} \Rightarrow (M, \mu) \underset{\checkmark\checkmark}{\in \underline{\Phi}}$.

$\boxed{\text{proof of (ii) for } \forall : \square}$

— · —

proof of (ii) for $\exists$: dual

— · —

proof of (i) : assume $\underline{\Psi}$ is closed.

let : $Y \xrightarrow{\pi} P$

$$X \xrightarrow{\xi} Y \xrightarrow{\pi} P \xrightarrow{m} M$$

$$\begin{tikzcd} (M, m\pi) \in \xi^* \underline{\Psi} \arrow[r, Leftrightarrow, "?"] \arrow[d, Leftrightarrow] & (P, \pi) \in \xi^* \underline{\Psi} \arrow[d, Leftrightarrow] \\ (M, m\pi\xi) \in \underline{\Psi} \arrow[r, Leftarrow] & (P, \pi\xi) \in \underline{\Psi} \end{tikzcd}$$

↑

by $\underline{\Psi}$ being closed

$\boxed{\text{(i) } \square}$

5. Initial embeddings of frames

Let Q, Λ be frames. We say that Q is an <u>initial segment</u> of Λ, $Q \leq \Lambda$, if Q is a full subcategory of Λ, and whenever $U \in Ob(Q)$, $K \in Ob(\Lambda)$ and there is an arrow $K \to U$ (in Λ) then $K \in Ob(Q)$.

More generally: the functor $F: Q \to \Lambda$ is an <u>initial embedding</u>, or simply <u>initial</u>, if F is one-to-one (injective) on objects, full and faithful, and whenever $U \in Ob(Q)$, $K \in Ob(\Lambda)$ and there is an arrow $K \to F(U)$ (in Λ), then there is (a necessarily unique) $V \in Ob(Q)$ such that $K = F(V)$.

If $F: Q \to \Lambda$ is initial, then F can be uniquely factored as

$$\begin{tikzcd}
Q \arrow[rr, "F"] \arrow[dr, "I"'] & \# & \Lambda \\
& Q' \arrow[ur, "F'"'] &
\end{tikzcd}$$

where I is an isomorphism of categories, and F' is the inclusion of an initial segment Q' of Λ to Λ.

Suppose $F: Q \to \Lambda$ is an initial embedding, $P, M \in \hat{\Lambda}$, $m: P \to M$. By restriction, we have

$$mF : PF \longrightarrow MF$$

$$\begin{tikzcd} Q^{op} \arrow[r, "F^{op}"] & \Lambda^{op} \arrow[r, bend left, "P", ""{name=U, below}] \arrow[r, bend right, "M"', ""{name=D, above}] & Set \arrow[Rightarrow, from=U, to=D, "m"] \end{tikzcd}$$

$$\begin{tikzcd} Q^{op} \arrow[r, bend left, "PF", ""{name=U, below}] \arrow[r, bend right, "MF"', ""{name=D, above}] & Set \arrow[Rightarrow, from=U, to=D, "mF"] \end{tikzcd}$$

Claim: If m is f.s., then mF is f.s.

This is essentially obvious – but here are some details: First of all, it suffices to consider the case when F is the inclusion of an initial segment, since for an isomorphism of categories, the assertion is obviously true.

Suppose $Q \leq \Lambda$, $F : Q \to \Lambda$ the inclusion. Let $U \in Ob(Q)$. I write

$$\hat{U}^{(Q)} = \hom_Q(-, U) : Q^{op} \to Set$$

$$\hat{U}^{(\Lambda)} = \hom_\Lambda(-, U) : \Lambda^{op} \to Set$$

$$\mathring{U}^{(Q)} \doteq \text{the subfunctor } \mathring{U} \text{ of } \hat{U}^{(Q)}$$

$$\mathring{U}^{(\Lambda)} \doteq \text{the subfunctor } \mathring{U} \text{ of } \hat{U}^{(\Lambda)}$$

Then we have:

$$\hat{U}^{(\Lambda)}(K) = \begin{cases} \hat{U}^{(Q)}(K) & \text{if } K \in Q \\ \emptyset & \text{if } K \in \Lambda - Q \end{cases}$$

