FOLDS / M. Makkai

Logic Department
Eötvös University
Budapest

March 2013

Technical notes

Contents:

Part 2 SYNTAX AND SEMANTICS

Part 3 FOLDS EQUIVALENCE

Part 1 GENERALITIES

Category: $\mathbb{C}$

has: <u>objects</u>;

elements of the class $Ob(\mathbb{C})\ (=\mathbb{C}_0)$

<u>arrows</u>

elements of the class $Arr(\mathbb{C})\ (=\mathbb{C}_1)$

<u>domain</u> (source) & <u>codomain</u> (target)

(objects) of every arrow: for $f \in Arr(\mathbb{C})$.

$$X \xrightarrow{f} Y \qquad \text{or} \qquad f: X \to Y$$

indicates $X = df$ ($=$ domain (f))

$Y = cf$ ($=$ codomain (f)).

<u>composition</u> of composable arrows:

given

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "f\cdot g"'] & & Z \end{tikzcd}$$

$$f\cdot g = g\circ f \quad = gf$$

<u>conditional</u> (partial) operation

$$(f, g) \longmapsto f\cdot g$$

subject to: $cf = dg$

(implicit) law : $d(f \cdot g) = df$

$c(f \cdot g) = cg$

identity arrow

$$X \in \mathbb{C}_0 \longmapsto X \xrightarrow{1_X} X$$

associative law :

$$\begin{tikzcd} & \cdot \arrow[r, "g"] & \cdot \arrow[dr, "h"] & \\ \cdot \arrow[ur, "f"] \arrow[urr, bend right, "f \cdot g"'] \arrow[rrr, bend right, "(f \cdot g) \cdot h"'] & & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[r, "g"] \arrow[drr, bend right, "g \cdot h"'] & \cdot \arrow[dr, "h"] & \\ \cdot \arrow[ur, "f"] \arrow[rrr, bend right, "f \cdot (g \cdot h)"'] & & & \cdot \end{tikzcd}$$

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

whenever $f \cdot g$ & $g \cdot h$ are well-defined
(which imply, before the equality,
that the two sides are parallel)

unit laws

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, "1_X"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\#"] & & Y \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dr, "1_Y"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\#"] & & Y \end{tikzcd}$$

(# indicates commutation)

Example for category:

Set

objects: sets $X, Y, \ldots$

arrows: functions $f: X \to Y$

composition:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "f \cdot g"'] & & Z \end{tikzcd}$$

$$f \cdot g = g \circ f$$

$x \in X$: $(g \circ f)(x) = g(f(x))$

Another example (the 'main' one...):

$\boxed{L_{cat}}$:

$$\begin{tikzcd} \underset{\sim}{I} \arrow[dr, "i"'] & \underset{\sim}{E} \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & \underset{\sim}{T} \arrow[dl, shift left=2, "t_0"] \arrow[dl, "t_1"] \arrow[dl, shift right=2, "t_2"'] \\ & \underset{\sim}{A} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & \\ & \underset{\sim}{O} & \end{tikzcd}$$

$j \underset{def}{=} di = ci$

$f_0 \overset{def}{=} de_0 = de_1$

$f_1 \overset{def}{=} ce_0 = ce_1$

$dt_0 = dt_2 \underset{def}{=} s_0$

$ct_0 = dt_1 \underset{def}{=} s_1$

$ct_1 = ct_2 \underset{def}{=} s_2$

$$Ob(L_{cat}) = \{ \underset{\sim}{O}, \underset{\sim}{A}, \underset{\sim}{I}, \underset{\sim}{E}, \underset{\sim}{T} \}$$

$$\begin{aligned} Arr(L_{cat}) = \{ & d, c \\ & i, j \\ & e_0, e_1, f_0, f_1 \\ & t_0, t_1, t_2, s_0, s_1, s_2 \} \\ & \cup \{ \text{identity arrows} \} \end{aligned}$$

$\boxed{\text{Functor}}$:

$\mathbb{C}$, $\mathbb{S}$ categories

$F : \mathbb{C} \longrightarrow \mathbb{S}$ functor

has : $Ob(\mathbb{C}) \xrightarrow{F} Ob(\mathbb{S})$

$Arr(\mathbb{C}) \xrightarrow{F} Arr(\mathbb{S})$

subject to :

$$\underset{\text{in } \mathbb{C}}{X \xrightarrow{f} Y} \quad \Rightarrow \quad \underset{\text{in } \mathbb{S}}{FX \xrightarrow{Ff} FY}$$

$$\underset{\text{in } \mathbb{C}}{f \cdot g = h} \quad \Rightarrow \quad \underset{\text{in } \mathbb{S}}{Ff \cdot Fg = Fh}$$

$$F(1_X) = 1_{FX}$$

<u>Example (s):</u>

functor

$$M : L_{cat} \longrightarrow Set$$

called an L_{cat} - <u>structure</u>

has sets and functions as follows:

