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Technical notes

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Part 1 GENERALITIES

Category: $\mathbb{C}$

has: objects ;

elements of the class $\text{Ob}(\mathbb{C}) (= \mathbb{C}_0)$

arrows

elements of the class $\text{Arr}(\mathbb{C}) (= \mathbb{C}_1)$

domain (source) & codomain (target)

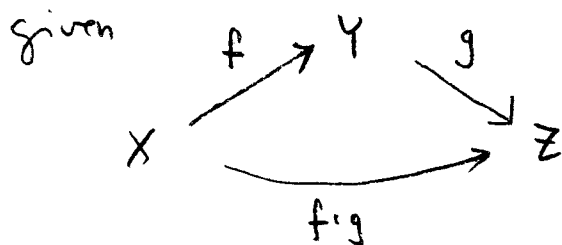
(objects) of every arrow: for $f \in \text{Arr}(\mathbb{C})$.

$$X \xrightarrow{f} Y \quad \text{or} \quad f: X \rightarrow Y$$

indicates $X = df (= \text{domain}(f))$

$Y = cf (= \text{codomain}(f))$.

composition of composable arrows:



$$f \cdot g = g \circ f = \underline{\underline{gf}}$$

conditional (partial) operation

$$(f, g) \mapsto f \cdot g$$

subject to: $cf = dg$

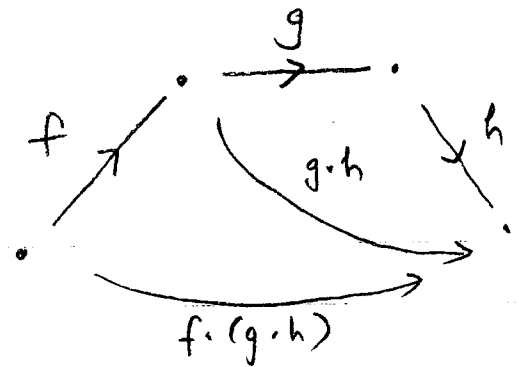
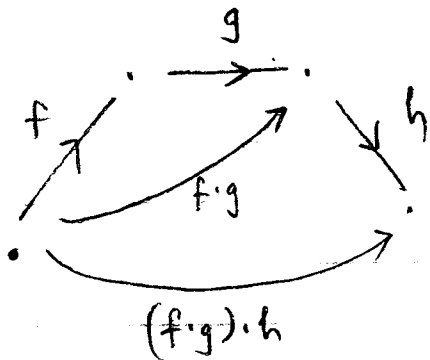
(implicit) law : $d(f \cdot g) = df$

$$c(f \cdot g) = cg$$

identity arrow

$$X \in \mathbb{C}_0 \quad \longmapsto \quad X \xrightarrow{1_X} X$$

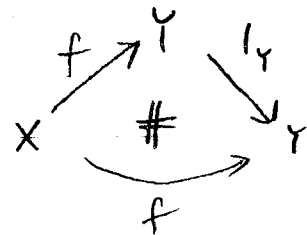
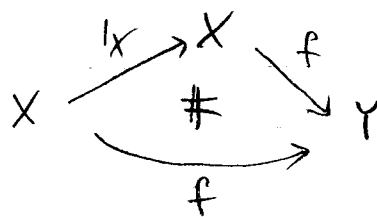
associative law:



$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

whenever $f \cdot g$ & $g \cdot h$ are well-defined
(which imply, before the equality,
that the two sides are parallel)

unit laws



($\#$ indicates commutation)

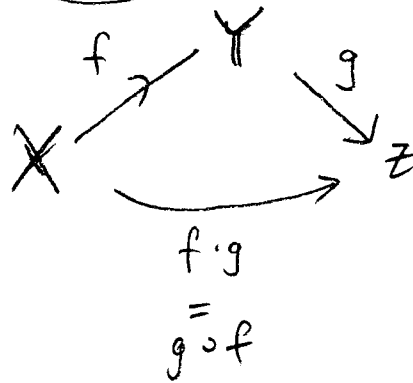
Example for category:

Set

objects: sets $X, Y, \dots$

arrows: functions $f: X \rightarrow Y$

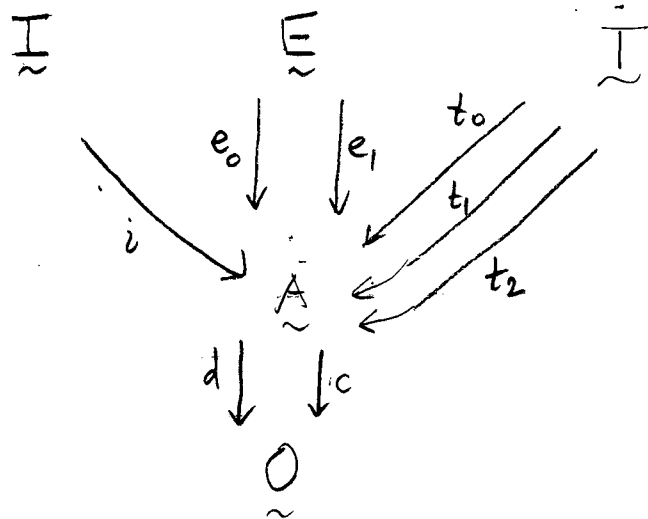
composition:



$$x \in X: (g \circ f)(x) = g(f(x))$$

Another example (the 'main' one...):

$\boxed{L_{cat}:}$



$$j \stackrel{\text{def}}{=} d \circ i = c \circ i$$

$$f_0 \stackrel{\text{def}}{=} d \circ e_0 = d \circ e_1$$

$$f_1 \stackrel{\text{def}}{=} c \circ e_0 = c \circ e_1$$

$$d \circ t_0 = d \circ t_2 \stackrel{\text{def}}{=} s_0$$

$$c \circ t_0 = d \circ t_1 \stackrel{\text{def}}{=} s_1$$

$$c \circ t_1 = c \circ t_2 \stackrel{\text{def}}{=} s_2$$

$$\text{Ob}(L_{\text{cat}}) = \{ \underline{0}, \underline{A}, \underline{I}, \underline{E}, \underline{T} \}$$

$$\begin{aligned} \text{Arr}(L_{\text{cat}}) = \{ & d, c \\ & i, j \\ & e_0, e_1, f_0, f_1 \\ & t_0, t_1, t_2, s_0, s_1, s_2 \} \\ & \cup \{ \text{identity arrows} \} \end{aligned}$$

Functor :

$\mathbb{C}$, $\mathbb{S}$ categories

$F: \mathbb{C} \longrightarrow \mathbb{S}$ functor

$$\text{has: } \text{Ob}(\mathbb{C}) \xrightarrow{F} \text{Ob}(\mathbb{S})$$

$$\text{Arr}(\mathbb{C}) \xrightarrow{F} \text{Arr}(\mathbb{S})$$

subject to:

$$X \xrightarrow[f \text{ in } \mathbb{C}]{} Y \quad \Rightarrow \quad FX \xrightarrow[f \text{ in } \mathbb{S}]{} FY$$

$$f \cdot g = h \text{ in } \mathbb{C} \quad \Rightarrow \quad Ff \cdot Fg = Fh \text{ in } \mathbb{S}$$

$$F(1_X) = 1_{FX}$$

Example (s):

