Part 3 FOLDS EQUIVALENCE

Introduction; L-equivalence A56

Fiberwise surjective maps A57

The example of $L = L_{cat}$ A67

Invariance of FOLDS statements under FOLDS equivalence A73

Introductory remarks to the next few sections A73.1

An example: axiomatizing the concept of category in FOLDS – up to FOLDS equivalence A78

Discussion of the category axioms A83

Proof of the theorem on the axiomatizability of the notion of category A89

Logical consequence A93.1

Sections A93.3

Category theory for L_{cat}-structures A94

Invariance of FOLDS statements under FOLDS equivalence

Let us use the following notation. Given M, N L-structures, C a context, $\alpha \in M(C)$ (see A46), $\beta \in N(C)$, we say that the <u>augmented</u> structure (M,α) is equivalent to the augmented structure (N,β), in notation

$$(M,\alpha) \simeq (N,\beta)$$

if there exist P, m, n and γ as shown:

$$\begin{tikzcd}
& P \arrow[dl, "m"'] \arrow[dr, "n"] & \\
M \arrow[r, phantom, "\#"] & {} \arrow[r, phantom, "\#"] & N \\
& C \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, "\gamma"] &
\end{tikzcd}$$

m, n: f.s. $\quad (*)$

We may write $(P, m, n, \gamma) : (M,\alpha) \simeq (N,\beta)$ for $(*)$; and $(P, m, n) : (M,\alpha) \simeq (N,\beta)$ if there is $\gamma \in P(C)$ making $(*)$ true.

Recall that our notion of 'formula' is relative to fixed, although arbitrary, context C (A35). For all formulas φ, the set of free variables of φ, $\mathrm{var}(\varphi)$, is a subcontext of C. In what follows, we consider C fixed, and suppress it from the notation. For instance, the <u>contexts</u> $\underset{\sim}{X}$, $\underset{\sim}{Y}$, ... are all subcontexts of C.

<u>Proposition</u> Let: $\underset{\sim}{X}$ a context, M, N L-structures, $\alpha \in M\underset{\sim}{X}$, $\beta \in N\underset{\sim}{X}$, and assume that

$$(M, \alpha) \simeq (N, \alpha)$$

Then, for any (FOLDS-) formula φ with $\mathrm{var}(\varphi) \subseteq \underset{\sim}{X}$,

$$M \models \varphi[\alpha] \iff N \models \varphi[\alpha]$$

(see bottom A 50 for the notation).

The <u>proof</u> is by induction on the complexity of the formula φ. We leave the verifications for the propositional part of the logic to the reader, and consider the quantifiers.

It is clearly sufficient to show that if $m: P \to M$ is f.s., and $\gamma: \underset{\sim}{X} \to P$, then

? (*) $$P \models \varphi[\gamma] \iff M \models \varphi[m\gamma].$$

Before turning to formulas, take any variable $x :: K$, and context $\underset{\sim}{X}$ such that $\mathrm{dep}(x) \subseteq \underset{\sim}{X}$. Let P be an L-structure, $\gamma: \underset{\sim}{X} \to P$, i.e., $\gamma \in P(\underset{\sim}{X})$.

The notation $P_\gamma[x]$, "the range of the variable under the interpretation γ", was introduced on A37 (with different letters); $P_\gamma[x] \subseteq PK$, and for $b \in PK$, $b \in P_\gamma[x] \iff \partial b = \gamma(\partial x) = \gamma \circ \partial x$

On the other hand, the notation $PK[\beta\cdot]$ comes from A60; it is meaningful for $\beta: \mathring{K} \to P$, and for $b \in PK$, $b \in PK[\beta] \iff \partial\beta = \beta$. Therefore,

$$\boxed{P_\gamma[x] = PK[\gamma \circ \partial x]}$$

Similarly, for $\alpha \in M(\underset{\sim}{X})$, $M_\alpha[x] = MK[\alpha \circ \partial x]$.

Therefore, the assumption that $m : P \to M$ is f.s. implies that the map

$$(*) \quad \begin{cases} PK[\gamma \circ \partial x] \longrightarrow MK[m \circ \gamma \circ \partial x] \\ \quad b \longmapsto mb \ (= m_K b) \end{cases}$$

is surjective.

Consider the case of $(*)$, A75,

$$\underbrace{P \models (\forall x \varphi)[\gamma]}_{(1)} \overset{?}{\Longleftrightarrow} \underbrace{M \models (\forall x \varphi)[m\gamma]}_{(2)}.$$

Let us write $\alpha = m\gamma$.

