(October 27)

1. We define an axiomatic system, called the *First-Order Theory of Abstract Sets* (FOTAS). Its syntax will be completely specified. Certain axioms will be given; but these may be extended by additional ones at a later time (as it is expected from the experience with the Morse-Kelly system). In FOTAS, we only have sets; no classes. However, we also have functions as a primitive notion.

We use:

◦ *set variables*: $X, Y, Z, \ldots$ [thus: capital letters are sets now; not "classes"; in fact they are "abstract sets"];

◦ *element variables*: $x, x', y, z, \ldots$. Each element-variable must be declared to be *typed* by a set-variable, thus: $x : X$. The intuitive meaning of $x : X$ is that x is an element of X ; however, $x : X$ is *not* a poposition, and, for instance, one *cannot* write $\neg(x : X)$. This, of course, is familiar from ordinary type theory. The only difference is that in $x : X$, X itself is a variable.

◦ The combinations $\forall x : X . \Phi(X, x)$, $\forall X . \forall x : X . \Phi(X, x)$, $\exists X . \forall x : X . \Phi(X, x)$ are well-formed, provided $\Phi(X, x)$ is a well-formed proposition (formula), with possibly other free variables. The first quantified formula says, of course, that "for all x in X , $\Phi(X, x)$ holds"; the third that "there exists X such that for all x in X , $\Phi(X, x)$ holds".

◦ Note however that $\forall X \Phi(X, x)$ (with $x : X$ as before) is not meaningful – and, correspondingly, it will be declared illegal (not well-formed) syntactically. Supposing that the free variable x actually occurs in $\Phi(X, x)$ (we'll see examples below), in $\forall X \Phi(X, x)$, we have lost its "grounding" as $x : X$: we have no particular X any more to relate x to: we cannot evaluate it at values of x .

◦ We have *function variables* $f, g, \ldots$. Each function variable has to be typed thus: $f : X \to Y$, with X and Y set-variables, possibly the same. Of course, the "meaning" of $f : X \to Y$ is that f is a function whose domain is X , and whose codomain is Y (in particular, the range of f is included in Y). The type of f is thus a dependent type: Arr(X, Y) ("arrow from X to Y "), and we have the variable declaration f :Arr(X, Y). We write $f : X \to Y$ synonymously to f :Arr(X, Y).

The upshot is that it is not possible to talk about functions in general; only ones with pre-assigned domain and codomain. We can use a quantifier over the collection of arrows from a given X to a given Y , but not over functions in general (directly, at least). Suppose we have a well-formed proposition (formula) $\Phi(X, Y, f)$, with possibly other free variables. Then the formula $\forall X . \exists Y . \exists f : \mathsf{Arr}(X, Y) . \Phi(X, Y, f)$ is meaningful and well-formed. For instance, if $\Phi(X,Y,f)$ is $\top$ (“identically true”), then $\forall X.\exists Y.\exists f:\mathsf{Arr}(X,Y).\top$ says that from every “object” (set), there is an “arrow” (function) to at least one object. The statement $\forall X.\exists f:\mathsf{Arr}(X,X).\top$ is also meaningful. It is not all-right to say, however, that $\forall X.\forall Y.f:\mathsf{Arr}(X,Y).\top$, which would, apparently, say that f is an arrow from every set to every set at the same time. The latter statement, even if it looks meaningful to you, is not expressible in our language (it is *not* simply false).

Let us note that, once the variable f is declared thus: $f:\mathrm{Arr}(X,Y)$ (equivalently: $f:X\to Y$), we may abbreviate $\forall X.\exists Y.\exists f:\mathsf{Arr}(X,Y).\top$ as $\forall X.\exists Y.\exists f.\top$, although this now looks ambiguous. We will have variable declarations and judgments separately, for a full description of a situation:, thus:

$$X:Set;\ Y:Set;\ f:X\to Y\ ::\ \forall X.\exists Y.\exists f.\top\,.$$

◦ For each set variable X, we have the *equality predicate* $=_X$: equality for elements of X. The grammar of $=_X$ is this: for variables $x_1:X$, $x_2:X$, the *atomic formula* $x_1 =_X x_2$ is well-formed, with free variables x_1 and x_2 (of course).