$$\begin{tikzcd}
M\underset{\sim}{I} \arrow[dr, "Mi"'] & M\underset{\sim}{E} \arrow[d, shift right, "Me_0"'] \arrow[d, shift left, "Me_1"] & M\underset{\sim}{T} \arrow[dl, shift right, "Mt_0"'] \arrow[dl, "Mt_1"] \arrow[dl, bend left, "Mt_2"] \\
 & M\underset{\sim}{A} \arrow[d, shift right, "Md"'] \arrow[d, shift left, "Mc"] & \\
 & M\underset{\sim}{O} &
\end{tikzcd}$$

such that :

$$\underbrace{(Md)(Mi)}_{\text{composition of functions}} = (Mc)(Mi) = Mj$$

etc.

In particular: suppose $\mathbb{C}$ is a <u>small</u> category: $Ob(\mathbb{C})$, $Arr(\mathbb{C})$ are objects of Set. We have the following L_{cat}-structure

$$\tilde{\mathbb{C}} : L_{cat} \longrightarrow Set$$

$$\tilde{\mathbb{C}}\,\underset{\sim}{O} \overset{def}{=} Ob(\mathbb{C}), \qquad \tilde{\mathbb{C}}\,\underset{\sim}{A} \overset{def}{=} Arr(\mathbb{C})$$

$$\begin{array}{ccc} & \tilde{\mathbb{C}}\,\underline{A} & \\ (\tilde{\mathbb{C}}d =)\, d & \downarrow \quad \downarrow & c (= \tilde{\mathbb{C}}c) \\ & \tilde{\mathbb{C}}\,\underline{O} & \end{array} \qquad \begin{array}{c} f \\ \swarrow \quad \searrow \\ df \qquad cf \end{array}$$

$$\tilde{\mathbb{C}}\,\underset{\sim}{I} \overset{def}{=} Ob(\mathbb{C}) = \tilde{\mathbb{C}}\,\underset{\sim}{O}$$

$$\begin{array}{ccc} \tilde{\mathbb{C}}\,\underset{\sim}{I} & & X \\ \downarrow i & & \downarrow \\ = \tilde{\mathbb{C}}\,\underset{\sim}{A} & & 1_X \end{array}$$

$$\tilde{\mathbb{C}}\,\underset{\sim}{E} \overset{\text{def}}{=} \tilde{\mathbb{C}}\,\underset{\sim}{A}$$

$$\begin{array}{c} \tilde{\mathbb{C}}\,\underline{E} \\ e_0 \Big\downarrow \quad \Big\downarrow e_1 \\ \tilde{\mathbb{C}}\,\underset{\sim}{A} \end{array}$$

: both identities

$\tilde{\mathbb{C}}\,\underset{\sim}{T}$ = set of commutative triangles in $\mathbb{C}$;

$$= \left\{ (X,Y,Z,f,g,h) : \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd},\ h = g \circ f \right\}$$

If $\tau = (X, Y, Z, f, g, h) \in \tilde{\mathbb{C}}\,T$,
$\qquad\quad s_0, s_1, s_2, t_0, t_1, t_2$

then $(\tilde{\mathbb{U}}t_0)\,\tau \underset{\text{abbrev.}}{=} t_0\tau = f$; e.t.c.

Important $\check{\mathbb{C}}$ is indeed a functor ; e.g.

$$(\tilde{\mathbb{C}}d)\circ(\tilde{\mathbb{C}}i) = (\tilde{\mathbb{C}}c)\circ(\tilde{\mathbb{C}}i)$$

because $X \in \tilde{\mathbb{C}}\,\underset{\sim}{I} \Rightarrow (\tilde{\mathbb{C}}i)(X) = 1_X$, and

$$(\tilde{\mathbb{C}}d)(1_X) = (\tilde{\mathbb{C}}c)(1_X) = X.$$

$\boxed{\text{FOLDS signature}}$

A category L is a (FOLDS-)signature (standing in for "similarity type" of model theory) if:

1) L is co-well founded:

<u>there is no</u> ω-type sequence

$$K_0 \xrightarrow[f_0]{} K_1 \xrightarrow[f_1]{} \cdots \xrightarrow[f_{n-1}]{} K_n \xrightarrow[f_n]{} \cdots$$

of <u>non-identity</u> arrows ($f_n \neq id_{K_n}$)

2) for every $K \in Ob(L)$, the fan-out of K

$$L|K \underset{def}{=} \{ f \in Arr(L) : dom(f) = X \;(\& f \neq id_X) \}$$

is a finite <u>set</u>.