(1) means: for all $b \in PK[\gamma \circ \partial x]$, $P \models \varphi[\gamma(b/x)]$

(2) means: for all $a \in MK[\alpha \circ \partial x]$, $M \models \varphi[\alpha(a/x)]$

1. Assume (2), to show (1). Take $b \in PK[\gamma \circ \partial x]$; let $a \underset{\text{def}}{=} mb$. For $\gamma' = \gamma(b/x)$, $\alpha' = \alpha(a/x)$ we have $\alpha' = m\gamma'$. $M \models \varphi[\alpha']$ (assumed) implies $P \models \varphi[\gamma']$ by the induction hypothesis. This shows that (2) implies (1)

2. Assume (1), to show (2). Take $a \in MK[\alpha \circ \partial x]$.

By (*), previous page, there is $b \in PK[\gamma \cup \partial x]$ such that $a = m\,b$.
With γ' and α' as before, $P \models \varphi[\gamma']$ (assumed) implies $M \models \varphi[\alpha']$ by the induction hypothesis. This shows that (1) implies (2).

We invite the reader to handle the case of the existential quantifier, $\exists x \varphi$, in a similar manner.

<u>Warning</u>: I do not mean to use the law $\exists x \equiv \neg \forall x \neg$ from classical logic!
The argument for '$\exists x \varphi$' is entirely self-contained, just like that for '$\forall x \varphi$', "<u>constructively valid</u>"!

Introductory remarks to the next few sections

In those sections, we reformulate familiar concepts such as category and functor in the spirit of FOLDS, and show that the reformulations are not essentially different from the original concepts.

Why would want to do such a thing? Why bother reformulating something that is familiar and good as it is?

The answer is that we gather experience in this way for the later higher-dimensional concepts. Independently from FOLDS, those concepts have been formulated in an explicit manner only in low dimensions such as 2 and 3, and even for those dimensions, they are increasingly complicated and ad-hoc looking. Our goal is to formulate and theorize about higher dimensional categories in the context of FOLDS. While doing so, we must clarify the connections of the new concepts to the established low dimensional notions.

An example: axiomatizing the concept of category in FOLDS – up to FOLDS equivalence.

Consider, again, the signature $L = L_{cat}$ (A5), and the operation $\mathbb{C} \longmapsto \tilde{\mathbb{C}}$ taking a small category $\mathbb{C}$ into an L-structure (A6).

We can consider those L-sentences (formulas without free variables), in any L-context whatsoever, that are true in all L-structures of the form $\tilde{\mathbb{C}}$;

$$Th_{cat} = \{ \varphi \in FOLDS_{\emptyset}(L) : \tilde{\mathbb{C}} \models \varphi \text{ for every small category } \mathbb{C} \}$$

($FOLDS_{\emptyset}(L)$ denotes the set of FOLDS-sentences over the signature L; φ with $var(\varphi) = \emptyset$)

For any set Σ of FOLDS sentences over L, $Mod(\Sigma)$ ($= Mod_L(\Sigma)$) is the class of all L-structures satisfying all "axioms" in Σ:

$$\mathrm{Mod}(\Sigma) = \{ M \in \mathrm{Str}(L) : M \models \varphi \text{ for all } \varphi \in \Sigma \}.$$

I claim that the following equality of classes of structures

Thm

$$\boxed{\mathrm{Mod}(\mathrm{Th}_{cat}) = \{ M \in \mathrm{Str}(L) : \text{there is a small category } \mathbb{C} \text{ such that } M \simeq_L \tilde{\mathbb{C}} \}}$$

Let the left-hand side be K_1; the right-hand side K_2.

The inclusion $K_2 \subseteq K_1$ is immediate by the invariance of FOLDS statements under FOLDS equivalence (A73).

For the reverse inclusion, $K_1 \subseteq K_2$, we consider $M \in K_1$, and we construct a small category $\mathbb{C}$ such that $M \simeq_L \tilde{\mathbb{C}}$. In fact, we will have a fiberwise surjective map $\boxed{m : M \longrightarrow \tilde{\mathbb{C}}}$; thus $(M, \mathrm{id}_M, m) : M \simeq_L \tilde{\mathbb{C}}$.

The assumption $M \in K_1$ means that M satisfies all FOLDS statements in Th_{cat}. We will find 12 elements in Th_{cat}, the FOLDS <u>axioms</u> for the notions of category, such that the construction of $\mathbb{C}$ and that of $m : M \longrightarrow \hat{\mathbb{C}}$ are made possible by the assumption that M satisfies the 12 axioms. For the sake of economy, we first list these 12 axioms, although it would be natural to introduce them gradually as the steps of the construction demands them. The axioms will be named by the following symbols:

I 1
E 1, E 2, E 3,
T 1, T 2,
IE 1, IE 2,
IT 1, IT 2,
TE 1, TE 2.

Each symbol reflects which of the three dimension-2 kinds: I, E, T is involved in it. In I1, I is involved; in TE1, both E and T are involved.
After the list, explanations will follow.