◦ For each pair of set variables X and Y, we have the equality predicate $=_{X,Y}$ for arrows $f:X\to Y$. The grammar of $=_{X,Y}$ is this: for variables $f:X\to Y$, $g:X\to Y$, the *atomic formula* $f =_{X,Y} g$ is well-formed, with free variables f and g (of course).

◦ We have an operation symbol *App(lication)* which works like this:

$$X:Set;\ Y:Set;\ f:X\to Y;\ x:X\ ::\ App(f,x):Y$$

We mean that in case the variables X,Y,f,x are declared as shown, we have the well-formed *term* $App(f,x)$ typed as a *term of type* Y : $App(f,x):Y$. We will abbreviate the term $App(f,x)$ as $f^{|}x$, or even $f(x)$, or fx. Of course, $App(f,x)$ signifies the (unique) value of the function f at the argument x.

With the usual iterated term-formation rules, as an example, we can then form the following typed term :

$$X\xrightarrow{f}Y\xrightarrow{g}Z\ ,x:X\ ::\ g(f(x)):Z\,.$$

Of course, $g(f(x))$ abbreviates $App(g, App(f, x))$.

The above essentially completes the specification of the language of FOTAS. We have explained the (restricted) use of the quantifiers. We have the connectives

$$\top\,(\text{true}),\ \perp(\text{false}),\ \wedge(\text{and}),\ \vee(\text{or}),\ \rightarrow(\text{implies})$$

used as usual (no restriction). $\neg\Phi$ abbreviates $\Phi \rightarrow \perp$.

I should add that in the formula $\forall x : X.\ x =_X x$, the variable X is a *free variable*.

For clarity's sake, let's review the natural *general (ensemblist) semantics* of the language of FOTAS (*a'la* Tarski). We are using set theory, in fact Morse-Kelly, in this specification.

A structure M for FOTAS consists of:

a collection (a set, or a class) S (= Set^M, the interpretation of the sort Set in the structure M) (the collection of "sets");

for each "set" , or *object* $X \in S$, a *set* $\mathrm{El}(X)$ (= $\mathrm{El}^M(X)$), the collection of "elements of X " (here we insist that $\mathrm{El}(X)$ be a set);

for each pair of objects $X, Y \in S$, a *set* $\mathrm{Arr}(X,Y)$ (or: $\hom^M(X,Y)$) of all *arrows* from X to Y;

for each object $X \in S$, the binary relation $=_X$ on the set $\mathrm{El}(X)$ is taken to be real equality. (This is the standard interpretation, within the framework of Fregean absolute equality. In another version, a "non-standard" one, $=_X$ is an arbitrary binary relation $=_X$ on the collection $\mathrm{El}(X)$);

similar specs for $=_{X,Y}$;

a ternary operation App = (App^M) that applies to any quadruple (X, Y, f, x) where $X \in S,\ Y \in S,\ f \in Arr(X,Y)$, and $x \in \mathrm{El}(X)$, and gives, as the output, a value denoted (naturally) as $f(x)$ in $\mathrm{El}(Y)$. In other words, for any X, Y, f, x as stated we have an actual function (carelessly denoted as) $f : \mathrm{El}(X) \rightarrow \mathrm{El}(Y)$.

Under the semantics, we have, by an obvious (implied-by-the-above) Tarskian truth-definition for formulas of FOTAS in general, resulting in a truth-value of any formula, at any *admissible* evaluation of its free variables. "Admissible" here means, for instance, that if we had the free variables x and X, with $x:X$ declared, then in evaluating x and X, we must have observed the condition that $x\in\mathrm{El}(X)$.

(Note also that there is a corresponding $\mathcal{E}$-valued semantics of FOTAS, for any "Set-like" universe (category) $\mathcal{E}$; for instance, for any topos $\mathcal{E}$.)

The two main axioms (the second being an axiom scheme, in fact) are

◦ *Function extensionality*:

$$f:X\to Y\,,\ g:X\to Y\,,\ x:X\ ::\quad \forall f.\forall g.(f=_{X,Y}g\ \leftrightarrow\ \forall x(f(x)=_Y g(x)))\,.$$

Several abbreviating devices may be used. Firstly, in an axiom, the initial universal quantifiers may be omitted (and considered being there, after all). Secondly, the subscripts of the equality signs may be omitted, since they can be uniquely restored from the context. We obtain the simplified statement $f=g\ \leftrightarrow\ \forall x(f(x)=g(x))$.