<u>Automatically</u>: L is skeletal ($K \cong K' \Rightarrow K = K'$) and 'one-way':

$$f : K \to K \quad \Rightarrow f = 1_K.$$

If L is a (FOLDS-) signature, then

$$Ob(L) = \dot{\bigcup_{n<\omega}} L_n$$

where for $K \in Ob(L)$

$K \in L_n \iff$ 1) $K \notin \bigcup_{m<n} L_m$

& 2) for all $p : K \longrightarrow K_p$, $p \neq id_K$, $K_p \in \bigcup_{m<n} L_m$.

We say

$$\dim(K) = n \iff K \in L_n$$

<u>Example</u>: In L_{cat},

$\dim(\underset{\smile}{O}) = 0$

$\dim(\underset{\smile}{A}) = 1$

$\dim(\underset{\smile}{I}) = \dim(\underset{\smile}{E}) = \dim(\underset{\sim}{T}) = 2$

Contexts

Let L be a (FOLDS-) signature

An (L-) <u>context</u> is a <u>finite</u> functor:

$$C : L \to \mathrm{Set}$$

meaning the set

$$\dot{\bigcup_{K \in Ob(L)}} C(K) \quad \text{is a finite set;}$$

disjoint union

we also <u>require</u> that $C(K) \cap C(K') = \emptyset$ for $K \neq K'$.

In particular, each $C(K)$ is finite

– and all but finitely many of the sets $C(K)$ are empty

(note: there are important <u>infinite</u> signatures).

Example for functor: <u>representable functor</u>:

Given any category $\mathbb{C}$ (with small hom-sets), and any object X in $\mathbb{C}$, we have the representable functor

$$\hat{X} \underset{\text{def}}{=} \hom_{\mathbb{C}}(X, -) : \mathbb{C} \longrightarrow \mathbf{Set}$$

defined by:

$$\hat{X}(U) = \hom_{\mathbb{C}}(X, U) = \{\text{all } X \to U\}$$

$(U \in \mathrm{Ob}(\mathbb{C}))$

$$\begin{tikzcd} U \arrow[d, "f"'] \\ V \end{tikzcd} \qquad \begin{tikzcd} \hat{X}(U) \arrow[d, "\hat{X}(f) = f\circ(-)"] \\ \hat{X}(V) \end{tikzcd} \qquad \begin{tikzcd} X \to U \arrow[d, mapsto] \\ X \to U \xrightarrow{f} V \end{tikzcd}$$

When $\mathbb{C} = \mathbb{L}$ a signature

$X = K \in \mathrm{Ob}(\mathbb{L})$ (a '<u>kind</u>')

then

$$\hat{K} = \hom_{\mathbb{L}}(K, -) \quad \text{is a}$$

context (finite functor); this is condition 2) in the def'n of "signature".

<u>Example</u> For $L = L_{cat}$

$$K = \underset{\sim}{T}$$

$\hat{T} : L \longrightarrow \mathrm{Set}$ has: picture:

$$\left.\begin{aligned} \hat{T}(\underline{O}) &= \{s_0, s_1, s_2\} \\ \hat{T}(\underline{A}) &= \{t_0, t_1, t_2\} \\ \hat{T}(\underset{\sim}{I}) = T(\underset{\sim}{E}) &= \emptyset \\ \hat{T}(\underset{\sim}{T}) &= \{1_{\underset{\sim}{T}}\} \end{aligned}\right\}$$

$$\begin{tikzcd} & s_1 \arrow[dr, "t_1"] & \\ s_0 \arrow[ur, "t_0"] \arrow[rr, bend right, "t_2"'] \arrow[rr, phantom, "1_{\underset{\sim}{T}}"] & & s_2 \end{tikzcd}$$

$\boxed{\text{Niches}}$: further examples for contexts:

given $K \in L$ (L : signature)

then $\overset{\circ}{K} : L \longrightarrow \mathrm{Set}$

is the following functor

$$\overset{\circ}{K}(U) = \begin{cases} \hom_L(K, U) & \text{if } \dim U < \dim K \\ \emptyset & \text{otherwise} \end{cases}$$

$\overset{\circ}{K}(U \xrightarrow{f} V) = (-) \circ f$ (see def^n of $\hat{K}$)

works since $U \xrightarrow{f} V \Rightarrow \dim V \leq \dim U$

$\mathring{K}$ is a <u>subfunctor</u> of $\hat{K}$:

for every $X \in Ob(\mathbb{L})$,

$$\mathring{K}(X) \subseteq \hat{K}(X)$$

and in fact

$$\mathring{K}(X) = \hat{K}(X)$$

for all X <u>except</u> for $X = K$,

when

$$\mathring{K}(K) = \emptyset \quad \& \quad \hat{K}(K) = \{1_K\}:$$

$\mathring{K}$ misses just one 'element' of $\hat{K}$

$\mathring{K}$ is called the (abstract) <u>K-niche</u>.

