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1. Well-foundedness

Let R be a Relation on the class X ($R \subseteq X \times X$). We say that the structure (X, R) is *well-founded* (wf) if the following holds true:

$$\forall Y \subseteq X \ \langle \{\forall x \in X \ [\forall y(yRx \to y \in Y) \ \to \ x \in Y]\} \ \Rightarrow \ Y = X \rangle \ .$$

In words: call a subclass Y of X *inductive* (with respect to the relation R) if the clause in $\{\ \ldots\}$ holds: $\forall x \in X \ [\forall y(yRx \to y \in Y) \ \to \ x \in Y]$. R is wf if the only inductive class is X itself.

[1.1] Suppose that (X, R) is wf. Let S be any sub-Relation (subclass) of R, and Z any class such that $S \subseteq Z \times Z$. Then (Z, S) is wf as well. (*Exercise*)

The main example for a wf Relation is the membership Relation $\in \ = \{(x, y) : x \in y\}$ on the class $\mathbb{V}$ of pure sets. Indeed, Y being an inductive subclass of $\mathbb{V}$ means that Y is a subclass of $\mathbb{V}$ that is closed under set-formation. Since $\mathbb{V}$ is a subclass of any class closed under set-formation, an inductive subclass of $\mathbb{V}$ must equal $\mathbb{V}$.

Suppose that (X, R) is wf, and let Y be a subclass of X ; $Y \subseteq X$. Let us call Y bottomless (with respect to R) if the following holds:

$$\forall x(x \in Y \to \exists y(y \in Y \wedge yRx)) \ .$$

[1.2] *If* (X, R) *is wf, then every* (R *-)bottomless class is empty*.

Proof Consider the complement of the bottomless Y in X : $\overline{Y} = \{x \in X : x \notin Y\}$. $\overline{Y}$ is inductive: assume

$$\forall y(yRx \to y \in \overline{Y}) \ , \qquad (2.1)$$

to see $x \in \overline{Y}$: if we had $x \in Y$, then, by bottomlessness, $\exists y(y \in Y \wedge yRx)$, contradicting (1). Therefore, $\overline{Y} = X$, which is to say that Y is empty.

[1.3] *If* (X, R) *is wf, then* $\neg xRx$ *for all* $x \in X$; $\neg(xRy \wedge yRx)$ *for all* $x, y \in X$.

Proof If xRx, then the class $\{x\}$ is bottomless: contradiction. If $xRy \wedge yRx$, then the class $\{x, y\}$ is bottomless.

2. The natural numbers

We define $0 \overset{def}{=} \emptyset = \{x : \bot\}$ (the empty set), $S(x) \overset{def}{=} x \cup \{x\}$.

We define the class $\mathbb{N}$ as $\mathbb{N} \overset{def}{=} \{x : \forall X\{[(0 \in X \wedge \forall y(y \in X \to Sy \in X)] \to x \in X\}$. The elements of $\mathbb{N}$ are called (von-Neumann) *natural numbers*.

Later on, we will adopt the axiom of infinity: $\mathbb{N}$ is a set. At this point, however, we do not need this axiom.

The letters m, n, p range over natural numbers. This means that the quantified expression $\forall nP(n)$ is to be read as $\forall x(x \in \mathbb{N} \to P(x))$ and $\exists nP(n)$ as $\exists x(x \in \mathbb{N} \wedge P(x))$.

The *principle of mathematical induction*:

$$P(0) \wedge \forall n(P(n) \to P(Sn)) \;\to\; \forall nP(n)$$

is a direct consequence of the definition: take $X = \{n : P(n)\}$ in the definition of $\mathbb{N}$.

[2.1] **(.1)** Every natural number is a pure set: $\mathbb{N} \subseteq \mathbb{V}$. (*exercise; hint*: use mathematical induction for the predicate $P(n) \equiv n \in \mathbb{V}$).

(.2) Every natural number is a transitive set (a class X is *transitive* if $\forall x \forall y((y \in x \wedge x \in X) \to y \in X)$).

(.3) $Sn \neq 0$ (obvious)

(.4) $Sm = Sn \to m = n$

Proof Suppose that $Sm = Sn$. This means $m \cup \{m\} = n \cup \{n\}$. Therefore, both of the following are true: (1) either $m \in n$ or $m = n$ (by $m \in n \cup \{n\}$), and (2) either $n \in m$ or $n = m$. Hence, either ($m \in n$ and $n \in m$), or $m = n$. However, by (.1) and [1.3], $m \in n$ and $n \in m$ is impossible. Therefore, $m = n$ follows.

Remark: (.2) and (.3) are (some of the) so-called Peano axioms.

(.5) $m = 0 \ \vee \ \exists n.m = Sn$ (obvious by induction).

The *order*-Relation $<$ on $\mathbb{N}$ is given by: $m < n \overset{def}{\leftrightarrow} m \in n$.

[2.2] (.1) $<$ is transitive: $m < n < p$ implies that $m < p$ (follows from [2.1.2])

(.2) $m < Sn \ \leftrightarrow \ m < n \vee m = n$ (obvious from the definitions)

3. A summary of the axioms

Y is a set $\overset{def}{\leftrightarrow} \exists Z.Y \in Z \leftrightarrow Y \in \mathbb{U} = \{x : x = x\}$

Lower-case variables range over sets.

Class comprehension schema: For any predicate $P(X,\vec{Y})$, we have
$\forall \vec{Y}.\exists Z.\forall x(x \in Z \leftrightarrow P(X,\vec{Y}))$.

