Formulas: φ, ψ denote formulas. Then the following are also formulas:

$\top$ ('true') ; $var(\top) := \emptyset$

$\bot$ ('false') ; $var(\bot) := \emptyset$

$\varphi \wedge \psi$: $var(\varphi \wedge \psi) := var(\varphi) \cup var(\psi)$

$\varphi \vee \psi$; $var(\varphi \vee \psi) := var(\varphi) \cup var(\psi)$

$\varphi \to \psi$; $var(\varphi \to \psi) := var(\varphi) \cup var(\psi)$

$\left.\begin{matrix}\forall\\ \exists\end{matrix}\right\} x\varphi$ provided $x \in var(\varphi)\uparrow$

(for the latter, see above)

and, if so,

$$var\left(\left.\begin{matrix}\forall\\ \exists\end{matrix}\right\} x\varphi\right) := \left(var(\varphi) \cup dep(x)\right) - \{x\}$$

Discussion. Contexts (subcontexts of C) are closed under union; the empty set is a context; and crucially, the proviso for the quantified formula $\forall x\varphi$ or $\exists x\varphi$ to be well-formed ($x \in var(\varphi)\uparrow$) makes the new set, $var(\forall x\varphi)$ ($= var(\exists x\varphi)$) also a context – if $var(\varphi)$ was one.

By induction, for every (well-formed) formula φ,

$var(\varphi)$ is a context.

Digression on the concept of truth.

After all these explanations, especially the one concerning the range of a variable (A28), we should be able to <u>read</u> formulas, <u>understand</u> their <u>meaning</u> in an interpretation. I will now give some examples to illustrate this. The formal truth-definition, which is still necessary (I think!), and will be given, will sound like a tautology - as usual, as everyone who has taught logic and tried to give the formal truth-definition, will know.

(this digression will be continued at a later time).

Examples for formulas, and their reading

Let C be the context of A21; all 'contexts' below are subcontexts of C.

	Formula φ	Its context: $var(\varphi)$.
φ_1	$\top$ (true)	$\emptyset$
φ_2	$\exists \tau.\, \top$	$\{X, Y, f, i\}$ (since $dep(\tau) = \{X, Y, f, i\}$)
φ_3	$\forall \varphi\, \exists \tau.\, \top$	$\{X, Y, f, i\}$ [$dep\, \varphi = \{Y, i\}$; $var(\forall \varphi\, \exists \tau.\, \top) = (var(\exists \tau.\, \top) \cup dep(\varphi)) - \{\varphi\}$]
φ_4	$\forall i\, \forall \varphi\, \exists \tau.\, \top$	$\{X, Y, f\}$
φ_5	$\forall f\, \forall i\, \forall \varphi\, \exists \tau.\, \top$	$\{X, Y\}$
φ_6	$\forall Y\, \forall f\, \forall i\, \forall \varphi\, \exists \tau.\, \top$	$\{X\}$
φ_7	$\forall X\, \forall Y\, \forall f\, \forall i\, \forall \varphi\, \exists \tau.\, \top$	$\emptyset$

φ_7 is an axiom for 'category' — more about them later. But now, let us see that, for any category $\mathbb{C}$,

? $\tilde{\mathbb{C}}$ satisfies φ_7 : φ_7 is true in $\tilde{\mathbb{C}}$:

$$\tilde{\mathbb{C}} \models \varphi_7$$

(for $\tilde{\mathbb{C}}$, see A6).

φ_7 is a <u>sentence</u>, meaning $var(\varphi_7) = \emptyset$.
It is easier to read the formula when, with each variable we show its type – the <u>kind</u> and the <u>boundary</u> – because then we see the <u>range</u> of the variable. For φ_7:

$$\forall X:\underline{O}\,.\ \forall Y:\underline{O}\,.\ \forall f:\underline{A}(X,Y).\ \forall i:\underline{A}(Y,Y).$$

$$\forall \varphi:\underline{I}(i)\,.\ \exists \tau:\underset{\sim}{I}(f,i,f)\,.\ T$$

<u>Abbreviations used</u>: the <u>type</u> of a variable x contains the variables in ∂x; when x is i, these are Y and i.

Since, however, $Y = di\ (=ci)$ in C, Y need not be displayed, since, in ∂x, if $y \in \partial x$ and $z = qy$, then $z \in \partial x$. This is why we wrote $\underline{I}(i)$ instead of $\underline{I}(Y,i)$.
We did similarly with $\underset{\sim}{I}(f,i,f)$.

By the way, the 'same' (equivalent) formula could have been written this way:

$$\forall X:\underline{O}.\ \forall Y:\underline{O}.\ \forall f:\underline{A}(X,Y).\ \forall i:\underline{A}(Y,Y)$$

$$[(\exists \varphi:\underline{I}(i).\,T) \to \exists \tau:\underset{\sim}{T}(f,i,f).\,T]$$

– just to show an example for the use of the implication.

What does this mean in an arbitrary L-structure M? ($L = L_{cat}$). For an easier language, let us write $\bar{O}$ for $M(\underline{O})$, $\bar{A}$ for $M(\underline{A})$, etc.