Natural transformation

Suppose categories and functors:

$$\mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{S}$$

A <u>natural</u> <u>transformation</u>

$$h : F \longrightarrow G$$

or:

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] & \mathbb{S} \arrow[from=F, to=G, Rightarrow, "h"] \end{tikzcd}$$

is a mapping of <u>objects</u> of $\mathbb{C}$ to <u>arrows</u> of $\mathbb{S}$ as follows:

for $X \in Ob(\mathbb{C})$

$$\underbrace{h(X) = h_X}_{\text{also written}} : FX \longrightarrow GX$$

<u>such that</u> every time $f : X \to Y$ in $\mathbb{C}$, we have

$$\begin{tikzcd} FX \arrow[r, "h_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\#"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "h_Y"'] & GY \end{tikzcd}$$

Commuting.

Ideology:

Let: L be a signature.

A Set-valued L-functor

$$M : L \to \mathrm{Set}$$

is said to be an <u>L-structure</u>

(and $M : L \to \mathcal{S}$, $\mathcal{S}$ any category, an $\mathcal{S}$-<u>valued</u> <u>L-structure</u>).

Basic notion of <u>semantics</u> of FOLDS.

But $C : L \to \mathrm{Set}$ finite, a context, is then also an L-structure – although a context is a <u>syntactical</u> object.

Typical point of categorical logic:

syntax and semantics <u>mix</u>.

This is a good thing, sometimes; ~~m~~ yields economical formulations.

Further words: given L-structure $M : L \to \mathbf{Set}$, and context $C : L \to \mathbf{Set}$, a natural transformation

$$\left\{ \begin{array}{l} \alpha : C \to M, \text{ that is:} \\ \begin{tikzcd} L \arrow[r, bend left, "C"{name=U}] \arrow[r, bend right, "M"'{name=D}] & \mathbf{Set} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd} \end{array} \right.$$

is an <u>evaluation</u> of C in M.

<u>Remarks</u> In 'usual' logic (say, Chang & Keisler: Model Theory), we have the satisfaction relation: $M \models \varphi[\alpha]$

structure

formula

evaluation (of free variables)

We have our <u>structures</u>; we don't have 'formulas' yet; but the evaluations (variable evaluations) we just introduced. They are more complicated than in 'usual' logic

— · —

Homomorphism, isomorphism

A natural transformation h between L-structures:

$$\left\{ \begin{array}{l} M \xrightarrow{h} N \\ \begin{tikzcd} L \arrow[r, bend left, "M"{name=U}] \arrow[r, bend right, "N"'{name=D}] \arrow[from=U, to=D, Rightarrow, "h"] & \mathrm{Set} \end{tikzcd} \end{array} \right.$$

is what corresponds to <u>homomorphism</u> in algebra, and (ordinary) model theory.

A homomorphism is a "structure preserving mapping".

<u>Example</u>: Given categories $\mathbb{C}$ and $\mathbb{D}$, functors $F : \mathbb{C} \longrightarrow \mathbb{D}$ are in a <u>bijective</u> correspondence with homomorphisms

$$\begin{tikzcd} L \arrow[r, bend left, "\tilde{\mathbb{C}}"{name=U}] \arrow[r, bend right, "\tilde{\mathbb{D}}"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\tilde{F}"] & \mathrm{Set} \end{tikzcd}$$

(for $\tilde{\mathbb{C}}$, $\tilde{\mathbb{D}}$: see p. A6)

Verify!

(If this was not so, something would be wrong with the "ideology".)

An <u>isomorphism</u>

$$h : M \xrightarrow{\cong} N$$

of L-structures is a homomorphism that has an inverse: there exists h^{-1}

$$\begin{tikzcd} M \arrow[loop left, "h^{-1}h \,=\, 1_M"] \arrow[r, shift left, "h"] & N \arrow[l, shift left, "h^{-1}"] \arrow[loop right, "1_N \,=\, hh^{-1}"] \end{tikzcd}$$

here: $1_M : M \longrightarrow M$, the <u>identity</u> homomorphism,

$$\begin{tikzcd} L \arrow[r, bend left, "M"{name=U}] \arrow[r, bend right, "M"'{name=D}] & Set \arrow[from=U, to=D, Rightarrow, "1_M"] \end{tikzcd}$$

has: $h_K : MK \longrightarrow MK = 1_{MK}$ (in Set)

$h : M \to N$ is an isomorphism

iff for all $K \in \mathbb{L}$, $h_K : MK \to MK$ is a bijection (an isomorphism in the category Set)