I1: $X:\underline{O} \Rightarrow i:\underline{A}(X,X),\ \varphi:\underset{\sim}{I}(i)$

E1: $X,Y:\underline{O};\ f:\underline{A}(X,Y) \Rightarrow \varepsilon:\underline{E}(f,f)$

E2: $X,Y:\underline{O};\ f,g:\underline{A}(X,Y);\ \varepsilon:\underline{E}(f,g) \Rightarrow \varepsilon':\underline{E}(g,f)$

E3: $X,Y:\underline{O};\ f,g,h:\underline{A}(X,Y);\ \varepsilon_1:\underline{E}(f,g);\ \varepsilon_2:\underline{E}(g,h)$

$$\Rightarrow \varepsilon_3:\underline{E}(f,h)$$

T1: $X,Y,Z:\underline{O};\ f:\underline{A}(X,Y);\ g:\underline{A}(Y,Z)$

$$\Rightarrow h:\underline{A}(X,Z);\ \tau:\underset{\sim}{T}(f,g,h)$$

T2: $X,Y,Z,W:\underline{O};\ f:\underline{A}(X,Y);\ g:\underline{A}(Y,Z);\ h:\underline{A}(Z,W);$

$i:\underline{A}(X,Z);\ j:\underline{A}(X,W);\ k:\underline{A}(Y,W);$

$\tau_1:\underset{\sim}{T}(f,g,i);\ \tau_2:\underset{\sim}{T}(i,h,j);\ \tau_3:\underset{\sim}{T}(g,h,k)$

$$\Rightarrow \tau_4:\underset{\sim}{T}(f,k,j)$$

IE1: $X:\underline{O};\ i,j:\underline{A}(X,X),\ \varphi:\underset{\sim}{I}(i),\ \psi:\underset{\sim}{I}(j) \Rightarrow \varepsilon:\underline{E}(i,j)$

$$\mathrm{IE2}: \quad X : \underline{O};\ i, j : \underline{A}(X, X),\ \varphi : \underline{I}(i),\ \varepsilon : \underset{\sim}{E}(i, j)$$
$$\Longrightarrow \quad \psi : \underset{\sim}{I}(j)$$

$$\mathrm{IT1}: \quad X, Y : \underline{O};\ i : \underline{A}(X, X);\ g : \underline{A}(X, Y);\ \varphi : \underset{\sim}{I}(i)$$
$$\Longrightarrow \quad \tau : \underset{\sim}{T}(i, g, g)$$

$$\mathrm{IT2}: \quad X, Y : \underline{O},\ f : \underline{A}(X, Y),\ i : \underline{A}(Y, Y),\ \varphi : \underset{\sim}{I}(i)$$
$$\Longrightarrow \quad \tau : \underset{\sim}{T}(f, i, f)$$

$$\mathrm{TE1}: \quad X, Y, Z : \underline{O};\ f, f' : \underline{A}(X, Y);\ \varepsilon_1 : \underset{\sim}{E}(f, f');$$
$$g, g' : \underline{A}(Y, Z);\ \varepsilon_2 : \underline{E}(g, g');\ h, h' : \underline{A}(X, Z);$$
$$\tau : \underset{\sim}{T}(f, g, h);\ \tau' : \underset{\sim}{T}(f', g', h')$$
$$\Longrightarrow \quad \varepsilon_3 : \underset{\sim}{E}(h, h')$$

$$\mathrm{TE2}: \quad X, Y, Z : \underline{O};\ f, f' : \underline{A}(X, Y);\ \varepsilon_1 : \underset{\sim}{E}(f, f');$$
$$g, g' : \underset{\sim}{A}(Y, Z);\ \varepsilon_2 : \underset{\sim}{E}(g, g');\ h, h' : \underline{A}(X, Z);$$
$$\tau : \underset{\sim}{T}(f, g, h);\ \varepsilon_3 : \underset{\sim}{E}(h, h')$$
$$\Longrightarrow \quad \tau' : \underset{\sim}{T}(f', g', h').$$

Discussion of the category axioms

1. First, I explain the (novel) kind of abbreviation used in the category axioms, A81, A82. Pick one, say T1. To read it, imagine the universal quantifier in front of each variable to the left of the symbol '$\Rightarrow$', and the existential quantifier in front of the variables to the right of '$\Rightarrow$' — and add (intuitively, rather superfluously, but syntactically correctly) the T (= true) propositional constant at the end. The result in case of T1 is:

$$\underbrace{\forall X; Y, Z : \underline{O}}_{\substack{\text{abbreviating}\\ \forall X:\underline{O}.\forall Y:\underline{O}.\,\forall Z:\underline{O}}} .\ \forall f : \underline{A}(X,Y).\ \forall g : \underline{A}(Y,Z).\ \exists h : \underline{A}(X,Z).\ \exists \tau : \underbrace{\underset{\sim}{T}(f,g,h)}_{\substack{\text{abbreviating}\\ \underset{\sim}{T}(X,Y,Z,f,g,h)\\ \text{as explained on A40}}}.\,T$$

See also A39 where the formula is preceded by the context, the context is not incorporated in the formula as we just did, and did also on A40. The example on A39, A40 is the same as IT2 on A82.

2. Each of the axioms expresses an intuitive truth about categories — i.e., each one is satisfied by $\tilde{\mathbb{C}}$, for any category $\mathbb{C}$: In other words, each is an element of Th_{cat}.