By extensionality, Z is unique; we write $Z = \{x : P(X,\vec{Y})\}$.

The *set-existence axioms* are:

Axiom of subset: $\forall x \,.\, Y \subseteq x \rightarrow Y$ is a set (a subclass of a set is a set)

Define $\oslash \overset{def}{=} \{x : \perp\}$.
Axiom of the empty set: $\oslash$ is a set. .

For sets x and y, define $\{x, y\} \overset{def}{=} \{u : u = x \ \vee \ u = y\}$.
Axiom of the pair-set: $\forall x \,.\, \forall y \,.\, \{x, y\}$ is a set.

For a set x, define $\bigcup x \overset{def}{=} \{ u : \exists y \,.\, u \in y \wedge y \in x \}$
Axiom of the union set: $\forall x \,.\, \bigcup x$ is a set.

For a set x, define $\mathcal{P}(x) \overset{def}{=} \{y : \forall z(z \in y \rightarrow z \in x)\}$
Axiom of the power set: $\forall x \,.\, \mathcal{P}(x)$ is a set.

The class $\mathbb{N}$ of the natural numbers was defined above.

Axiom of infinity: $\mathbb{N}$ is a set.

A *Relation* is a class all whose elements are ordered pairs.

$\mathrm{Dom}(R) \overset{def}{=} \{x : \exists y\ .\ \langle x, y\rangle \in R\}$, $\mathrm{Range}(R) \overset{def}{=} \{y : \exists x\ .\ \langle x, y\rangle \in R\}$. A Function is a Relation R such that $\forall x \forall y_1 \forall y_2 (\langle x, y_1\rangle \in R \wedge \langle x, y_2\rangle \in R) \rightarrow y_1 = y_2)$

Axiom of replacement: If R is a Function, and $\mathrm{Dom}(R)$ is a set, then $\mathrm{Range}(R)$ is a set.

(The axiom of choice will be considered later.)

4. Transitive models of set theory

Let Φ be any formula in the language of classes. All variables, free or bound, in Φ are class-variables (the set-variables, which are a device of abbreviation, are not used). Given any variable X not occurring in Φ either as a free or a bound variable, we let $\Phi[X]$ denote the formula, with the single free variable X, obtained by relativizing each quantifier in Φ to subclasses of X. This means replacing each $\forall Y...$ in Φ by $\forall Y(Y \subseteq X \rightarrow ...)$, and $\exists Y...$ by $\exists Y(Y \subseteq X \wedge ...)$.

Let us abbreviate $\forall Y(Y \subseteq X \rightarrow ...)$ by $\forall Y \subseteq X...$, and $\exists Y(Y \subseteq X \wedge ...)$ by $\exists Y \subseteq X...$

Note that if we have a set-quantifier $\forall y$ …, with y a set- variable (as usual), this means $\forall Y((\exists U(Y \in U)) \rightarrow ...)$. After relativizing to subclasses to X, it becomes $(\forall Y \subseteq X)((\exists U \subseteq X)(Y \in U) \rightarrow ...)$, which is equivalent to $(\forall Y \subseteq X)(Y \in X \rightarrow ...)$.

Now, from now on, we assume that the class X is transitive: $y \in x \in X$ implies $y \in X$. Thus, $Y \in X$ implies that Y is a set, and $Y \subseteq X$. Therefore, the phrase $(\forall Y \subseteq X)(Y \in X \rightarrow ...)$ is equivalent to $\forall y(y \in X \rightarrow ...)$.

Another remark. Frequently, we can re-write formulas by using the abbreviations $\forall u \in Y...$ for $\forall u(u \in Y \rightarrow ...)$, and $\exists u \in Y...$ for $\exists u(u \in Y \wedge ...)$.

We conclude that, with X transitive, the set-quantifier $\forall y$, after relativizing to subclasses of X, becomes $\forall y \in X...$, and similarly, $\exists y$ becomes $\exists y \in X...$

Moreover, if our original formula Φ contains the bounded quantifier $\forall u \in v...$, or $\exists u \in v...$, then in $\Phi[X]$ the quantifier remains the same: the reason is that

$\forall u((u \in X \wedge u \in v) \rightarrow ...)$ is the same as $\forall u(u \in v \rightarrow ...)$, with the understanding that $v \in X$, since X is transitive; similarly for $\exists u \in v...$.

Consider the example of the power-set axiom as Φ (this is a senence, without free variables):

$$\forall y \exists z \forall u(u \in z \leftrightarrow \forall v \in u.v \in y)$$

(I have re-written the phrase $u \subseteq y$, that is, $\forall v(v \in u \rightarrow v \in y)$, as $\forall v \in u.v \in y$).

Then $\Phi[X]$ is (equivalent to)

$$(\forall y \in X)(\exists z \in X)(\forall u \in X)(u \in z \leftrightarrow \forall v \in u.v \in y) .$$

Let us examine what this means (of course, it may or may not be true, depending on what X is). The set z said to exist has to satisfy that, for u in X, u is in z iff $u \subseteq y$; that is, $z \cap X = \mathcal{P}(y) \cap X$. But since z is to be in X, and X is transitive, $z \cap X = z$. Thus, it is required that $z = \mathcal{P}(y) \cap X$. In conclusion: the truth of $\Phi[X]$, the power-set axiom for the transitive structure $(X, \in\restriction X)$, is to say that $\mathcal{P}(y) \cap X$ is an element of X; $\mathcal{P}(y) \cap X \in X$.