$\boxed{\text{The functor category } [\mathbb{C}, \mathbb{D}]}$

Given any two categories, $\mathbb{C}$ and $\mathbb{D}$, the functors $\mathbb{C} \to \mathbb{D}$ and the natural transformations $\mathbb{C} \overset{\longrightarrow}{\underset{\longrightarrow}{\downarrow}} \mathbb{D}$ form a category – the former as objects, the latter as arrows. The composition of nt's (natural transformations)

$$\begin{tikzcd}[column sep=huge]
\mathbb{C} \arrow[r, shift left=5, "F"] \arrow[r, "G" description] \arrow[r, shift right=5, "H"'] \arrow[r, phantom, shift left=2.5, "\downarrow h"] \arrow[r, phantom, shift right=2.5, "\downarrow k"] \arrow[r, phantom, shift left=8, "kh"] & \mathbb{D}
\end{tikzcd}$$

$X \in \mathrm{ob}\,\mathbb{C}$: $(kh)_X = k_X \circ h_X$

Note: diagram: $X \xrightarrow{f} Y$ in $\mathbb{C}$ generates:

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "h_X"'] \arrow[dd, bend right=60, "(kh)_X"'] \arrow[dr, phantom, "\#"] & FY \arrow[d, "h_Y"] \arrow[dd, bend left=60, "(kh)_Y"] \\
GX \arrow[r, "Gf"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\#"] & GY \arrow[d, "k_Y"] \\
HX \arrow[r, "Hf"'] & HY
\end{tikzcd}$$

$$\Longrightarrow \quad \begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "(kh)_X"'] \arrow[dr, phantom, "\#"] & FY \arrow[d, "(kh)_Y"] \\
HX \arrow[r, "Hf"'] & HY
\end{tikzcd}$$

'#' says: 'commutes'.

this was the proof that kh is indeed an nt.

Said category is denoted: $[\mathbb{C}, \mathbb{D}]$

(or $\mathbb{D}^{\mathbb{C}}$; or $\mathrm{Hom}(\mathbb{C}, \mathbb{D})$)

For L a signature,

$$\mathrm{Str}(L) \underset{\mathrm{def}}{=} [L, \mathrm{Set}]$$

the <u>category</u> of <u>L-structures</u>.

Part 2 SYNTAX AND SEMANTICS

Example of a context over L_{cat}

Recall L_{cat} from A3, the definition of 'context' from A10. Let $L = L_{cat}$. We define a context $\mathcal{C} = \mathcal{C}_{2\text{-nd-unit-law}} : L \longrightarrow \mathrm{Set}$ related to the "2nd unit law for categories",

$1_Y \circ f = f$, in the situation

$$\begin{tikzcd} & Y \arrow[dr, "1_Y"] & \\ X \arrow[ur, "f"] \arrow[rr, "f"'] & \# & Y \end{tikzcd}$$

In the context $\mathcal{C}$, there will be six entities – symbols; but <u>arbitrary</u> symbols! – involved

Besides X, Y, f and $i = 1_Y$, two more: φ: that testifies that i is an (the) identity arrow on Y; and τ: testifying that the triangle commutes

Here is the definition of the functor

$$C : L \longrightarrow Set ::$$

$C(\underline{O}) = \{X, Y\}$

$C(\underline{A}) = \{f, i\}$; $df = X$, $cf = Y$, $di = Y$, $ci = Y$

$C(\underline{I}) = \{\varphi\}$; $\underline{i}\varphi = i$, $\underline{j}\varphi = Y$

$C(\underset{\sim}{T}) = \{\tau\}$; $s_0\tau = X$, $s_1\tau = Y$, $s_2\tau = Y$,
$t_0\tau = f$, $t_1\tau = i$, $t_2\tau = f$;

$C(\underline{E}) = \emptyset$

When giving the definition above of the effect of C on an arrow $\underline{a}$ in L, instead of "$C(\underline{a})$" we just write "$\underline{a}$". Thus; "$df = X$" really means "$C(d)(f) = X$", etc.

We also omitted the underlinings from most of the notations of the arrows of L — not so for $\underline{i}$, $\underline{j}$!

Of course, one needs to check that the definition gives a functor; e.g.,

$C(d) \circ C(t_0) = C(s_0)$; i.e $d(t_0\, \tau) = s_0\, \tau$; etc.

In what follows, we will use this example to illustrate the concepts to come.

Abbreviated notation for contexts

Let $C : L \to \mathbf{Set}$ be a context over the signature L (see A10). We want to think of the context as a set of variables, organized (structured) in a certain way. We take the set

$$|C| = \dot{\bigcup_{K \in Ob(L)}} C(K)$$

by definition, this is a finite set. Its elements are called variables – variables of the

an assigned __kind__ : $K \in ob(L)$ such that $x \in C(K)$; we write $x :: K$ for : x is of the kind K.
(again, suppressing C from the notation).

For any __proper__ (non-identity arrow) $p : K \to K_p$
($K_p \underset{def}{=}$ codomain (p); $\dim K_p < \dim K$),

and $x :: K$, we let

$$p\,x \underset{def}{=} C(p)(x)$$

$$(\text{note: } C(p) : C(K) \longrightarrow C(K_p)$$
$$x \longmapsto (Cp)(x) = p\,x\,)$$

(This is the notation that was used in the specification of the example above; C is suppressed.)