I1: existence of the identity arrow:

for every object X, we have $1_X : X \to X$

(the statement just made is meant as the reason why $\tilde{\mathbb{C}} \models I1$)

E1: reflexivity of the equality relation (on parallel arrows!)

E2: symmetry of equality

E3: transitivity of equality

(Abraham Lincoln, in the Spielberg film "Lincoln", quoted 'Common Notion 1' from Euclid's Elements, Book 1: "Things which equal the same thing also equal one another".)

T1: existence of composite of composable arrows:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] & & Z \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "z"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & {} & Z \end{tikzcd}$$

<u>Historical note:</u> Peter Freyd introduced a diagrammatic notation for first order statements about categories; see the book by him & A. Scedrov: "Categories, Allegories". Freyd's work is an important precursor to FOLDS. Later, I will say more about this.

An alternative, very useful, abbreviated notation, shown first for T1:

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\boxed{z}"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "\boxed{h}"'] & {} & Z \end{tikzcd}$$

This notation removes the repetitions from the previous one. To read it, quantify all non-boxed variables (X, Y, Z, f, g) universally, the $\boxed{\text{boxed}}$ ones, <u>afterwards</u>, existentially: $\forall X\ \forall Y\ \forall Z\ \forall f\ \forall g\ \exists h\ \exists \tau .\ T$

T2: associativity of composition:

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[d, phantom, "\tau_1"] & Z \arrow[dr, "h"] & \\
X \arrow[ur, "f"] \arrow[urr, bend right, "i"'] \arrow[rrr, bend right=40, "j"'] & & {} \arrow[d, phantom, "\tau_2"] & W \\
 & & {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[drr, bend right, "k"'] \arrow[d, phantom, "\boxed{\tau_4}"] & Z \arrow[dr, "h"] \arrow[d, phantom, "\tau_3"] & \\
X \arrow[ur, "f"] \arrow[rrr, bend right=40, "j"'] & {} & {} & W
\end{tikzcd}$$

This still uses repetition — but that is because we must draw in 2 dimensions. The figure is a 3-dimensional one, properly speaking: we have the tetrahedron

$$\begin{tikzcd}
 & Y \arrow[rr, "k"] \arrow[dr, "g"] & & W \\
X \arrow[ur, "f"] \arrow[rr, bend right, "i"'] \arrow[urrr, dashed, "j"] & & Z \arrow[ur, "h"'] &
\end{tikzcd}$$

with the four faces filled with $\tau_1, \tau_2, \tau_3, \tau_4$ — the notation of which facts would be awkward.

The formulation of the associative law may be non-standard: instead of saying $(f \cdot g) \cdot h = f \cdot (g \cdot h)$

we say:

"if $j = (f\cdot g)\cdot h$, then $j = f\cdot(g\cdot h)$"

or:

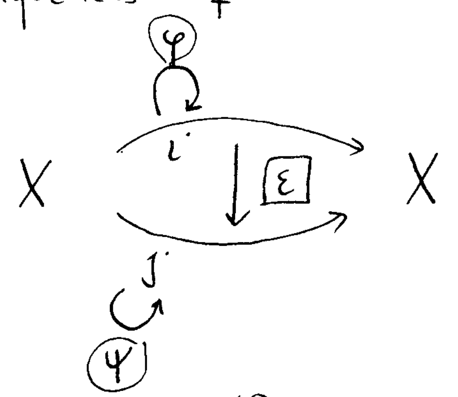
$$i =_{\tau_1} f\cdot g \ \ \& \ \ j =_{\tau_2} i\cdot h \ \ \& \ \ k =_{\tau_3} g\cdot h \ \Rightarrow$$

$\Rightarrow$ there is (witness) τ_4 for

$$j =_{\tau_4} f\cdot k .$$

It was not necessary to express associativity exactly like this; we could have followed the original idea more closely; in that case, we would have two arrows $j_1, j_2 : X \to W$, and would have involved equality: the kind $\underline{E}$.

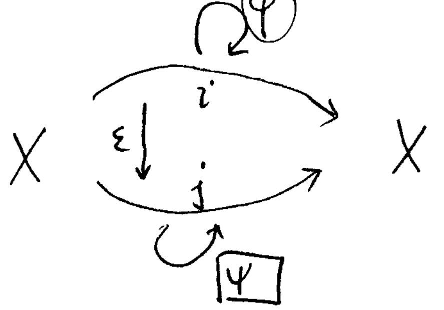
IE 1: uniqueness of the identity arrow



$i =_{\varphi} id_X$

$j =_{\psi} id_X$

IE 2:



equality axiom

if $i =_{\varphi} id_X$, and $i =_{\varepsilon} j$, then there is ψ s.t. $j =_{\psi} id_X$

IT1: 1-st unit law:
(left)

$$\begin{tikzcd} \varphi & X \arrow[dr, "g"] & \\ X \arrow[ur, "i"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\boxed{\tau}"] & & Y \end{tikzcd}$$