With an abuse of language, as is usually done in mathematics, we use the same symbol for the variable and the element of M it denotes.
We write $k: U \to V$ for $k \in \bar{A}(U,V)$.
Our formula reads:

for all X in $\bar{O}$. for all Y in $\bar{O}$. for all $f: X \to Y$

for all $i: Y \to Y$. for all φ in $\bar{I}(i)$.

there is τ in $\bar{T}(f,i,f)$

[you don't have to say: such that TRUE!]

When $M = \tilde{\mathbb{C}}$, this means

<u>for all</u> objects X, Y, and arrows

$$X \xrightarrow{f} Y \xrightarrow{i} Y$$

1 $\rightsquigarrow$ $\begin{cases} \text{and } \underline{\text{for all}} \text{ evidence } \varphi \text{ for } i = id_Y \\ \underline{\text{for all}}\ \varphi \text{ such that } i =_{\varphi} id_Y \end{cases}$

2 $\rightsquigarrow$ <u>there is</u> (evidence) τ such that $i \circ f =_{\tau} f$

An evidence for $i =_{\varphi} id_Y$ means (literally now) that $\varphi = Y$, and $i = id_Y$ (see the definition of $\tilde{\mathbb{C}}$!)

and τ is <u>the</u> commutative triangle

$$\begin{tikzcd} & Y \arrow[dr, "i = id_Y"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

if the latter is commutative — <u>which it is</u>, <u>because</u> $\tilde{\mathbb{C}}$ is a category, and the 2-nd unit law <u>holds</u> in it.

Counterexample:

take the well-formed formula

$\varphi_2 :=: \exists \tau. T$ with its context: $\{X, Y, f, i\}$,

The following is not well-formed:

$$* \qquad \forall X. \exists \tau\, T$$

Its 'context' should be $\{Y, f, i\}$, which is not a context. — But, is this "reasonable"?

$\forall X. \exists \tau\, T$ talks about an 'object' Y, and two arrows f and i. It seems to say that if $X = \mathrm{dom}(f)$, then $i \circ f =_{\tau} f$ — thus, it seems to say the same as $\exists \tau. T$ did, which, however, says something about Y, f, i and X. So, in a way, $\forall X. \exists \tau\, T$ is not absolutely meaningless — just ungrammatical! And since we want a language that is not just expressive, but also grammatically clarified, we reject

$\forall X.\ \exists z\, T$ as a formula. The grammatical rule that was violated is this: when a formula talks <u>about</u> a variable x (an element denoted by the variable x), then it also, automatically, talks about all variables x <u>depends</u> on. In our case, f <u>depends</u> on both X and Y; therefore, if f is a free variable in a formula, so must X be.

The category-theoretical intuition involved is this: you cannot talk about an arrow in a category <u>in general</u>; you can only talk about an arrow with a previously specified domain and codomain.

Restriction? Yes! The restricted language will have stronger invariance properties than arbitrary first-order formulas: an arbitrary first order statement about a category is not necessarily invariant under category equivalence; a FOLDS statement is.

Simplest example for an elementary (MacLane), i.e., first-order statement about a category:

(*) "for all objects X and Y, $X = Y$".

It is true for the one-object categories, which are also called monoids. However, (*) is not invariant under category equivalence. See later. Of course, (*) is not in FOLDS (over L_{cat}) – and cannot be written equivalently in FOLDS.

Formal semantics

1) As in the section "Formulas", we fix a context $\mathbb{C}$, and use the terminology and notation fixed there and previously, from A23 on.

For a context $\underset{\sim}{X}$ [sub context of $\mathbb{C}$], and L-structure M, we write

$$\boxed{M(\underset{\sim}{X}) \overset{def}{=} \hom(\underset{\sim}{X}, M)},$$

the 'extension' of the context $\underset{\sim}{X}$ in M; an element of $M(\underset{\sim}{X})$ is an <u>evaluation</u> of $\underset{\sim}{X}$ in M, an $\underset{\sim}{X}$-<u>element</u> of M. An element of $M(\underset{\sim}{X})$ is a natural transformation $\Phi : \underset{\sim}{X} \to M$.

Let us write

$$|M| \overset{def}{=} \bigsqcup_{K \in Ob(L)} M(K) \overset{def}{=}$$

$$= \{(K, a) : K \in Ob(L),\ a \in MK\}$$

↑
disjoint sum

$|M|$ is the <u>underlying set</u> of the structure M. For simplicity, we usually assume (as we always can, "up to isomorphism"), that M is <u>separated</u> in the sense that $MK \cap MK' = \emptyset$ for $K \neq K'$,

and take $|M| \underset{def}{=} \dot{\bigcup}_{K \in Ob(L)} MK$, just as with

contexts in place of structures. Then, as before, $\Phi \in M(\underset{\sim}{X})$, (an evaluation of $\underset{\sim}{X}$ in M,) can be identified with a function $\Phi : |\underset{\sim}{X}| \to |M|$; and $M(\underset{\sim}{X})$ can be identified with the set of those functions $\Phi : |\underset{\sim}{X}| \to |M|$ for which

$$\boxed{\Phi(px) = (Mp)(\Phi x)}$$

whenever $x :: K$ in $\underset{\sim}{X}$, and $p : K \to K_p$ in L (on the left, px is short for $\underset{\sim}{X}(p)$ as before).