Thus, we have

$$x :: K \quad \Longrightarrow \quad p\,x :: K_p$$

We write $\boxed{\underline{dep}(x) = \{p\,x : p \in L/K\}}$ $(x :: K)$

x __depends__ on each $y \in dep(x)$.

The fact that C is a <u>functor</u> is expressed in this way:

every time $K \xrightarrow{p} K_p \xrightarrow{q} K_q$,

and $x :: K$,

we have $\boxed{q(px) = (qp)x}$;

thus, we can write 'qpx' without fear of error.

Now, suppose we have two contexts C, D over L, and map $\Phi : C \to D$;

since C, D are functors, Φ is a natural transformation; let's call Φ a <u>change of variables</u>. Φ can be regarded as a mapping

$$\Phi : |C| \longrightarrow |D|$$

such that $x :: K$ in $C \Rightarrow \Phi x :: K$ in D

and $px = y$ in $C \Rightarrow p(\Phi x) = \Phi y$ in D

i.e. $\boxed{\Phi(px) = p(\Phi x).}$

An <u>example</u> is the important context

$$\overset{\circ}{K} : L \longrightarrow Set$$

$(K \in Ob(L))$; see A 12

Now, $|\overset{\circ}{K}| = \underbrace{K\|L}_{\text{new notation}} \overset{\text{def}}{=}$

$$\overset{\text{def}}{=} \{ p \in Arr(L) : dom(p) = K \;\&\; p \neq id_K \}$$

Let us write K_p for codomain (p).

In the context $\overset{\circ}{K}$, $p :: K_p$. Moreover,

when $K_p \xrightarrow{q} K_q$, an arrow in L with domain $= K_p$, i.e., $K \xrightarrow{p} K_p \xrightarrow{q} K_q$,

then

$$p :: K_p \quad \Rightarrow \quad qp :: K_q$$

in the just-introduced sence of the notation 'qp'

<u>and</u> also in the sense of composition : $qp = q \circ p : K \to K_q$.

Suppose

$$L \overset{D}{\underset{C}{\rightrightarrows}} Set$$

are contexts. We say that D is a <u>subcontext</u> of C (in symbols: $D \subseteq C$) if there is a map $i: D \to C$ whose components $i_K : DK \to CK$ are all set-inclusions: for $x \in DK$, $i_K(x) = x$.

In other words, $D \subseteq C$ if $DK \subseteq CK$ (DK is a subset of CK) <u>and</u>, for $K \xrightarrow{p} K_p$, $x \in DK$, we have that $px \in DK_p$ in the sense of D is the same as $px \in CK_p$ in the sense of C. Also, for a context C, and (separate) sets DK, one for each $K \in Ob(L)$, a subset of CK, the sets DK, $K \in Ob(L)$, <u>determine</u> a subcontext iff for each $K \xrightarrow{p} K_p$, and $x \in DK$, px (<u>in the sense of</u> C) is in the set DK_p.

In other words, a <u>subcontext</u> of a fixed context C is a subset D of C that is closed under the operation of multiplying with any 'face-operator' p

$$\underline{x \in D \quad \&} \quad px \text{ is meaningful}$$

$$(\text{dom } p = K,\ x :: K)$$

$$\underline{\quad \Longrightarrow \quad}$$

$$\underline{px \in D}.$$

Note the immediate consequence: subcontexts are closed under union: $D_1 \subseteq_{sc} C$, $D_2 \subseteq_{sc} C$

$$\Rightarrow D_1 \cup D_2 \subseteq_{sc} C$$

Also, the empty set $\emptyset$ is a subcontext.

Boundaries

Let us fix a context (of variables): C (over a fixed signature L). Let K be a kind in L. With $\mathring{K}$ the K-niche, we call a change-of-variables $\boxed{\beta : \mathring{K} \to C}$ a <u>boundary</u> (maybe, the word is not so good: "boundary <u>of</u> something" is the natural use; we <u>will</u> have this too, but right now, β is a boundary that may be empty; a boundary without being a boundary of something).