(note that if $K \in \Lambda - Q$ and $U \in Q$, then $\hom_\Lambda(K, U) = \emptyset$)

and

$$\mathring{U}^{(\Lambda)}(K) = \begin{cases} \mathring{U}^{(Q)}(K) & \text{if } K \in Q \\ \emptyset & \text{if } K \in \Lambda - Q \end{cases}$$

It follows that, $\boxed{\text{for } \boxed{U \in Ob(Q),}}$

the diagrams

$$\begin{tikzcd}
\mathring{U}^{(Q)} \arrow[r, "\beta"] \arrow[d, "i_U^{(Q)}"'] \arrow[dr, phantom, "\#"] & PF \arrow[d, "mF"] \\
\hat{U}^{(Q)} \arrow[r, "\hat{a}"'] & MF
\end{tikzcd} \quad \text{in } \widehat{Q}$$

are in a bijective correspondence with the diagrams

$$\begin{tikzcd}
\mathring{U}^{(\Omega)} \arrow[r, "\beta^*"] \arrow[d, "i_U^{(\Omega)}"'] \arrow[dr, phantom, "\#"] & P \arrow[d, "m"] \\
\hat{U}^{(\Omega)} \arrow[r, "\hat{a}^*"'] & M
\end{tikzcd} \quad \text{in } \widehat{\Omega},$$

the bijection effected by restriction from the second to the first, and likewise,

the diagrams

$$\begin{tikzcd}
\mathring{U}^{(Q)} \arrow[r, "\beta"] \arrow[d, "i"'] & PF \arrow[d, "mF"] \\
\hat{U}^{(Q)} \arrow[r, "\hat{a}"'] \arrow[ur, "b"] & MF
\end{tikzcd} \quad \# \quad \#$$

are bijectively related to the diagrams

$$\begin{tikzcd}
\mathring{U}^{(\Lambda)} \arrow[r, "\beta^*"] \arrow[d, "i"'] \arrow[dr, phantom, "\#" near start] & P \arrow[d, "m"] \\
\hat{U}^{(\Lambda)} \arrow[r, "\hat{a}^*"'] \arrow[ur, "b^*"'] \arrow[ur, phantom, "\#" very near end, shift right=4] & M
\end{tikzcd}$$

The Claim follows.

6. The category of augmented frames (signatures)

Let $\underline{\Lambda}_1 = (\Lambda_1, X_1)$, $\underline{\Lambda}_2 = (\Lambda_2, X_2)$ be augmented frames: Λ_1 is a frame (face-structure category), $X_1 : \Lambda_1^{op} \to Set$ is a finite functor; similarly for $\underline{\Lambda}_2$.

A *morphism* $\underline{F} : \underline{\Lambda}_1 \to \underline{\Lambda}_2$ is $\underline{F} = (F, \xi)$, where $F : \Lambda_1 \to \Lambda_2$ is an initial embedding, and $\xi : X_1 \to X_2 F$:

$$\begin{tikzcd}
\Lambda_1^{op} \arrow[dr, "X_1"] \arrow[dd, "F^{op}"'] & \\
{} \arrow[r, phantom, "\Downarrow \xi"] & Set \\
\Lambda_2^{op} \arrow[ur, "X_2"'] &
\end{tikzcd}$$

We can compose these morphisms:

$$\begin{tikzcd}
(\Lambda_1, X_1) \arrow[r, "{(F_1, \xi_1)}"'] & (\Lambda_2, X_2) \arrow[r, "{(F_2, \xi_2)}"'] & (\Lambda_3, X_3)
\end{tikzcd}$$

gives rise to

$$\begin{tikzcd}
(\Lambda_1, X_1) \arrow[rr, "{(F_2 F_1 , \underbrace{(\xi_2 F_1)\xi_1}_{\xi_3})}"'] & & (\Lambda_3, X_3)
\end{tikzcd}$$