IT2: 2-nd (right) unit law:

$$\begin{tikzcd} & Y \arrow[dr, "i"] & \varphi \\ X \arrow[ur, "f"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\boxed{\tau}"] & & Y \end{tikzcd}$$

TE1: uniqueness of composite

TE2: equality axiom of composite

these share the same diagram:

$$\begin{tikzcd} & Y \arrow[dr, shift left=2, "g"] \arrow[dr, shift right=2, "g'"'] \arrow[dr, phantom, "\varepsilon_2"] & \\ X \arrow[ur, shift left=2, "f"] \arrow[ur, shift right=2, "f'"'] \arrow[ur, phantom, "\varepsilon_1"] \arrow[rr, shift left=2, "h"] \arrow[rr, shift right=2, "h'"'] \arrow[rr, phantom, "\Downarrow \varepsilon_3"] & & Z \end{tikzcd}$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=4, "\tau"] & & Z \end{tikzcd}$$

$$\begin{tikzcd} & Y \arrow[dr, "g'"] & \\ X \arrow[ur, "f'"] \arrow[rr, "h'"'] \arrow[rr, phantom, shift left=4, "\tau'"] & & Z \end{tikzcd}$$

In TE1, $\boxed{\varepsilon_3}$ is boxed;

In TE2, $\boxed{\tau'}$ is boxed.

Proof of the theorem on A79: FOLDS axiomatizability of the concept of category

We follow the notation and outline on A78 – A80.

Thus, we have $M \in K_1$ given to us. We construct $\mathbb{C}$ and $m: M \to \check{\mathbb{C}}$.

$$\mathrm{Ob}(\mathbb{C}) \underset{\mathrm{def}}{=} M(\underline{O})$$

Next, let $X, Y \in M(\underline{O})$, $M\underline{A}(X,Y) \overset{\mathrm{def}}{=} \{f \in M\underline{A} : (Md)f = X, (Mc)f = Y\}$

$$= \{f \mid X \xrightarrow{f} Y\}$$

(as usual)

Define relation $\sim_{X,Y}$, abbreviated $\sim$, on the set $M\underline{A}(X,Y)$ as follows: for $X \underset{g}{\overset{f}{\rightrightarrows}} Y$,

$f \sim g \underset{\mathrm{def}}{\iff}$ there is $\varepsilon \in M\underline{E}(f,g)$. Using the equality axioms E1, E2, E3, assumed to be

true in M, we see that $\sim_{X,Y}$ is an equivalence relation on $M\underline{A}(X,Y)$.
For $f \in M\underline{A}(X,Y)$, $[f]$ denotes the equivalence class $\{g : f \sim g\}$.

We define

$$\mathrm{Arr}(\mathbb{C}) \underset{\mathrm{def}}{=} \{(X,Y,a) : X,Y \in M\underline{O},\ a=[f] \text{ for some } f\in M\underline{A}(X,Y)\}$$

(d = domain; c = codomain)

$$d(X,Y,a) \overset{\mathrm{def}}{=} X$$
$$c(X,Y,a) \overset{\mathrm{def}}{=} Y$$

The $\mathbb{C}$-arrow $a=[i]$, $X \xrightarrow{a} X$ in $\mathbb{C}$, $X \xrightarrow{i} X$ in M, <u>is</u>, by definition, <u>an identity arrow</u> on X iff there is $\varphi \in M\underline{I}(i)$. By IE2 in M, this concept is well-defined: the definition is independent of the representative i of a. By I1, for every $X \in \mathrm{Ob}(\mathbb{C})$, there is at least one identity arrow on X; by IE1, there is at most one. We denote the unique identity arrow on X by id_X.

Given $a = [f] : X \to Y$, $b = [g] : Y \to Z$, we say that $c = [h] : X \to Z$ <u>is a composite of a and b</u> if there exist $\tau \in M\underset{\sim}{T}(X, Y, Z, f, g, h)$.

T1 in M ensures that for any

$$X \xrightarrow{a} Y \xrightarrow{b} Z$$

in $\mathbb{C}$, a composite of a & b exists. TE2 ensures that the concept "composite of a and b" is well-defined: independence from representatives. TE1 ensures that the composite is unique. Of course, we write $a \cdot b$, equivalently, $b \circ a$ or ba, for <u>the</u> composite.

The axioms T2, IT1, IT2 in M give the associative law, and the two unit laws in $\mathbb{C}$.

The definition of $\mathbb{C}$ is complete.

We define the natural transformation $m : M \to \tilde{\mathbb{C}}$ (for $\tilde{\mathbb{C}}$, see A6).

$$m_{\underline{O}} : M\underline{O} \longrightarrow \tilde{\mathbb{C}}\underline{O} = Ob(\mathbb{C}) = M\underline{O}$$

is defined as the identity : $m_{\underline{O}} = id_{M\underline{O}}$

Since $\underline{O}$ is of dimension 0 in $L = L_{cat}$, m being f.s. at $\underline{O}$ means that $m_{\underline{O}}$ is a surjective function; this is true.