◦ *Function comprehension*:

Given any formula $\Phi(X,Y,x,y)$ with the free variables $x:X$ and $y:Y$, and possibly other free variables, the following is an axiom:

$$\forall x.\exists!y.\Phi(X,Y,x,y)\ \to\exists(f:X\to Y).\forall x.\forall y.(f(x)=y\ \leftrightarrow\ \Phi(X,Y,x,y))\,.$$

($\exists!y.\Phi(y)$ abbreviates $\exists y.\forall y'(\Phi(y')\leftrightarrow y'=y)$ as usual; of course, $y':Y$.)

Uniqueness of f in comprehension is assured by extensionality.

The effect of the axioms is this. Suppose we have any model M of the axioms so far. Suppose further that we have a formula $\Phi(X,Y,x,y;\vec{a})$, with $x:X$ and $y:Y$, and with the free variables denoted $\vec{a}$ are all given (admissible!) values (parameters) in M, also denoted by $\vec{a}$. The formula $\Phi(X,Y,x,y;\vec{a})$ provides for a *definable (with parameters)* relation $R\subseteq\mathrm{El}(X)\times\mathrm{El}(Y)$. If this relation R is functional, that is, R *is* a function with domain $\mathrm{El}(X)$ in the usual set-theoretic sense, then there is a unique arrow f, an element of $\hom^M(X,Y)$, $f:X\to Y$, that denotes R, i.e., such that $f(x)=y$ iff xRy ($x\in\mathrm{El}(X)$, $y\in\mathrm{El}(Y)$) (remember that $f(x)=y$ abbreviates $App^M(f,x)=y$).

As an example, let $X\xrightarrow{f}Y\xrightarrow{g}Z$, $x:X$, $z:Z$, and consider the formula $g(f(x))\ =\ z$ (with free variables *all* the displayed variables). Suppose all the five variables in $X \xrightarrow{f} Y \xrightarrow{g} Z$ are given (appropriate) values (simply: assume that we have objects and arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ in M). I claim that it is obvious that $g(f(x)) = z$ defines a functional relation from X to Z : for all $x \in \mathrm{El}(X)$, there is a unique $z \in \mathrm{El}(Z)$ such that $g(f(x)) = z$. Therefore, there is a well-defined arrow $h: X \to Z$ such that $h(x) = z$ iff $g(f(x)) = z$, for all $x \in \mathrm{El}(X)$, $z \in \mathrm{El}(Z)$; that is, $h(x) = g(f(x))$ for all $x \in \mathrm{El}(X)$. We denote h as $g \circ f$, and call it the *composite* of f and g .

The formula $x =_X y$ ($x : X, y : X$) defines, for any given object ("set") X , the identity function $\mathrm{id}_X : X \to X$.

Extensionality shows that the associative law for composition: in case $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, with all entities in M , then $h \circ (g \circ f) =_\partial (h \circ g) \circ f$: both sides define the function $F : \mathrm{El}(X) \to \mathrm{El}(W)$ for which $F(x) = h(g(f(x)))$ for all $x \in \mathrm{El}(X)$.

We have shown that every model M of FOTAS gives rise to what is called a *concrete category*. The category has objects the elements of Set^M , i.e., what we called objects above; and arrows what we called arrows above. Denoting this category by M too, we have the faithful functor $F : M \to \mathrm{Set}$, with Set the category of sets and functions, where $F(X) = \mathrm{El}^M(X)$, and, for $f : X \to Y$ in M , $F(f)$ the function that we wrote as $f : \mathrm{El}(X) \to \mathrm{El}(Y)$.

It is far from true, however, that every concrete category appears as a model of FOTAS. The axiom scheme of function comprehension will give rise to arrows that would not be there without it.

Remark of apology: there are some inconsistencies in the fonts used (and I was too lazy to correct it). For instance, I write sometimes App , and sometimes App for the same thing. There is no danger of confusion though. There are only two meanings of App : the syntactical one (a symbol), and its interpreted version, App^M , regardless of fonts used.