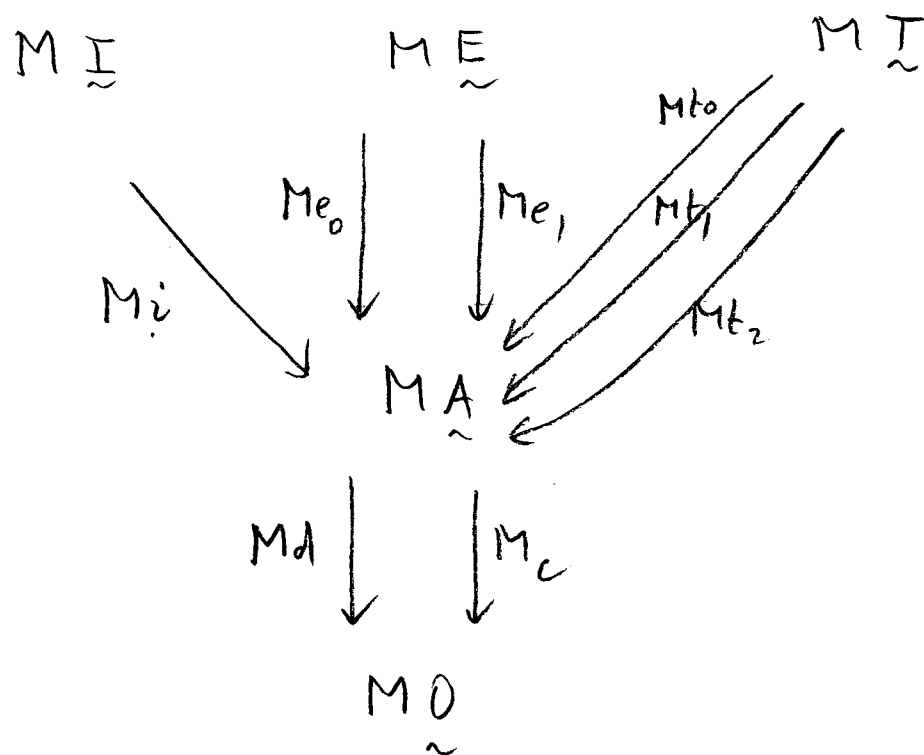
functor

$$M : L_{cat} \longrightarrow Set$$

called an L_{cat} -structure

has sets and functions as

follows:



such that :

$$(M_d)(M_i) = (M_c)(M_i) = M_j$$

etc.

composition of functions

In particular: suppose $\mathcal{C}$ is
 a small category: $\text{Ob}(\mathcal{C}), \text{Arr}(\mathcal{C})$
 are objects of Set . We have the
 following L_{cat} -structure

$$\tilde{\mathcal{C}} : L_{\text{cat}} \longrightarrow \text{Set}$$

$$\tilde{\mathcal{C}} \underline{0} \stackrel{\text{def}}{=} \text{Ob}(\mathcal{C}), \quad \tilde{\mathcal{C}} \underline{A} \stackrel{\text{def}}{=} \text{Arr}(\mathcal{C})$$

$$\begin{array}{ccc}
 & \tilde{\mathcal{C}} \underline{A} & \\
 (\tilde{\mathcal{C}} d =) d \downarrow & & \downarrow c (= \tilde{\mathcal{C}} c) \\
 & \tilde{\mathcal{C}} \underline{0} & \\
 & \downarrow & \\
 & \tilde{\mathcal{C}} \underline{I} & \\
 & \downarrow & \\
 & \tilde{\mathcal{C}} \underline{A} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & f & \\
 \downarrow & & \downarrow \\
 df & & cf
 \end{array}$$

$$\tilde{\mathcal{C}} \underline{I} \stackrel{\text{def}}{=} \text{Ob}(\mathcal{C}) = \tilde{\mathcal{C}} \underline{0}$$

$$\begin{array}{ccc}
 \tilde{\mathcal{C}} \underline{I} & & X \\
 \downarrow i & & \downarrow \\
 = \tilde{\mathcal{C}} \underline{A} & & 1_X
 \end{array}$$

$$\tilde{\mathbb{C}} \stackrel{\text{def}}{=} \tilde{\mathbb{C}}_{\tilde{A}}$$

$$\tilde{\mathbb{C}} \stackrel{\text{def}}{=} \tilde{\mathbb{C}}$$

$$\begin{array}{ccc} e_0 \downarrow & \downarrow e_1 & : \text{ both identities} \\ \tilde{\mathbb{C}}_{\tilde{A}} & & \end{array}$$

$\tilde{\mathbb{C}}_{\tilde{T}} =$ set of commutative triangles in $\tilde{\mathbb{C}}$

$$= \left\{ (X, Y, Z, f, g, h) : \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow h & \nearrow g \\ & & Z \end{array}, h = g \circ f \right\}$$

If $\tau = (X, Y, Z, f, g, h) \in \tilde{\mathbb{C}}_{\tilde{T}}$,
 $s_0, s_1, s_2, t_0, t_1, t_2$

then $(\tilde{\mathbb{C}} t_0) \tau = t_0 \tau = f$; e.t.c.
 abbrev.

Important $\tilde{\mathbb{C}}$ is indeed a functor; (e.g.)

$$(\tilde{\mathbb{C}} d) \circ (\tilde{\mathbb{C}} i) = (\tilde{\mathbb{C}} c) \circ (\tilde{\mathbb{C}} i)$$

because $X \in \tilde{\mathbb{C}}_{\tilde{I}} \Rightarrow (\tilde{\mathbb{C}} i)(X) = 1_X$, and

$$(\tilde{\mathbb{C}} d)(1_X) = (\tilde{\mathbb{C}} c)(1_X) = X.$$

FOLDS signature

A category L is a (FOLDS-)signature
(standing in for "similarity type"
of model theory) if:

1) L is ω -wellfounded:

there is no ω -type sequence

$$K_0 \xrightarrow{f_0} K_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} K_n \xrightarrow{f_n} \dots$$

of non-identity arrows ($f_n \neq \text{id}_{K_n}$)

2) for every $K \in \text{Ob}(L)$, the
fan-out of K

$$L/K \stackrel{\text{def}}{=} \left\{ f \in \text{Arr}(L) : \text{dom}(f) = X \right. \\ \left. (\& f \neq \text{id}_X) \right\}$$

is a finite set.