To define $m_{\underline{A}} : M\underline{A} \longrightarrow \tilde{\mathbb{C}}\underline{A}$, note that for $K = \underline{A}$, $M\mathring{K} = M\mathring{\underline{A}}$ is the set of pairs (X, Y), $X \in M\underline{O}$, $Y \in M\underline{O}$ (X associated with $d : \underline{A} \to \underline{O}$, Y with $\underline{c} : \underline{A} \to \underline{O}$).

In general terms, to define, for $m : P \to M$ (reverting to the notation on A 60), the map $m_K : PK \to MK$ if $m_{K'}$ for lower-dimensional K''s has already been defined, it is necessary and sufficient to define $m_\beta : PK[\beta] \to MK[m\beta]$ for every $\beta \in P\mathring{K}$. Accordingly, we define

$$m_{X,Y} : M\underline{A}(X,Y) \longrightarrow \tilde{\mathbb{C}}\underline{A}(X,Y)$$

(note: $mX = X$, etc.)

for each pair (X,Y), $X, Y \in M\underline{O}$; and we do this in the obvious way:

$$M\underline{A}(X,Y) \xrightarrow{m_{X,Y}} \tilde{\mathbb{C}}\underline{A}(X,Y) = \hom_{\mathbb{C}}(X,Y)$$

$$f \longmapsto [f]$$

(this is all right, since

$$X \xrightarrow{f} Y \text{ in } M \quad \Rightarrow \quad X \xrightarrow{[f]} Y \text{ in } \mathbb{C} \;)$$

The requirement that m be f.s. at the kind $\underline{A}$ is equivalent to saying that $m_{X,Y}$ be surjective for all X, Y — and this is right.

We will leave it to the reader to complete the definition of $m : M \to \tilde{\mathbb{C}}$, by specifying, in obvious way, the maps

$$m_\rho : MK[\rho] \longrightarrow \tilde{\mathbb{C}}[m\rho]$$

for $K \in \{\underline{I}_1, \underline{E}, \underset{\sim}{I}\}$ and $\rho \in M\mathring{K}$ — which, being each surjective, ensures that $m : M \to \tilde{\mathbb{C}}$ is f.s.

This completes the proof of the <u>theorem</u> on the FOLDS-axiomatizability of the concept of category, Theorem A79.

Logical consequence

Let us introduce the notion of logical consequence in the usual way: for L a (FOLDS-) signature, Σ a set of (FOLDS-) sentences (first-order, as introduced on A35 & A46), over L, φ a single such sentence, we say that φ is a (logical) consequence of Σ, in symbols $\Sigma \models \varphi$, if all models of Σ are models of φ:

for all L-structures M,
if $M \models \Sigma$ (meaning: $M \models \sigma$ for all $\sigma \in \Sigma$)
then $M \models \varphi$

(see top A54 for the last piece of notation).

Corollary Th_{cat} [see: A78] is finitely axiomatizable; for Σ_{cat} the set of the 12 category axioms, $\Sigma_{cat} = \{I1, \dots, TE2\}$, we have that for all $\varphi \in Th_{cat}$,

$$\boxed{\Sigma_{cat} \models \varphi.}$$

Proof. Let $\varphi \in Th_{cat}$. Suppose $M \models \Sigma_{cat}$.
By the proof of the theorem (!), there is small category $\mathbb{C}$ such that $M \simeq_L \tilde{\mathbb{C}}$.
By definition of Th_{cat}, $\tilde{\mathbb{C}} \models \varphi$. By invariance (A 74), it follows that $M \models \varphi$.
This proves the corollary.

— · —

By a theory in FOLDS, we mean a pair $T = (L, \Sigma)$ where L is a (FOLDS-) signature, Σ is a set of (finitary first-order FOLDS-) sentences over L.

A model of T is an L-structure M for which $M \models \Sigma$. To say that T is finitely axiomatizable is to say that there is a finite set $\Sigma' \subseteq \underbrace{L_0[\text{FOLDS}]}_{\text{set of all FOLDS sentences over } L}$

such that $\text{Mod}(T) = \underbrace{\text{Mod}(L, \Sigma')}_{\text{class of models of } (L, \Sigma')}$.

Sections

A useful lemma, to be used later, is a special property of f.s. maps: they have sections.

Thm. Let L be any FOLDS signature, $m : P \twoheadrightarrow M$ an f.s. map in $Str(L)$. Then there is a section of m: there is $s : M \to P$, such that $m \circ s = 1_M$.

proof (outline). The construction of s will use the axiom of choice (and a recursion.) Let $L \upharpoonright n$ be the full subcategory of L on the objects with dimension $< n$; $L \upharpoonright 0$ is the empty category. Recursively, we define

$$s_n : \underbrace{M \upharpoonright (L \upharpoonright n)}_{M_n \text{ for short}} \longrightarrow \overbrace{P \upharpoonright (L \upharpoonright n)}^{P_n \text{ for short}}$$