We have the *empty context* $\underset{\sim}{X} = \emptyset$, which is the functor $L \xrightarrow{\emptyset} Set$ all whose values $\emptyset(K)$ are the empty set $\emptyset$; $\emptyset$ is a subcontext of C. The set $M(\emptyset)$ is the singleton-set $\{!_\emptyset\}$, $!_\emptyset : \emptyset \to M$, for which $(!_\emptyset)_K = (\emptyset \xrightarrow{!} MK)$.

An example: if $L = L_{cat}$, $\underset{\sim}{X}$ is the context we called C on A 21 and A 27 (and C is any 'encompassing' context containing $\underset{\sim}{X}$), and $M :: L_{cat} \to \mathbf{Set}$, then $M(\underset{\sim}{X})$ is the set of all 'systems' of elements X, Y, f, i, φ, z that 'fit' together as in the picture bottom A27: e.g., $(M\underline{d})(f) = X$, $(M\underline{i})(\varphi) = i$, e.t.c.

2) Suppose $\underset{\sim}{X}$ is a context, x is a variable, $x :: K$, such that $\mathrm{dep}(x) \subseteq \underset{\sim}{X}$. x itself may or may not belong to $\underset{\sim}{X}$. Let a be an element of the range of x in M, $a \in M_\alpha[x]$, and let α be an evaluation of $\underset{\sim}{X}$, $\alpha \in M(\underset{\sim}{X})$.

The set $\underset{\sim}{X} \cup \{x\}$ is (determines) a context.

We can define an evaluation of it,

$$\alpha(a/x) : \underset{\sim}{X} \cup \{x\} \longrightarrow M$$

by putting

$$\boxed{\alpha(a/x)(u) = \begin{cases} \alpha(u) & \text{if } u \in \underset{\sim}{X} - \{x\} \\ a & \text{if } u = x \end{cases}}$$

We see that $\alpha(a/x)$ is well-defined exactly because $a \in M_\alpha[x]$.

~~Iff~~ $x \notin \underset{\sim}{X}$, then $\alpha(a/x)$ extends α; $\alpha(a/x)$ is the extension of α that fills the niche $\overset{\circ}{K}$ by the element a in M.

If $x \in \underset{\sim}{X}$, then $\alpha(x) \in MK$, filling the $\overset{\circ}{K}$-niche already; in this case, $\alpha(a/x)$ replaces $\alpha(x)$ by a.

3) For any (L-) formula φ, and any context $\underset{\sim}{X}$ such that $var(\varphi) \subseteq \underset{\sim}{X}$, we define the <u>$\underset{\sim}{X}$-extension of φ in M</u>, the set $M[\underset{\sim}{X} : \varphi]$, a subset of $M(\underset{\sim}{X})$:

$$M[\underset{\sim}{X} : \varphi] \subseteq M(\underset{\sim}{X})$$

recursively in φ:

$$M[\underset{\sim}{X} : \top] = M(\underset{\sim}{X})$$

TRUE

$$M[\underset{\sim}{X} : \bot] = \emptyset \quad (\subseteq M(\underset{\sim}{X}))$$

$$M[\underset{\sim}{X} : \varphi \wedge \psi] = M[\underset{\sim}{X} : \varphi] \cap M[\underset{\sim}{X} : \psi]$$

$$M[\underset{\sim}{X} : \varphi \vee \psi] = M[\underset{\sim}{X} : \varphi] \cup M[\underset{\sim}{X} : \psi]$$

$$M[\underset{\sim}{X} : \forall x \varphi] =$$

$$= \{\alpha \in M[\underset{\sim}{X}] : \text{for all } a \in M_\alpha[x], \text{ we have } \alpha(a/x) \in M[\underset{\sim}{X} \cup \{x\} : \varphi]\}$$

$$M[\underset{\sim}{X} : \exists x \varphi] =$$

$$= \{\alpha \in M[\underset{\sim}{X}] : \text{there exists } a \in M_\alpha[x] \text{ such that } \alpha(a/x) \in M[\underset{\sim}{X} \cup \{x\} : \varphi]\}$$

4) <u>Discussion</u>

4.1) We can use the following traditional notation: when $\mathrm{var}(\varphi) \subseteq \underset{\sim}{X}$, $\alpha \in M(\underset{\sim}{X})$, we write $M \models \varphi[\alpha]$ and read: "φ is true in M at α" if:

$$M \models \varphi[\alpha] \overset{\text{def}}{\iff} \alpha \in M[\underset{\sim}{X} : \varphi]$$

Then, for instance, we have

$$M \models (\varphi \wedge \psi)[\alpha] \iff M \models \varphi[\alpha] \text{ and } M \models \psi[\alpha]$$

and

$$M \models (\forall x \varphi)[\alpha] \iff \text{for all } a \in M_\alpha[x], \quad M \models \varphi[\alpha(a/x)].$$

4.2) It is important to check that the clauses of the definition are "meaningful": e.g., for the formula $\forall x \varphi$, <u>if</u>, as it is assumed on the left-hand-side, $\mathrm{var}(\forall x \varphi) \subseteq \underset{\sim}{X}$, <u>then</u> $\mathrm{var}(\varphi) \subseteq \underset{\sim}{X} \cup \{x\}$.