Let us see some boundaries in the example

$C = C_{2nd\text{-}unit\text{-}law}$ [A21]

First of all, a more complete picture for C:

$$\begin{tikzcd}
 & Y \arrow[dr, "i"] \arrow[dr, phantom, bend left=50, "\varphi"] \arrow[d, phantom, "\tau"] & \\
X \arrow[ur, "f"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

(we have added τ and φ to the picture)

The following is a boundary in $\mathcal{C}$:

$$\begin{tikzcd} & Y \arrow[dr, "i"] & \\ X \arrow[ur, "f"] \arrow[rr, bend right, "f"'] & & Y \end{tikzcd}$$

This is the map $\underline{\beta : |\mathring{T}| \longrightarrow |\mathcal{C}|}$ for which (compare

$$\begin{tikzcd} & s_1 \arrow[dr, "t_1"] & \\ s_0 \arrow[ur, "t_0"] \arrow[rr, bend right, "t_2"'] & & s_2 \end{tikzcd}$$

the picture for $|\mathring{T}|$) has:

$$\boxed{\beta s_0 = X, \ \beta s_1 = Y, \ \beta s_2 = X, \ \beta t_0 = f, \ \beta t_1 = i, \ \beta t_2 = f}$$

Another, "not visible", boundary in $\mathcal{C}$

$$\beta' : |\mathring{\underline{A}}| \longrightarrow |\mathcal{C}|$$

is:

$$X \dashrightarrow X$$

$$\beta' d = X, \quad \beta' c = X.$$

The β above is $\underline{\text{filled}}$; it's the boundary of τ;

β' is <u>empty</u>; there is no arrow $X \dashrightarrow X$ in $\mathcal{C}$ – all according to the following general definition:

given $x \in |\mathcal{C}|$, the boundary of x, ∂x, is given by

$$\boxed{(\partial x)p \underset{\text{def}}{=} px}$$

In more detail: let $x :: K$; for $p \in |\mathring{K}| = L/K$, $p : K \to K_p$, we have px in $\mathcal{C}$; $px = (\mathcal{C}p)x$ (reminders!). The fact that ∂x is in fact a map $\partial x : \mathring{K} \to \mathcal{C}$ is verified by

$$\begin{array}{ccc} (\partial x)(qp) & \overset{?}{=} & q((\partial x)p) \\ \| & & \| \\ (qp)x & \underset{\checkmark}{=} & q(px) \end{array}$$

follows A25

(in the equation): $\Phi(p\,x) = p\,(\Phi\,x)$

the corresponding things: $\partial x\,(q\,p) = q\,((\partial x)\,p\,)$

In the <u>example</u>, the boundary β is the boundary of τ: $\beta = \partial\tau$. The boundary β' is empty: there is no x such that $\partial x = \beta'$.

In general, any number of x's may have the same boundary; we could add another τ' to the context with the same boundary β as τ.

The range of a variable in an interpretation

We now come to the main idea of FOLDS: the range of a variable.

Given a variable $x::K$, and an <u>interpretation</u>, meaning an L-structure M, and an evaluation $\alpha : D \to M$, (D a sub context of the fixed C), that evaluates at least the variables in ∂x (it does not need to evaluate x itself!), (i.e.: $\mathrm{dep}(x) \subseteq D$), we define the <u>range</u> of x, $M_\alpha[x]$ as a certain subset of MK, $M_\alpha[x] \subseteq MK$.

The meanings of the quantifiers $\forall x$, $\exists x$ will then be specified by saying: (under the given interpretation (M, α),)

"$\forall x$" means "for all $a \in M_\alpha[x]$"

"$\exists x$" means "there exists $a \in M_\alpha[x]$".

1) Let us start with the following data:

$\mathbb{L} \xrightarrow{M} \mathbf{Set}$: L-structure;

$K \in Ob(\mathbb{L})$, $a \in MK$.

We can define the boundary of a,

$\partial a : \mathring{K} \longrightarrow M$, exactly as the boundary

$\partial x : \mathring{K} \to C$ was defined for a variable x:

$(\partial a)_u : \mathring{K}[u] \to Mu$ is given by

$$K \xrightarrow{f} u \longmapsto (Mf)a \in Mu$$

$$\left(\text{note: } \begin{array}{ccc} MK & \xrightarrow{Mf} & Mu \\ x & \longmapsto & (Mf)a \end{array}\right)$$

$$\boxed{(\partial a)f = (Mf)a}$$

(: 'contexts' and 'structures' are, both, functors $L \longrightarrow Set$!)

2) Now, let us have

$$L \xrightarrow{M} Set$$

$K \in Ob(L)$, $x :: K$ in C $(x \in C(K))$,

and an evaluation $\alpha : D \longrightarrow M$

for a subcontext $D \subseteq C$ such that

$\partial x : \mathring{K} \longrightarrow C$ is 'in' D, in the sense that

we have:

$$\begin{tikzcd} \mathring{K} \arrow[r, "\partial x"] \arrow[rr, bend right=40, "\partial x"'] & D \arrow[r, "incl."'] \arrow[d, phantom, "\#"] & C \\ & {} & \end{tikzcd}$$

equivalently, for each $p \in L|K$, px (in the sense of C) is in D. We can take the composite:

$$\begin{tikzcd}[column sep=huge] L \arrow[r, bend left=50, "\mathring{K}"{name=U}] \arrow[r, "D"{name=Mi, description}] \arrow[r, bend right=50, "M"'{name=B}] & Set \arrow[from=U, to=Mi, Rightarrow, "\partial x"] \arrow[from=Mi, to=B, Rightarrow, "\alpha"] \end{tikzcd}$$

$$\begin{tikzcd} L \arrow[r, bend left=40, "\mathring{K}"] \arrow[r, bend right=40, "M"'] \arrow[r, phantom, "\Downarrow \alpha(\partial x)"] & \mathrm{Set} \end{tikzcd}$$

$$\alpha(\partial x) \underset{\text{def}}{=} \alpha \circ \partial x ;$$

$\alpha(\partial x)$ is the <u>interpretation</u> of the variable-boundary ∂x in M. Given any $a \in MK$, from 1), we have

$$\begin{tikzcd} L \arrow[r, bend left=40, "\mathring{K}"] \arrow[r, bend right=40, "M"'] \arrow[r, phantom, "\Downarrow \partial a"] & \mathrm{Set} \end{tikzcd}$$

Therefore, it is meaningful to define:

$$M_\alpha[x] \overset{\text{def}}{=} \{ a \in MK : \partial a = \alpha(\partial x) \}$$

that is, for $a \in MK$,

$$\boxed{a \in M_\alpha[x] \iff \partial a = \alpha(\partial x).}$$

In words: the range of the variable $x :: K$ under the interpretation (M, α) is the set of all $a \in MK$ whose boundary is the α-image of the boundary of x.

<u>Example</u>: Returning to our previous example, let $X = \tau$, $D \subseteq C$ the subcontext that misses τ but nothing else in C.

Let $M : L (= L_{cat}) \longrightarrow \mathrm{Set}$. For any $a \in M\underset{\sim}{T}$, $\partial a : \overset{\circ}{\underset{\sim}{T}} \longrightarrow M$, or rather

$$|\partial a| : |\overset{\circ}{\underset{\sim}{T}}| \longrightarrow |M|$$

$$\left(= \dot{\bigsqcup_{K \in L}} MK\right)$$

is given as $(\partial a)s_0 = (Ms_0)a \in M\underline{O}$, ...

$$(\partial a)t_2 = (Mt_2)a \in M\underline{A}$$

(see bottom A31).

Therefore, by $\boxed{\text{box}}$ A31, $\partial x = \beta$ in $\boxed{\text{box}}$ A28, $M_\alpha[x]$ is the set of all $a \in MK$ such that

$\partial a = \alpha(\partial x) = \alpha(\beta)$, that is,

$(Ms_0)(a) = \alpha(X)$, $(Ms_1)(a) = (Ms_2)(a) = Y$,

$(Mt_0)(a) = (Mt_2)(a) = f$, $(Mt_1)(a) = i$

that is, a fits into the α-image of the triangle on top of A 28, i.e., into

$$\begin{tikzcd} & \alpha Y \arrow[dr, "\alpha i"] & \\ \alpha X \arrow[ur, "\alpha f"] \arrow[rr, "\alpha f"'] \arrow[rr, phantom, bend left=20, "a"] & & \alpha Y \end{tikzcd}$$

As expected!

Formulas

Let us fix a context $C : L \to Set$. All 'formulas' will use variables from C. We could make C a "large infinite" context as well (any functor $L \to Set$, in fact) – but this is not important.

The letters $\underset{\sim}{X}$, $\underset{\sim}{Y}$, ... (script X, Y, ...) will always be subcontexts of C. We will not write $|\underset{\sim}{X}|$ even when we mean it; we will just write $\underset{\sim}{X}$.

Recursively, we define 'formula' (C-formula), and "set $\boxed{Var(\varphi)}$ of free variables of formula φ."

<u>Auxiliary definition:</u> for context $\underset{\sim}{X}$ (subcontext of C), let us write:

$$\underset{\sim}{X}\uparrow \overset{\text{def}}{=} \{y \in |C| : \text{for all } x \in \underset{\sim}{X},\ y \notin \mathrm{dep}(x)\}$$

$\underset{\sim}{X}\uparrow$ consists of the variables on which <u>no</u> variable in $\underset{\sim}{X}$ depends. When $y \in \underset{\sim}{X}\uparrow$, y may or may not belong to $\underset{\sim}{X}$ – if it does, it is on the <u>top</u> of $\underset{\sim}{X}$, in the sense of the (partial) order of dependence: – writing $y \prec x$ to mean $y \in \mathrm{dep}(x)$, $\prec$ is an irreflexive partial order.: $y \in \underset{\sim}{X}\uparrow$ iff either y is a $\prec$-maximal element of $\underset{\sim}{X}$, or else $y \notin \underset{\sim}{X}$. Also, $y \in \underset{\sim}{X}\uparrow$ if and only if the set $\underset{\sim}{X} - \{x\}$ is a context.