(Compare:

$$\begin{tikzcd}
\Lambda_1^{op} \arrow[d, "F_1"'] \arrow[drr, "X_1"] & & \\
\Lambda_2^{op} \arrow[d, "F_2"'] \arrow[rr, "X_2"] & \Uparrow \xi_1 & Set \\
\Lambda_3^{op} \arrow[urr, "X_3"'] & \Downarrow \xi_2 &
\end{tikzcd}$$

$\xi_1 : X_1 \longrightarrow X_2 F_1$

$\xi_2 : X_2 \longrightarrow X_3 F_2$

$(\xi_2 F_1 : X_2 F_1 \longrightarrow X_3 F_1 F_2)$

$$\begin{tikzcd}
X_1 \arrow[r, "\xi_1"] \arrow[rr, bend right, "\xi_3"'] & X_2 F_1 \arrow[r, "\xi_2 F_1"] & X_3 F_1 F_2
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda_1^{op} \arrow[dd, "F_2 F_1"'] \arrow[dr, "X_1"] & \\
\Downarrow \xi_3 & Set \\
\Lambda_3 \arrow[ur, "X_3"'] &
\end{tikzcd}$$

)

This composition is associative, and has identities $(\Lambda, X) \xrightarrow[(\mathrm{Id}_{\Lambda},\, \mathrm{id}_X)]{} (\Lambda, X)$.

We have the category $\underline{\mathrm{Aug}\,\mathrm{Fr}}$ of augmented frames.

The morphism $(F, f) : (\Lambda_1, X_1) \longrightarrow (\Lambda_2, X_2)$ can be factored as the composite of "simpler" morphisms:

$$\begin{tikzcd}
(\Lambda_1, X_1) \arrow[rr, "{(F,\, f)}"] \arrow[dr, "{(\mathrm{Id}_{\Lambda_1},\, f)}"'] & & (\Lambda_2, X_2) \\
& (\Lambda_1, X_2 F) \arrow[ur, "{(F,\, \mathrm{id}_{X_2 F_1})}"'] &
\end{tikzcd}$$

We will work with the full subcategory $\underline{\mathrm{Aug}\,\mathrm{Fr}}_{\leq \aleph_0}$ of $\underline{\mathrm{Aug}\,\mathrm{Fr}}$ whose objects (Λ, X) have Λ a <u>countable</u> frame: $\mathrm{Ob}(\Lambda)$ is a countable (possibly finite) set.

$\boxed{\text{7. Reducts and coreducts}}$

Let $\underline{Q} = (Q, Y)$, $\underline{\Lambda} = (\Lambda, X)$

be augmented frames, $\underline{F} = (F, \xi) : \underline{Q} \longrightarrow \underline{\Lambda}$

a morphism (see above). For an

augmented structure $\underline{M} = (M, \alpha)$ over $\underline{\Lambda}$,

→ the reduct $\underline{M} \upharpoonright \underline{F}$ is the a.s. over $\underline{Q}$

defined by

$$\underline{M} \upharpoonright \underline{F} = (MF, (\alpha F)\xi)$$

(compare:

$$\begin{tikzcd} Q^{op} \arrow[r, "F"] & \Lambda^{op} \arrow[r, bend left=40, "X"] \arrow[r, bend right=40, "M"'] \arrow[r, phantom, "\downarrow\alpha"] & \mathrm{Set} \end{tikzcd}$$

$$\begin{tikzcd} Q^{op} \arrow[r, bend left=60, "Y"] \arrow[r, "XF" description] \arrow[r, bend right=60, "MF"'] \arrow[r, phantom, shift left=4, "\downarrow\xi"] \arrow[r, phantom, shift right=4, "\downarrow\alpha F"] & \mathrm{Set} \end{tikzcd}$$

)

Note: if $(P, m, n, \gamma) : \underline{M} \sim \underline{N}$

then $(P, mF, nF, (\gamma F)\xi) : \underline{M} \upharpoonright \underline{F} \sim \underline{N} \upharpoonright \underline{F}$