(here: $L \upharpoonright n \xrightarrow[\varphi_n = \text{inclusion}]{} L \xrightarrow{P} Set$; $P \upharpoonright (L \upharpoonright n) \overset{def}{=} P \circ \varphi_n$;

similarly for $M\lceil(L\lceil n)$). For $n=0$, s_0 is the empty map. Suppose s_n has been defined, satisfying

$$m_n \circ s_n = 1_{M_n};$$

here $m_n \overset{\text{def}}{=} m\varphi_n$:

$$\left\{\begin{array}{l} L\lceil n \xrightarrow{\varphi_n} L \underset{M}{\overset{P}{\rightrightarrows}} Set \quad (\downarrow m) \\[2ex] L\lceil n \underset{M\circ\varphi_n}{\overset{P\circ\varphi_n}{\rightrightarrows}} Set \quad (\downarrow m\varphi_n) \end{array}\right.$$

We define s_{n+1}. s_{n+1} will be defined so that it extends s_n: $s_{n+1}\restriction(L\restriction n) = s_n$. We only need to define $s_{n+1}(a)$ for $a \in MK$, $\dim K = n$.

∂a was defined on A31; use φ_n from bottom last page.

Have:

$$L\restriction n \xrightarrow{\varphi_n} L \ \underset{M}{\overset{\mathring{K}}{\rightrightarrows}} \ \mathrm{Set} \;; \qquad \downarrow \partial a$$

compose:

$$\begin{tikzcd} L\restriction n \arrow[r, shift left=2, "\mathring{K}\circ\varphi"] \arrow[r, shift right=2, "M\circ\varphi = M_n"'] & \mathrm{Set} \end{tikzcd} \qquad \downarrow (\partial a)\varphi, \qquad \downarrow s_n \qquad \xrightarrow[P_n]{}$$

to get $\beta' \underset{\text{def}}{=} s_n \circ (\partial a)\varphi : \mathring{K}\circ\varphi \longrightarrow P_n$. Since for all $p \in L|K$, $K_p \in \mathrm{Ob}(L\restriction n)$, it is easy to see that there is a unique β: $\mathring{K} \to P$ such that $\beta' = \beta\varphi$. (β is the same function as β').

Applying m_n to both sides of

$$s_n \circ (\partial a)\varphi = \beta\varphi$$

we get
$$\underbrace{m_n \circ s_n}_{= id} \circ (\partial a)\varphi = (\underbrace{m\varphi}_{m_n}) \circ (\beta\varphi)$$

i.e. $(\partial a)\varphi = (m\varphi) \circ (\beta\varphi)$

from which $\partial a = m\beta$ easily follows (all non-empty values of the functor $\mathring{K} : L \longrightarrow Set$ are also values of the restriction $\mathring{K} \circ \varphi : L \upharpoonright n \longrightarrow Set$). Since m is f.s., there is $b \in PK$ such that $m_K b = a$ and $\partial b = \beta$. We put $s_{n+1}(a) = b$.

We leave the verification of the correctness of our recursion to the reader. $s : M \to P$ is defined so that $s \circ \varphi = s_n$ for all n. $\boxed{\square \text{ Thm}}$

To lighten somewhat the burden of the abstraction of the last proof, let us consider the case $L = L_{cat}$. We have $m: P \longrightarrow M$ f.s. The construction of $s: M \to P$ is by recursion on the dimension of the argument of s in M.

For $X \in M\underline{0}$, $m_{\underline{0}}: P\underline{0} \to M\underline{0}$ is surjective; therefore we can pick (axiom of choice) $sX \in P\underline{0}$ such that $m(sX) = X$. Suppose we have done this simultaneously for all $X \in M\underline{0}$. Next, we have to define $s(f)$ for $X \xrightarrow{f} Y$ in M

We have:

$$\begin{tikzcd} \bar{X} \arrow[d, "m"'] & \bar{Y} \arrow[d, "m"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and the f.s. property says that there is $\bar{f}$,

$\bar{f} : \bar{X} \to \bar{Y}$ in P such that

$$\begin{tikzcd}
\bar{X} \arrow[rr, "\bar{f}"] & & \bar{Y} \\
& {} \arrow[d, "m"] & \\
X \arrow[rr] & {} & Y
\end{tikzcd}$$

We define $sf = \bar{f}$, simultaneously for all $f \in M\underline{A}$, I think, the idea is now clear for the rest.

Category theory for L_{cat}-structures

We develop a language talking about L_{cat}-structures that imitates the language of category theory. This will be found useful in the next several sections.

We fix the FOLDS signature L_{cat}; we will refer to it as L.

We call an L-structure a pre-category. By a category, we mean a pre-category that is a model of Th_{cat}, i.e., a model of the 12 axioms, $\{I1, \ldots, TE2\}$.

$M, N, P, M', \ldots$ will denote pre-categories; when we assume of a pre-category that it is a category, this will be carefully stated.

Any map $F: M \to N$ is called a functor (of pre-categories; F is, actually, a natural transformation!)

Let us temporarily fix a pre-category M; objects and arrows are elements of $\underline{MO}$, resp., of $\underline{MA}$.