4.3) When, in particular, $\underset{\sim}{X} = \mathrm{var}(\varphi)$, then $M[\mathrm{var}(\varphi) : \varphi]$ makes sense. In fact, this special case of the interpretation is "enough"; this will be expressed in the so-called 'Substitution theorem' stated below.

Suppose $\underset{\sim}{Y} \subseteq \underset{\sim}{X}$ are contexts, M a structure. Then we have a projection-, or restriction-map

$$\pi : M[\underset{\sim}{X}] \longrightarrow M[\underset{\sim}{Y}]$$

defined by $\pi(\alpha) = \alpha \circ \underline{\text{incl.}}$, where

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathbb{L} \arrow[r, bend left=60, "\underset{\sim}{Y}", ""{name=Y, below}] \arrow[r, "\underset{\sim}{X}" near start, ""{name=X}] \arrow[r, bend right=60, "M"' near start, ""{name=M, above}] & \mathbf{Set}
\arrow[from=Y, to=X, Rightarrow, "\underline{\text{incl.}}"]
\arrow[from=X, to=M, Rightarrow, "\alpha"]
\end{tikzcd}$$

with $\underline{\text{incl}} : \underset{\sim}{Y} \to \underset{\sim}{X}$ the "inclusion" natural transformation. $\pi(\alpha)$ is also written as $\alpha \restriction \underset{\sim}{Y}$, "$\alpha$ restricted to $\underset{\sim}{Y}$".

Assume, in addition, that $\mathrm{var}(\varphi) \subseteq \underset{\sim}{Y}$; thus, $M[\underset{\sim}{X} : \varphi]$, $M[\underset{\sim}{Y} ; \varphi]$ are well-defined.

We have the following commutative diagram of sets:

$$\begin{tikzcd} M[\underset{\sim}{X}:\varphi] \arrow[r] \arrow[d, "\text{inclusion}"'] \arrow[dr, phantom, "\text{p.b.}"] & M[\underset{\sim}{Y}:\varphi] \arrow[d, "\text{inclusion}"] \\ M(\underset{\sim}{X}) \arrow[r, "\pi"'] & M(\underset{\sim}{Y}) \end{tikzcd}$$

which, in fact, is a pullback:

$$M[\underset{\sim}{X}:\varphi] = \underbrace{\pi^{-1}}_{\text{inverse-image}}(M[\underset{\sim}{Y}:\varphi])$$

that is, for $\alpha \in M(\underset{\sim}{X})$,

$$\boxed{\alpha \in M[\underset{\sim}{X}:\varphi] \iff \alpha \restriction \underset{\sim}{Y} \in M[\underset{\sim}{Y}:\varphi]}$$

('substitution theorem').

This fact can easily be verified by induction on the complexity of the formula φ. When, in particular, $\underset{\sim}{Y} = var(\varphi)$, we obtain that

$$M \models \varphi[\alpha] \iff M \models \varphi[\alpha \restriction var(\varphi)].$$

By convention, when $var(\varphi) = \emptyset$ — when we say that φ is a <u>sentence</u> —, we write $M \models \varphi$ for $M \models \varphi[!_\emptyset]$.

Of course, now $M \models \varphi$ iff $M \models \varphi[\alpha]$;

this holds for any meaningful evaluation

$\alpha : \underset{\sim}{X} \to M$.

<u>Examples</u>: Suppose $K \in Ob(L)$, $\dim K = 0$,

that is, $dep(K) = \emptyset$. Let $::x::K$.

We have the following sentences (among others):

$$\top, \quad \bot, \quad \forall x.\top, \quad \exists x.\bot.$$

The reader is invited to verify the following:

$M \models \top$ always

$M \not\models \bot$ always

$M \models \forall x.\bot \iff M(K) = \emptyset$

$M \models \exists x.\top \iff M(K) \neq \emptyset$

4.4) The semantics can be extended to involve "structures" $M : L \longrightarrow \mathcal{S}$ 'in' categories $\mathcal{S}$ more general than Set — and in fact, it is important to do so for the theory. For instance, $\mathcal{S}$ can be any (elementary) topos, or, more generally, any Heyting pretopos. When one restricts the formula φ to be in a special class ('coherent', 'regular'), $\mathcal{S}$ can be taken to have just the corresponding structure ('coherent', resp. 'regular') for the definitions on A 50 to make sense.