(Compare:

$$\begin{tikzcd}
& P \arrow[dl, "m"'] \arrow[dr, "n"] & \\
M \arrow[r, phantom, "\#"] & \arrow[r, phantom, "\#"] & N \\
& X \arrow[ul, "\alpha"] \arrow[uu, "\gamma"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& PF \arrow[dl, "mF"'] \arrow[dr, "nF"] & \\
MF \arrow[r, phantom, "\#"] & \arrow[r, phantom, "\#"] & NF \\
\arrow[r, phantom, "\#"] & XF \arrow[ul, "\alpha F"] \arrow[uu, "\gamma F"] \arrow[ur, "\beta F"'] \arrow[r, phantom, "\#"] & {} \\
& Y \arrow[u, "\xi"'] \arrow[uul, bend left, "(\alpha F)_\xi"] \arrow[uur, bend right, "(\beta F)_\xi"'] &
\end{tikzcd}$$

)

remember that m, n are f.s.

$\Rightarrow mF, nF$ are f.s.

Hence:

$$\underline{M} \underset{\underline{\Lambda}}{\sim} \underline{N} \quad \Rightarrow \quad \underline{M} \restriction \underline{F} \underset{\underline{Q}}{\sim} \underline{N} \restriction \underline{F}$$

— . —

With $\underline{F} : \underline{Q} \to \underline{\Lambda}$ as before, let $\underline{\Phi}$ be a class over $\underline{Q}$. The <u>coreduct</u> of $\underline{Q}$ along F, $\underline{\Phi} 1 F$, is the class over $\underline{\Lambda}$ for which

$$\underline{M} \in \underline{\Phi} 1 \underline{F} \underset{\text{def}}{\Longleftrightarrow} \underline{M} \restriction \underline{F} \in \underline{\Phi}$$

It immediately follows that

$$\underline{\Phi} \text{ is closed } \Rightarrow \underline{\Phi} \upharpoonright \underline{F} \text{ is closed}$$

8. Generalized FOLDS

A *generalized first-order logic with dependent sorts*, a GFOLDS for short, is a collection of classes of augmented structures over countable augmented signatures, satisfying conditions given below. With $\mathcal{L}$ the given GFOLDS, for any countable augmented signature $\underline{L}$, I denote by $\mathcal{L}_{\underline{L}}$ the collection of augmented structures over $\underline{L}$ in $\mathcal{L}$. Thus, $\mathcal{L}$ is the disjoint union of all $\mathcal{L}_{\underline{L}}$, with $\underline{L}$ ranging over countable augmented signatures – the empty class is counted as many times as there are $\underline{L}$'s. The conditions are 1) to 4) given below.

[For an augmented frame $(\mathcal{A}, X)$, $\mathcal{L}_{(\mathcal{A}, X)}$ is the same as $\mathcal{L}_{(\mathcal{A}^{op}, X)}$].

① Every class in $\mathcal{L}$ is closed under span equivalence. For any $\underline{L}$ (as above), and $\Phi \in \mathcal{L}_{\underline{L}}$, $\underline{M} \in \Phi$ & $\underline{M} \underset{\underline{L}}{\sim} \underline{N}$ imply that $\underline{N} \in \Phi$.

② $\mathcal{L}_{\underline{L}}$ (any $\underline{L}$) is a Boolean subalgebra of the collection of all classes over $\underline{L}$. That is,

$\underset{\sim}{T}_{\underline{L}}$, the class of all augmented structures over $\underline{L}$, belongs to $\mathcal{L}_{\underline{L}}$;

whenever $\Phi, \Psi \in \mathcal{L}_{\underline{L}}$, we have that $\Phi \cap \Psi$, $\Phi \cup \Psi$, $\underset{\sim}{T}_{\underline{L}} - \Phi$ (complement) belong to $\mathcal{L}_{\underline{L}}$ as well

③ $\mathcal{L}$ is closed under first-order FOLDS quantification. For any countable signature L, X, Y *finite* L-contexts (finite functors L→Set) and a *monomorphism* (!) $\xi : X \to Y$;

$$\Phi \in \mathcal{L}_{(L,Y)} \implies \forall_{\xi} \Phi \in \mathcal{L}_{(L,X)}$$

$$\left(\text{and } \exists_{\xi} \Phi \in \mathcal{L}_{(L,X)}\right)$$

④ $\mathcal{L}$ is closed under coreduction.