We say that f is *equal* to g, in symbols $f \sim g$ if f & g are parallel, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ for suitable X and Y; and there is (at least one) $\varepsilon \in M\underset{\sim}{I}(f,g)$ $(= M\underset{\sim}{I}(X,Y,f,g))$.

Note that being *equal* is an equivalence relation.

$X \xrightarrow{i} X$ is an *identity arrow* if there is $\varphi \in M\underset{\sim}{I}(i)$.

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

commutes if there is $\tau \in M\underset{\sim}{T}(f,g,h)$.

$$\begin{tikzcd} Y \arrow[r, "h"] & W \\ X \arrow[u, "f"] \arrow[r, "g"'] & Z \arrow[u, "i"'] \end{tikzcd}$$

commutes if there is $j : X \to W$

such that

$$\begin{tikzcd} Y \arrow[r, "h"] & W \\ X \arrow[u, "f"] \arrow[ur, "j"] \arrow[r] \arrow[ur, phantom, shift left=3, "\#"] \arrow[ur, phantom, shift right=3, "\#"] & Z \arrow[u, "i"'] \end{tikzcd}$$

(where the symbol $\#$ says: "the triangle commutes").

Exercise: Suppose M is a category.

Then:

1) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, there always is

an $h : X \longrightarrow Z$ such that

(*)
$$\begin{tikzcd} & Y \arrow[dr] \arrow[d, phantom, "\#"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd}$$
commutes.

2) Let us write $h \sim f \cdot g$ for saying: (*) commutes.
Note that $h' \sim h \sim f \cdot g$ implies $h' \sim f \cdot g$
(use 'equality axiom'!).

3) Suppose $h' \sim f \cdot g$ and $h \sim f \cdot g$.
Then $h' \sim h$.

Definition $X \xrightarrow{i} Y$ is an isomorphism if there is
$i^{-1} : Y \to X$ such that $i \circ i^{-1} : Y \to Y$
and $i^{-1} \circ i : X \to X$ are both identity arrows;
sorry; I should have said: there are commutative Δ's:

$$\begin{tikzcd} & Y \arrow[dr, "i^{-1}"] \arrow[d, phantom, "\#"] & \\ X \arrow[ur, "i"] \arrow[rr, "k"'] & {} & X \end{tikzcd}, \quad \begin{tikzcd} & X \arrow[dr, "i"] \arrow[d, phantom, "\#"] & \\ Y \arrow[ur, "i^{-1}"] \arrow[rr, "l"'] & {} & Y \end{tikzcd}$$
with k, l identity arrows.

Exercise. Again, suppose M is a category.

Then, of course, for every $X \in M\underline{O}$, there is (at least one) identity arrow $X \xrightarrow{i} X$ on X.

Let us write (symbolically!) $i \sim 1_X$ for saying: i is an identity arrow on X.

1) Note this:

$$i \sim 1_X \Rightarrow (j \sim i \iff j \sim 1_X).$$

2) $X \xrightarrow{i} Y$ being an isomorphism can be written as $i i^{-1} \sim 1_Y$ & $i^{-1} i \sim 1_X$. Show that if $i \sim j$, and i is an isomorphism, then so is j.

Exercise Now, M is an arbitrary precategory, $F : M \to N$. Show that F preserves commutative triangles, identity arrows, and isomorphisms.

Given functors $M \overset{F}{\underset{G}{\rightrightarrows}} N$, a natural transformation $h: F \to G$ is a family $\langle h_X \rangle_{X \in M\underline{O}}$, $h_X : FX \to GX$ in N, such that for all $X \xrightarrow{f} Y$ in M, the square

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "h_X"'] \arrow[dr, phantom, "\#"] & FY \arrow[d] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

commutes.

Suppose $F \xrightarrow{h} G \xrightarrow{k} H$ are natural transformations. $F \xrightarrow{\ell} H$ is a composite of h and k, $\ell \sim kh$, if for all $X \in M\underline{O}$, $\ell_X \sim k_X h_X$ (composition in N).

Exercise Assume that N is a category, $M \overset{F, G, H}{\rightrightarrows} N$ are functors, $F \xrightarrow{h} G \xrightarrow{k} H$ are nat. transf's. Then the composite kh exists, and it is unique (i.e., $\ell: F \to H$ exists s.t. $\ell \sim kh$; and if $\ell' \sim kh$ too,

then $\ell \sim \ell'$ in the sense that
$\ell_X \sim \ell'_X$ for all $X \in M\underline{0}$).

Consider the situation

$$P \xrightarrow{H} M \underset{G}{\overset{F}{\underset{\downarrow h}{\rightrightarrows}}} N \xrightarrow{K} Q$$

Exercise. Formulate what hH and Kh as in

$$P \underset{GH}{\overset{FH}{\underset{\downarrow hH}{\rightrightarrows}}} N, \quad M \underset{KG}{\overset{KF}{\underset{\downarrow Kh}{\rightrightarrows}}} Q$$

should be, and show they exist, essentially uniquely, when P, M, N, Q are categories.