Part 3: FOLDS EQUIVALENCE

$\boxed{\text{Introduction}}$ Let us fix an arbitrary FOLDS signature L. M, N, P will denote L-structures:

$$L \overset{M}{\underset{P}{\overset{N}{\rightrightarrows}}} \mathrm{Set}.$$

We will define when $M \simeq_L N$: "M and N are (L-)equivalent". This notion will generalize (weaken) the notion of isomorphism; of course

$$\underbrace{M \cong N}_{M \,\&\, N \text{ are isomorphic}} \iff \exists: \; gf \overset{(=)}{=} \mathrm{id}_M \circlearrowright M \underset{g}{\overset{f}{\rightleftarrows}} N \circlearrowleft \mathrm{id}_N \overset{(=)}{=} fg$$

The notion will be based on the notion of a fiberwise-surjective map $P \xrightarrow{m} M$ between L-structures (of course, m is a nat. transf.), to be defined below. We will then define

$$M \simeq_L N \overset{\text{def}}{\iff} \exists: \quad
\begin{array}{ccc} & P & \\ {}^{m}\swarrow & & \searrow^{n} \\ M & & N \end{array}$$

with m and n fiberwise surjective

Fiberwise surjective maps

(1) We will define when a map

$$m: P \longrightarrow M$$

of L-structures:

$$\begin{tikzcd} L \arrow[r, bend left, "P"] \arrow[r, bend right, "M"'] \arrow[r, phantom, "\downarrow m"] & Set \end{tikzcd}$$

is fiberwise surjective (f.s. for short). This will mean something stronger than saying that

$m_K : PK \longrightarrow MK$ is a surjective (onto) map of sets for all $K \in Ob(L)$. I note that the notion is naturally generalized to apply to $\mathbb{S}$-valued semantics

$$\begin{tikzcd} L \arrow[r, bend left, "P"] \arrow[r, bend right, "M"'] \arrow[r, phantom, "\downarrow m"] & \mathbb{S} \end{tikzcd}$$

when very little is required of $\mathbb{S}$: $\mathbb{S}$ is to be a <u>regular category</u>. (This generalization is not only possible, but also <u>important</u>.) However, for now, we stick to the Set-valued case: $\mathcal{S} = \mathbf{Set}$ as above.

2) Let us return to the section "The range of of a variable in an interpretation" (A30), in particular to the subsection 1), A31.

We have, as there, M & K, but no $a \in MK$ yet. An arbitrary evaluation $\beta \in \hom(\mathring{K}, M) = M(\mathring{K})$ is called a <u>boundary</u>, or K-<u>boundary</u>, in M. This is the <u>same</u> definition as that of 'boundary' in the formal context (A27) (the economy of category theory!), but now we are in the purely semantic context. On A31, we have, for any $a \in MK$, the boundary <u>of</u> a, $\partial a \in \hom(\mathring{K}, M)$.

We define, for given $\beta \in M(\overset{\circ}{K})$, the

<u>fiber</u> over β of MK:

$$\boxed{MK[\beta] \underset{\text{def}}{=} \{ a \in MK : \partial a = \beta \}.}$$

The set MK is the disjoint union of all

its fibers:

$$MK = \dot{\bigcup_{\beta \in M(\overset{\circ}{K})}} MK[\beta]$$

We see the mapping

$$MK \xrightarrow{\partial} M\overset{\circ}{K} :$$

for $\beta \in M\overset{\circ}{K}$:

$$MK[\beta] \xrightarrow{\partial} \{\beta\} \subseteq M\overset{\circ}{K},$$

$$MK[\beta] = \partial^{-1}(\beta) = \text{inverse image of } \beta \in M\overset{\circ}{K}.$$

(3) Let $m : P \to M$ be a morphism $Str(L)$: m is a natural transformation:

$$\begin{tikzcd} L \arrow[r, bend left, "P"] \arrow[r, bend right, "M"'] \arrow[r, phantom, "\downarrow m"] & Set \end{tikzcd}$$

Given $\beta \in P\mathring{K}$, we have $m\beta \overset{def.}{=} m \circ \beta \in M\mathring{K}$

$$\begin{tikzcd} L \arrow[r, "\mathring{K}"] \arrow[r, "P"] \arrow[r, "M"'] & Set \end{tikzcd} \quad \downarrow\beta,\ \downarrow m,\ m\beta$$

and an induced map

$$\begin{array}{ccc} PK[\beta] & \ni & b \\ \downarrow m_\beta & & \downarrow \\ MK[m\beta] & & m_K(b) \end{array}$$

(it is easy to check that $m_K(b) \in MK$ is indeed in the set $MK[m\beta]$).

<u>Definition</u> $m : P \to M$ is <u>fiberwise surjective</u> <u>at</u> K if $m_\beta : PK[\beta] \to MK[m\beta]$ is surjective for all $\beta \in P\mathring{K}$; it is <u>fiberwise surjective</u> if it is f.s. at <u>all</u> $K \in Ob(L)$.

Essentially repeating what we said in the Introduction, we define:

<u>Definition</u> An L-equivalence between M and N is a triple (P, m, n) such that $P \in \mathrm{Str}(L)$, and $m: P \to M$, $n: P \to N$ are fiberwise surjective.