For every morphism $\underline{F} : \underline{Q} \to \underline{\Lambda}$ of countable augmented frames,

$$\underline{\Phi} \in \mathcal{L}_{\underline{Q}} \Rightarrow \underline{\Phi} \uparrow \underline{F} \in \mathcal{L}_{\underline{\Lambda}} .$$

— · —

end of def of 'GFOLDS'

The classical model-theoretical conditions on $\mathcal{L}$ (not necessarily holding for an arbitrary GFOLDS $\mathcal{L}$):

Cc: Countable compactness: Given countable augmented signature $\underline{L}$, and classes $\underline{\Phi}_n \in \mathcal{L}_{\underline{L}}$, $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} \underline{\Phi}_n = \emptyset$ implies that there is $m \in \mathbb{N}$ such that $\bigcap_{n \leq m} \underline{\Phi}_n = \emptyset$.

CdLS: Countable downward Löwenheim-Skolem property:

For $\underline{L}$ and $\underline{\Phi}_n \in \mathcal{L}_{\underline{L}}$, $n \in \mathbb{N}$ as above $\bigcap_{n \in \mathbb{N}} \underline{\Phi}_n \neq \emptyset$ implies that there is a countable $\underline{M} \in \bigcap_{n \in \mathbb{N}} \underline{\Phi}_n$. ($\underline{M} = (M, \alpha)$ is countable means that $M : L \to Set$ is a countable functor: $\coprod_{K \in Ob(L)} M(K)$ is a countable set).

Thm <u>Lindström for FOLDS.</u>

For a GFOLDS $\mathcal{L}$,

$\mathcal{L}$ satisfies $\underline{\underline{C_c}}$ and $\underline{\underline{CdLS}}$ if and only if $\mathcal{L}$ is the minimal GFOLDS: for $\underset{\sim}{\text{any}}$ GFOLDS $\mathcal{L}'$, $\mathcal{L} \subseteq \mathcal{L}'$.

<u>NB</u>: For any collection $\mathcal{L}_0$ of classes such that every $\underline{\Phi}$ in $\mathcal{L}_0$ is closed under span-equivalence, there is a least GFOLDS $\langle \mathcal{L}_0 \rangle$ generated by $\mathcal{L}_0$. The conditions 2), 3) and 4) are closure conditions; $\langle \mathcal{L}_0 \rangle$ is the closure of $\mathcal{L}_0$ under these. $\langle \mathcal{L}_0 \rangle$ also satisfies the main FOLDS condition 1), since each of the closure steps 2), 3), 4)

generate closed classes from closed classes. For 2), this is trivial.
For 3), this is proved in §4., "Quantification";
for 4), in §7. "Reducts and coreducts".
The minimal GFOLDS is the collection
$\mathcal{L}_{\omega\omega} \overset{\text{def}}{=} \langle \emptyset \rangle$ generated by the empty collection $\mathcal{L}_0 = \emptyset$.

— ' —

The statement as well as the proof (to be given in Part 2 of these notes) of the Lindström theorem for FOLDS are given without any use of traditional syntax (formulas and the like), and even without categorical-logical formulations of syntax versus semantics. We can make the connections to traditions easily, however. The classes in $\mathcal{L}_{\omega\omega}$, the <u>minimal</u> GFOLDS, are exactly the classes

$$\mathrm{Mod}_L[X:\varphi]$$

Where:

- L is a countable FOLDS signature;
- X is an L-context, construed as a finite context of typed variables, with the dependent typing rules given by L;
- φ is a (finitary) FOLDS formula over L such that the free variables of φ are all in X;
- and $\mathrm{Mod}_L[X : \varphi]$ is the class of all augmented structures (M, α), $M : L \to \mathrm{Set}$, $\alpha : X \to M$ for which

$$M \models \varphi[\alpha]$$

"M satisfies φ at α".