Automatically: L is skeletal ($K \cong K' \Rightarrow K = K'$)

and 'one-way':

$$f: K \rightarrow K \Rightarrow f = 1_K.$$

If L is a (FOLDS-) signature,
then

$$\text{Ob}(L) = \bigcup_{n < \omega} L_n$$

where for $K \in \text{Ob}(L)$

$$K \in L_n \iff 1) K \notin \bigcup_{m < n} L_m$$

& 2) for all $p: K \rightarrow K_p$, $p \neq \text{id}_K$

$$K_p \in \bigcup_{m < n} L_m.$$

We say

$$\dim(K) = n \iff K \in L_n$$

Example: In L_{cat} ,

$$\dim(\emptyset) = 0$$

$$\dim(\underline{A}) = 1$$

$$\dim(\underline{I}) = \dim(\underline{E}) = \dim(\underline{T}) = 2$$

Contexts

Let L be a (FOLDS-) signature

An $(L-)$ context is a finite functor:

$$C : L \rightarrow \text{Set}$$

meaning the set

$$\bigcup_{K \in \text{Ob}(L)} C(K) \quad \text{is a finite set;}$$

disjoint union

we also require that $C(K) \cap C(K') = \emptyset$ for $K \neq K'$.

In particular, each $C(K)$ is finite

— and all but finitely many of the sets $C(K)$ are empty

(note: there are important infinite signatures).

Example for functor: representable functor:

Given any category $\mathbb{C}$ (with small hom-sets), and any object X in $\mathbb{C}$, we have the representable functor

$$\hat{X} \stackrel{\text{def}}{=} \text{hom}_{\mathbb{C}}(X, -) : \mathbb{C} \longrightarrow \text{Set}$$

defined by:

$$\hat{X}(u) = \text{hom}_{\mathbb{C}}(X, u) = \{ \text{all } X \rightarrow u \}$$

$$(u \in \text{Ob}(\mathbb{C}))$$

$$\begin{array}{ccc}
 u & \hat{X}(u) & X \rightarrow u \\
 f \downarrow & \downarrow \hat{X}(f) = f \circ (-) & \downarrow \\
 v & \hat{X}(v) & X \rightarrow u \xrightarrow{f} v
 \end{array}$$

When $\mathbb{C} = L$ a signature

$X = K \in \text{Ob}(L)$ (a 'kind'))

then

$\hat{K} = \text{hom}_L(K, -)$ is a

context (finite functor); this is

condition 2) in the def'n of "signature".

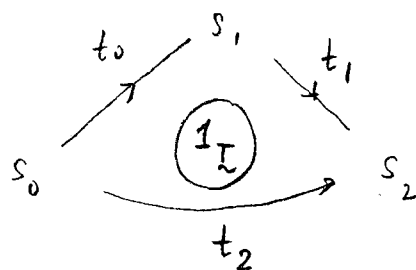
Example For $L = L_{cat}$

$$K = \underline{I}$$

$\hat{T} : L \longrightarrow \text{Set}$ has:

$$\left. \begin{aligned} \hat{T}(\underline{O}) &= \{s_0, s_1, s_2\} \\ \hat{T}(\underline{A}) &= \{t_0, t_1, t_2\} \\ \hat{T}(\underline{I}) &= T(\underline{E}) = \emptyset \\ \hat{T}(\underline{I}) &= \{1_{\underline{I}}\} \end{aligned} \right\}$$

picture:



Niches : further examples for contexts:

given $K \in L$ (L : signature)

then $\overset{o}{K} : L \longrightarrow \text{Set}$

is the following functor

$$\overset{o}{K}(U) = \begin{cases} \text{hom}_L(K, U) & \text{if } \dim U < \dim K \\ \emptyset & \text{otherwise} \end{cases}$$

$$\overset{o}{K}(U \xrightarrow{f} V) = (-) \circ f \quad (\text{see definition of } \overset{o}{K})$$

works since $U \xrightarrow{f} V \Rightarrow \dim V \leq \dim U$

$\overset{\circ}{K}$ is a subfunctor of $\hat{K}$:

for every $X \in \text{Ob}(L)$,

$$\overset{\circ}{K}(X) \subseteq \hat{K}(X)$$

and in fact

$$\overset{\circ}{K}(X) = \hat{K}(X)$$

for all X except for $X = K$,

when

$$\overset{\circ}{K}(K) = \emptyset \quad \& \quad \hat{K}(K) = \{1_K\}:$$

$\overset{\circ}{K}$ misses just one 'element' of $\hat{K}$

$\overset{\circ}{K}$ is called the (abstract) K -niche.

Natural transformation

Suppose categories and functors:

$$\begin{array}{ccc} \textcircled{1} & \xrightarrow{F} & \textcircled{2} \\ & \xrightarrow{G} & \end{array}$$

A natural transformation

$$h: F \rightarrow G$$

or:

$$\textcircled{1} \quad \begin{array}{ccc} & F & \\ \xrightarrow{\quad} & \downarrow h & \xrightarrow{\quad} \\ & G & \end{array} \quad \mathcal{B}$$

is a mapping of objects of $\mathcal{C}$
to arrows of $\mathcal{B}$ as follows:

for $X \in \text{Ob}(\mathcal{C})$

$$h(X) = h_X : FX \rightarrow GX$$

also written

such that every time $f: X \rightarrow Y$ in $\mathcal{C}$,

we have

$$\begin{array}{ccccc} FX & \xrightarrow{h_X} & GX \\ Ff \downarrow & \# & \downarrow Gf \\ FY & \xrightarrow{h_Y} & GY \end{array}$$

Commuting.

Ideology:

Let: L be a signature.

A Set-valued L -functor

$$M : L \rightarrow \text{Set}$$

is said to be an L -structure

(and $M : L \rightarrow \mathcal{S}$, $\mathcal{S}$ any category, an $\mathcal{S}$ -valued L -structure).

Basic notion of semantics of FOLDS.

But $C : L \rightarrow \text{Set}$ finite, a context, is then also an L -structure — although a context is a syntactical object.

Typical point of categorical logic:

syntax and semantics mix.

This is a good thing, sometimes;

mayfields: economical formulations.

Further words: given L -structure

$M : L \rightarrow \text{Set}$, and context

$C : L \rightarrow \text{Set}$, a natural transformation

$$\left\{ \begin{array}{ccc} & \alpha : C \rightarrow M, \text{ that is:} & \\ & \downarrow & \\ L & \xrightarrow{\quad C \quad} & \text{Set} \\ & \xrightarrow{\quad M \quad} & \end{array} \right.$$

is an evaluation of C in M .

Remarks In 'usual' logic (say, Chang & Keisler:

Model Theory), we have the satisfaction

$$\text{relation: } M \models \varphi[\alpha]$$

$\nearrow$ structure $\uparrow$ formula $\nwarrow$ evaluation (of free variables)

We have our structures; we don't have 'formulas' yet; but the evaluations (variable evaluations) we just introduced. They are more complicated

than in 'usual' logic

Homomorphism, isomorphism

A natural transformation h between L -structures:

$$\left\{ \begin{array}{ccc} M & \xrightarrow{h} & N \\ L & \begin{array}{c} \xrightarrow{M} \\ \downarrow h \\ \xrightarrow{N} \end{array} & \text{Set} \end{array} \right.$$

is what corresponds to homomorphism in algebra, and (ordinary) model theory.

A homomorphism is a "structure preserving mapping".

Example: Given categories $\mathbb{C}$ and $\mathbb{D}$,

functors $F : \mathbb{C} \longrightarrow \mathbb{D}$ are

in a bijection correspondence with

homomorphisms

$$L \begin{array}{ccc} \tilde{\mathbb{C}} & \xrightarrow{\quad} & \\ \downarrow \tilde{F} & & \\ \tilde{\mathbb{D}} & \xrightarrow{\quad} & \text{Set} \end{array}$$

(for $\tilde{C}, \tilde{D}$: see p. A6)

Verify!

(If this was not so, something would be wrong with the "ideology".)