Notation: $(P, m, n) : M \simeq_L N$. (*)

We say that M & N are L-equivalent, $M \simeq_L N$, if there exists an L-equivalence (*)

4) I will slightly reformulate the notion now

Consider the following diagram, based on $m: P \to M$:

$$\begin{tikzcd} PK \arrow[r, "\partial"] \arrow[d, "(m_K =)\ m"'] & P\mathring{K} \arrow[d, "m\ (= m_{\mathring{K}})"] \\ MK \arrow[r, "\partial"'] & M\mathring{K} \end{tikzcd} \qquad (**)$$

It is certainly commutative:

$$\begin{tikzcd} b \in PK \arrow[r, mapsto] \arrow[d, mapsto] & \partial b \in P\mathring{K} \arrow[d, mapsto] \\ mb \in MK \arrow[r, mapsto] & \partial(mb) \quad m\,\partial b \end{tikzcd}$$

this follows from m being a natural transformation. m being f.s. at K is equivalent to saying that: for any given $\beta \in P\mathring{K}$ and $a \in MK$ such that $\partial a = m\beta$:

$$
\begin{tikzcd}
 & & P\mathring{K} \arrow[d, "m"] & \ni \beta \arrow[dd, mapsto] \\
 & MK \arrow[r, "\partial"] & M\mathring{K} & \\
a \arrow[rr, mapsto] \arrow[ur, phantom, "\in"] & & \partial a & = m\beta
\end{tikzcd}
$$

there is at least one $b \in PK$ such that $mb = a$ & $\partial b = \beta$:

$$
\begin{tikzcd}
b \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \beta \\
 & PK \arrow[r, "\partial"] \arrow[d, "m"'] \arrow[ul, phantom, "\ni"] & P\mathring{K} \arrow[ur, phantom, "\ni"] & \\
 & MK & & \\
a & & &
\end{tikzcd}
$$

This says something about the diagram (∗∗) on A61, namely, that *that* diagram satisfies one-half, the 'existence' part, of the definition of 'pullback'. It's worth while giving a final categorical formulation, since this is the one that works for "arbitrary" semantical categories in place of Set, in particular, for any regular category $\mathcal{S}$.

Consider the following expansion of (∗∗), A61:

$$\begin{tikzcd}
PK \arrow[rr, "\partial"] \arrow[dr, "p"] \arrow[dd, "m"'] \arrow[ddr, phantom, "\#", near start] & & P\mathring{K} \arrow[dd, "m"] \\
 & Q \arrow[ur, "\pi_1"'] \arrow[dl, "\pi_0"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[u, phantom, "\#"] & \\
MK \arrow[rr, "\partial"'] & & M\mathring{K}
\end{tikzcd}$$

Here

$$\begin{tikzcd} & & P\mathring{K} \arrow[d, "m"] \\ & Q \arrow[ur, "\pi_1"] \arrow[dl, "\pi_0"'] & \\ MK \arrow[rr] & & M\mathring{K} \end{tikzcd}$$

is constructed, in "any" category (certainly, in a regular category) as a pullback, with the appropriate universal property. When we are in Set, Q is the set of all pairs (a, β) where $a \in MK$, $\beta \in P\mathring{K}$, and $\partial a = m\beta$. The arrow $p: PK \to Q$ is uniquely determined by the condition that the two triangles marked # are commutative. When we are in Set, p maps an element $\bar{a} \in PK$ to the pair $(m\bar{a} \in MK, \ \partial\bar{a} \in P\mathring{K})$; that this pair is in Q follows from the commutativity of the square $(**)$. Now, the definition:

$$\left\{\begin{array}{l} m: P \to M \text{ is f.s. at } K \\ \qquad \text{iff} \quad p: PK \to Q \text{ is surjective.} \\ \qquad\qquad\qquad\qquad\qquad \text{(regular epi)} \end{array}\right.$$

This equivalent version of the definition "talks about all $\beta \in P\mathring{K}$ at the same time", as opposed to the original version. Its great advantage is its generality: it works when P, M are $\mathcal{S}$-valued structures:

$$\begin{tikzcd} L \arrow[r, bend left, "P"{name=A}] \arrow[r, bend right, "M"'{name=B}] & \mathcal{S} \arrow[from=A, to=B, phantom, "\downarrow m"] \end{tikzcd}$$

for a regular category $\mathcal{S}$. (One has to re-define $P\mathring{K}$, $M\mathring{K}$, etc, in an appropriate way, but that uses only the finite-limit structure of $\mathcal{S}$, which is assumed in a regular category.)

I can indicate why this may turn out important. Consider the following:

$$
\begin{tikzcd}
L \arrow[r, shift left=1.5, "P"] \arrow[r, shift right=1.5, "M"'] \arrow[r, phantom, "\downarrow m"] \arrow[rr, bend left=60, "FP"] \arrow[rr, bend right=60, "FM"'] & \mathcal{S} \arrow[r, "F"] \arrow[r, phantom, shift right=6, "\Downarrow Fm"] & \mathrm{Set}
\end{tikzcd}
$$

We have considered a functor $F : \mathcal{S} \to \mathrm{Set}$ and the composites $FP : L \to \mathrm{Set}$, $FM : L \to \mathrm{Set}$ and $Fm : FP \to FM$. We have produced ordinary, Set-valued, structures out of the $\mathcal{S}$-valued ones, P and M. Now, if F is a regular functor, that is, preserves the finite-limit structure, together with *surjectivity* (*regular epimorphisms*), then from the fact that $m : P \to M$ is f.s. in the above-stated general sense, it follows that $Fm : FP \to FM$ is f.s. in the original, detailed definition for

Set-valued structures. In the applications, F is the variable, arbitrary 'model' (Set-valued model!) of the 'theory' $\mathbb{S}$; it produces two other Set-valued models, FP and FM, via the two 'formal' models P and M, and it produces an f.s. ("equivalence") between FP and FM, via the fixed 'formal' f.s. arrow ("equivalence") m. The applications I have in mind concern statements of the kind that two (FOLDS-) definitions of a concept are "*equivalent*".

$\boxed{\text{The example of } L = L_{cat}}$

For L_{cat}, see A3; for $\tilde{\mathbb{C}}$, see A6

Let us take $K = \underset{\sim}{I}$ (in L), and imagine arbitrary L-structures P, M, and homomorphisms $m: P \to M$. The diagram (**), A61, now;

with pictures for the typical elements in the four sets, as well as their images under the maps:

"$\partial b = \beta$" is depicted in the notation

$$b = \begin{tikzcd} & \bar Y \arrow[dr, "\bar g"] & \\ \bar X \arrow[ur, "\bar f"] \arrow[rr, bend right, "\bar h"'] \arrow[rr, phantom, "b"] & & \bar Z \end{tikzcd} \in P\underset{\sim}{I} \qquad \beta = \begin{tikzcd} & \bar Y \arrow[dr, "\bar g"] & \\ \bar X \arrow[ur, "\bar f"] \arrow[rr, bend right, "\bar h"'] & & \bar Z \end{tikzcd} \in P\overset{\circ}{\underset{\sim}{I}}$$

$\bar f = (Pt_0)(b)$

etc.

possibly empty (non-comm) triangle in $\underline{P}$

$$\begin{tikzcd} P\underset{\sim}{I} \arrow[r, "\partial"] \arrow[d, "m"'] & P\overset{\circ}{\underset{\sim}{I}} \arrow[d, "m"] \\ M\underset{\sim}{I} \arrow[r, "\partial"'] & M\overset{\circ}{\underset{\sim}{I}} \end{tikzcd}$$

$$a = \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, "a"] & & Z \end{tikzcd} \in M\underset{\sim}{I} \qquad \alpha = \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & & Z \end{tikzcd} \in M\overset{\circ}{\underset{\sim}{I}}$$

$f = (Mt_0)(a)$, etc.

possibly empty triangle in M

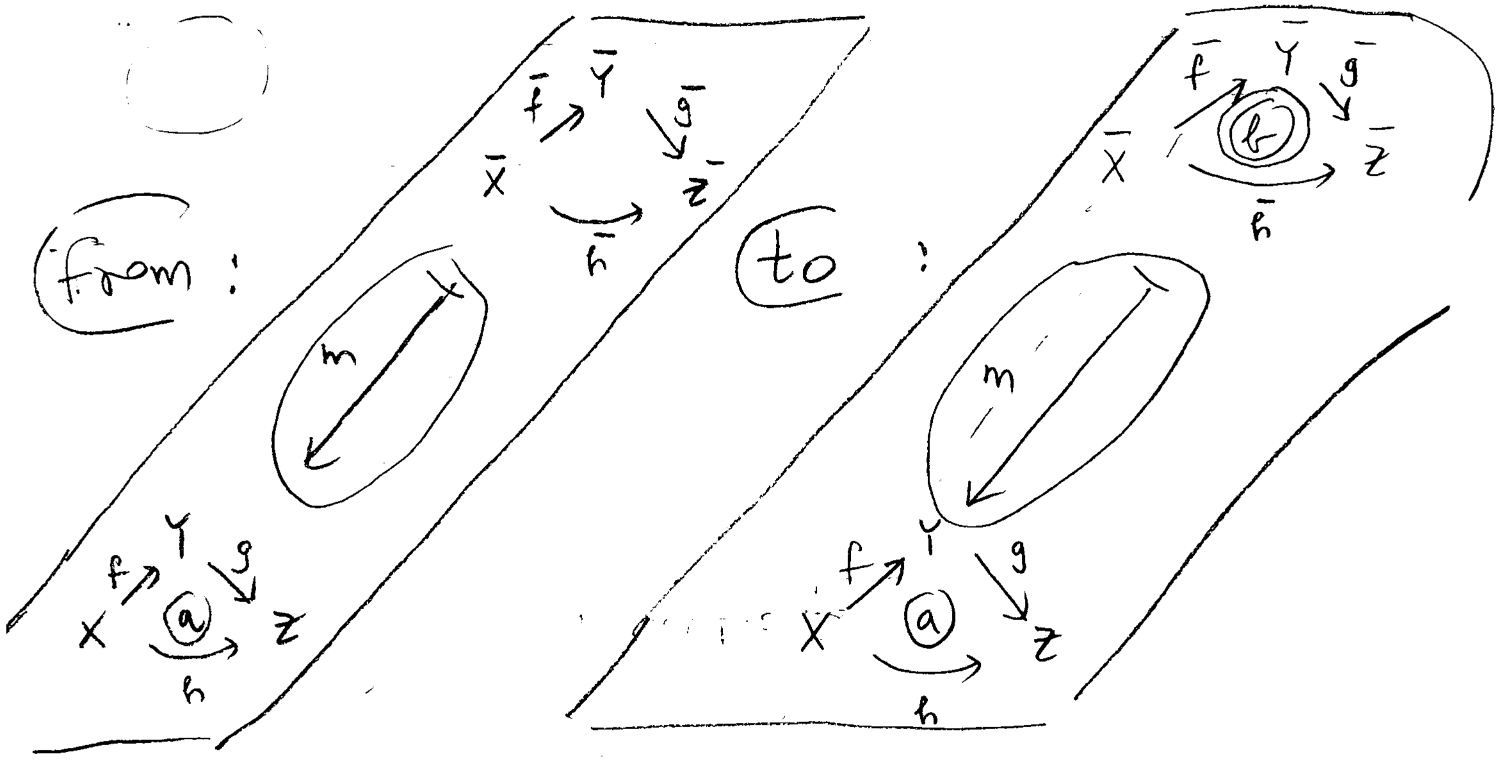
$\underset{\sim}{\text{If}}$ $\boxed{\partial a = \alpha}$, then, of course, α is not empty: a is a $\underset{\sim}{\text{filler}}$ for it.