An isomorphism

$$h: M \xrightarrow{\cong} N$$

of L -structures is a homomorphism

that has an inverse: there exists h^{-1}

$$\begin{array}{ccc} \begin{array}{c} \text{h}^{-1} \\ \text{h} \end{array} \bigcirc \begin{array}{c} \text{=} \\ \text{I}_M \end{array} \bigcirc M & \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{h^{-1}} \end{array} & N \bigcirc \begin{array}{c} \text{I}_N = \\ \text{h h}^{-1} \end{array} \end{array}$$

here: $\text{I}_M: M \rightarrow M$, the
identity homomorphism,

$$L \begin{array}{c} \xrightarrow{M} \\ \downarrow \text{I}_M \\ \xrightarrow{M} \end{array} \text{Set}$$

has: $h_K: MK \rightarrow MK = \text{I}_{MK} \text{ (in Set)}$

$h: M \rightarrow N$ is an isomorphism

(iff) for all $K \in L$, $h_K: MK \rightarrow MK$ is a bijection (an isomorphism in the category Set)

The functor category $[C, D]$

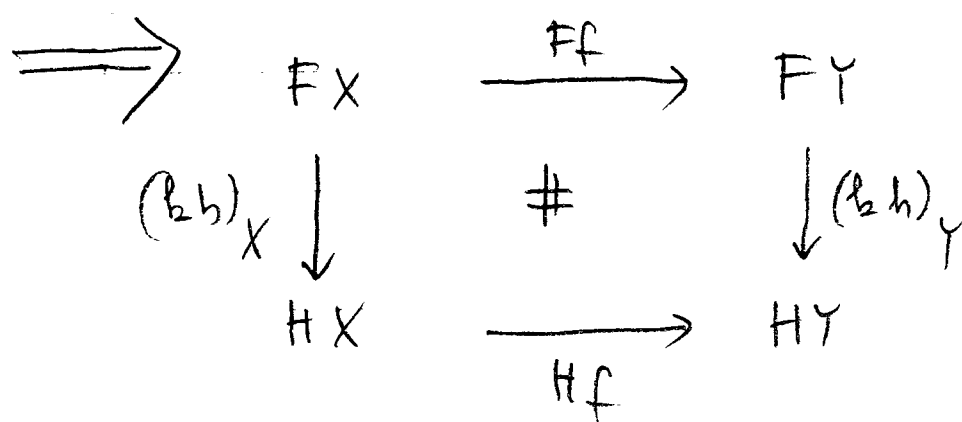
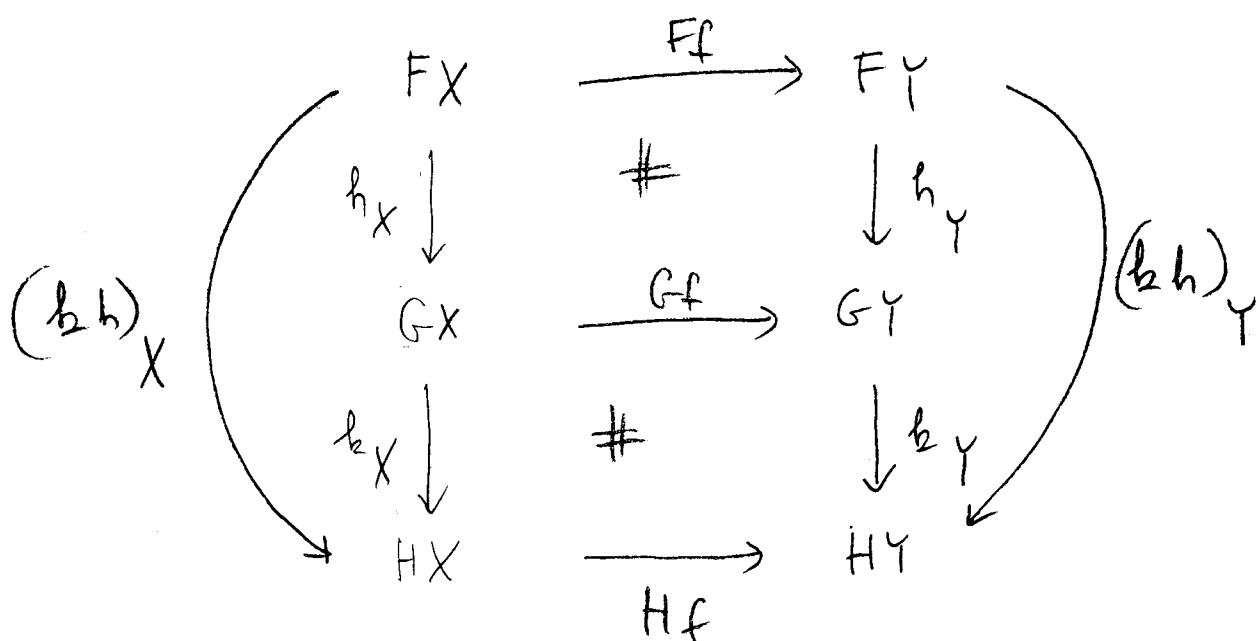
Given any two categories, C and D , the functors $C \rightarrow D$ and the natural transformations $C \xrightarrow{\downarrow} D$ form a category — the former as objects, the latter as arrows. The composition of nt's (natural transformations)