$$\begin{array}{ccc} \bar f & b & P\underset{\sim}{I} \\ \downarrow & \downarrow & \downarrow m \\ f & a & M\underset{\sim}{I} \end{array}$$

$\boxed{mb = a}$ implies that $m\bar f = f$, etc.

To say that m is f.s. at $\underset{\sim}{I}$, is to say that: <u>whenever</u> $a \in M\underset{\sim}{I}$, a "commutative triangle", and β is an "empty triangle" in P whose image under m is the boundary of a, ∂a, <u>then</u> there is at least one b in $P\underset{\sim}{I}$ whose boundary is the given 'shell' β, and whose m-image is a.

This can be <u>abbreviated</u> in this way: it is allowed to go:



Now suppose also a map $n : P \to N$

(which, eventually, I also assume f.s.)

Then we start with

$$
\begin{tikzcd}[column sep=small]
 & & & \bar{Y} \arrow[dr, "\bar{g}"] & & & \\
 & & \bar{X} \arrow[ur, "\bar{f}"] \arrow[rr, bend right, "\bar{h}"'] \arrow[dl, "m"'] & & \bar{Z} \arrow[dr, "n"] & & \\
 & Y \arrow[dr, "g"] & & & & \hat{Y} \arrow[dr, "\hat{g}"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] \arrow[r, phantom, "a"] & & Z & & \hat{X} \arrow[ur, "\hat{f}"] \arrow[rr, bend right, "\hat{h}"'] & & \hat{Z}
\end{tikzcd}
$$

As a first step, we fill in the top with $\bar{a} = b$ as before; as the second step, we get, as the n-image of $\bar{a}$, the filler $\hat{a}$ of the lower right triangle:

$$\begin{tikzcd}[column sep=small]
 & & & \bar{X} \arrow[r, "\bar{f}"] \arrow[rr, bend right, "\bar{h}"'] \arrow[dr, phantom, "\bar{a}"] & \bar{Y} \arrow[r, "\bar{g}"] & \bar{Z} & & & \\
 & & & & & & & & \\
X \arrow[r, "f"] \arrow[rr, bend right, "h"'] \arrow[dr, phantom, "a"] & Y \arrow[r, "g"] & Z & & & & \hat{X} \arrow[r, "\hat{f}"] \arrow[rr, bend right, "\hat{h}"'] \arrow[dr, phantom, "\hat{a}"] & \hat{Y} \arrow[r, "\hat{g}"] & \hat{Z}
\end{tikzcd}$$

$$\bar{a} \overset{m}{\longmapsto} a \qquad \bar{a} \overset{n}{\longmapsto} \hat{a}$$

Assuming n to be f.s., we can go the other way: from any given $\hat{a}$, we get first $\bar{a}$, then a.

We have looked at one of the five 'kinds': $\underset{\sim}{I}$ from $\{\underline{O}, \underline{A}, \underline{I}, \underline{E}$ and $\underset{\sim}{I}\}$, of L_{cat}. Similar imagery (even simpler ones), apply to the other four as the kind K in the definition of equivalence. For instance, since $\dim \underline{O} = 0$, the f.s.-at-$\underline{O}$ property of $m: P \to M$ is just to say that $m_{\underline{O}} : P\underline{O} \to M\underline{O}$

is a surjective function.

The following proposition is stated here because its formulation fits into the present context. Its proof will be given later when we discuss the FOLDS axioms for "category".

Proposition

$$\mathbb{X} \simeq \mathbb{A} \iff \tilde{\mathbb{X}} \simeq_{L_{cat}} \tilde{\mathbb{A}} \qquad (*)$$

↑ usual category equivalence

Reminder: for categories $\mathbb{X}$, $\mathbb{A}$, $\mathbb{X} \simeq \mathbb{A}$ means the existence of the following situation:

$$\begin{tikzcd} \mathbb{X} \arrow[loop left, "FG \cong 1_{\mathbb{X}}"] \arrow[r, shift right, "G"'] & \mathbb{A} \arrow[l, shift right, "F"'] \arrow[loop right, "GF \cong 1_{\mathbb{A}}"] \end{tikzcd} \qquad (**)$$