$$C \begin{array}{c} \xrightarrow{F} \\ \begin{array}{ccc} G & \downarrow h & \\ H & \downarrow k & \end{array} \\ \xrightarrow{\quad} \end{array} \Bigg\} kh \longrightarrow D$$

$$X \in \text{ob } C: (kh)_X = k_X \circ h_X$$

Note: diagram:

$X \xrightarrow{f} Y$ in $\mathbb{C}$ generates:



'#' says : commutes.

this was the proof that bh is indeed an nt.

Said category is denoted: $[\mathbb{C}, \mathbb{D}]$
(or $\mathbb{D}^{\mathbb{C}}$; or $\text{Hom}(\mathbb{C}, \mathbb{D})$)

For L a signature,

$$\text{Str}(L) \stackrel{\text{def}}{=} [L, \text{Set}]$$

the category of L -structures.

Part 2 SYNTAX AND SEMANTICS

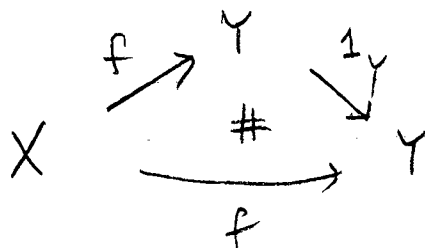
Example of a context over L_{cat}

Recall L_{cat} from A3, the definition of 'context' from A10. Let $L = L_{\text{cat}}$. We define

a context $C = C_{\text{2nd-unit-law}} : L \rightarrow \text{Set}$

related to the "2nd unit law for categories",

$1_Y \circ f = f$, in the situation



In the context C , there will be six entities - symbols; but arbitrary symbols! - involved

besides X, Y, f and $i = 1_Y$, two more: φ : that testifies that i is an (the) identity arrow on Y ; and τ : testifying that the triangle commutes

Here is the definition of the functor

$$C: L \longrightarrow \text{Set} ::$$

$$C(\underline{0}) = \{X, Y\}$$

$$C(\underline{A}) = \{f, i\}; \quad d f = X, \quad c f = Y, \quad d i = Y, \quad c i = Y$$

$$C(\underline{\Pi}) = \{\varphi\}; \quad \underline{i} \varphi = i, \quad \underline{j} \varphi = Y$$

$$C(\underline{I}) = \{\tau\}; \quad s_0 \tau = X, \quad s_1 \tau = Y, \quad s_2 \tau = Y, \\ t_0 \tau = f, \quad t_1 \tau = i, \quad t_2 \tau = f;$$

$$C(\underline{E}) = \emptyset$$

When giving the definition above of the effect of C on an arrow $\underline{a}$ in L , instead of " $C(\underline{a})$ " we just write " $\underline{a}$ ".

Thus, " $d f = X$ " really means " $C(d)(f) = X$ ", etc.

We also omitted the underlinings from most of the notations of the arrows of L — not so for $\underline{i}, \underline{j}, \underline{1}$.

Of course, one needs to check that the definition gives a functor; e.g.,

$$C(d) \circ C(t_0) = C(s_0); \text{ i.e. } d(t_0 z) = s_0 z \text{ etc.}$$

In what follows, we will use this example to illustrate the concepts to come.

Abbreviated notation for contexts

Let $C : L \rightarrow \text{Set}$ be a context over the signature L (see A.10). We want to think of the context as a set of variables, organized (structured) in a certain way. We take the set

$$|C| = \bigcup_{K \in \text{Ob}(L)} C(K)$$

by definition, this is a finite set. Its elements are called variables - variables of the

$$x_1, x_2, \dots, x_n \in |C|$$

an assigned kind : $K \in \text{ob}(L)$ such

that $x \in C(K)$; we write $x :: K$ for: x is of the kind K .

(again, suppressing C from the notation).

For any proper (non-identity arrow) $p : K \rightarrow K_p$

($K_p \stackrel{\text{def}}{=} \text{codomain}(p)$; $\dim K_p < \dim K$),

and $x :: K$, we let

$$p x \stackrel{\text{def}}{=} C(p)(x)$$

(note: $C(p) : C(K) \rightarrow C(K_p)$)

$$x \longmapsto (C(p)(x) = p x)$$

(This is the notation that was used in the

specification of the example above; C is suppressed.)

Thus, we have

$$x :: K \Rightarrow p x :: K_p$$

We write $\text{dep}(x) = \{ p x : p \in L \mid K \}$ ($x :: K$)

x depends on each $y \in \text{dep}(x)$.

The fact that C is a functor is expressed in this way:

$$\text{every time } K \xrightarrow{p} K_p \xrightarrow{q} K_q,$$

and $x :: K$,

we have

$$\boxed{q(px) = (qp)x};$$

thus, we can write ' $qp x$ ' without fear of error.

Now, suppose we have two contexts

C, D over L , and 'map $\Phi : C \rightarrow D$;

since C, D are functors, Φ is a natural transformation; let's call Φ a change of variables. Φ can be regarded as a

mapping

$$\Phi : |C| \longrightarrow |D|$$

such that $x :: K \text{ in } C \Rightarrow \Phi x :: K \text{ in } D$

and

$$\text{i.e. } px = y \text{ in } C \Rightarrow p(\Phi x) = \Phi y \text{ in } D$$

$$\underline{\text{i.e. } \boxed{\Phi(px) = p(\Phi x)}}.$$

An example is the important context

$$\overset{\circ}{K} : L \longrightarrow \text{Set}$$

($K \in \text{Ob}(L)$) ; see A 12

$$\text{Now, } |\overset{\circ}{K}| = \underbrace{K|L}_{\text{new notation}} \stackrel{\text{def}}{=} \{ p \in \text{Arr}(L) : \text{dom}(p) = K \text{ \& } p \neq \text{id}_K \}$$

$$\stackrel{\text{def}}{=} \{ p \in \text{Arr}(L) : \text{dom}(p) = K \text{ \& } p \neq \text{id}_K \}$$

Let us write K_p for $\text{domain}(p)$.

In the context $\overset{\circ}{K}$, $p :: K_p$. Moreover,

when $K_p \xrightarrow{q} K_q$, an arrow in L with
domain $= K_p$, i.e., $K \xrightarrow{p} K_p \xrightarrow{q} K_q$,

then

$$p :: K_p \Rightarrow qp :: K_q \text{ in the sence of}$$

in the just-introduced sence of the notation 'qp'

and also in the sence of composition : $qp = q \circ p : K \rightarrow K_q$.

Suppose

$$L \begin{array}{c} \xrightarrow{D} \\ \xrightarrow{C} \end{array} \text{Set}$$

are contexts. We say that D is a subcontext of C if there is a map

$i: D \rightarrow C$ in symbols: $D \subseteq C$ whose components $i_K: DK \rightarrow CK$

are all set-inclusions: for $x \in DK$, $i_K(x) = x$.

In other words, $D \subseteq C$ if $DK \subseteq CK$ (DK is a subset of CK) and, for $K \xrightarrow{p} K_p$,

$x \in DK$, we have that $px \in DK_p$ in the sense of D is the same as $px \in CK_p$ in the sense of C . Also, for a context C , and

(separate) sets DK , one for each $K \in \text{Ob}(L)$, a subset of CK , the sets $DK, K \in \text{Ob}(L)$,

determine a subcontext iff for each $K \xrightarrow{p} K_p$,

and $x \in DK$, px (in the sense of C) is in the set DK_p .

In other words, a subcontext of a fixed context C is a subset D of C that is closed under the operation of multiplying with any 'face-operator' p

$$\underline{x \in D \text{ \& } px \text{ is meaningful}}$$

$$(, \text{dom } p = K, x :: K)$$

$$\underline{\Rightarrow}$$

$$\underline{px \in D.}$$

Note the immediate consequence: subcontexts

are closed under union: $D_1 \subseteq_{sc} C, D_2 \subseteq_{sc} C$

$$\Rightarrow D_1 \cup D_2 \subseteq_{sc} C$$

Also, the empty set $\emptyset$ is a subcontext.

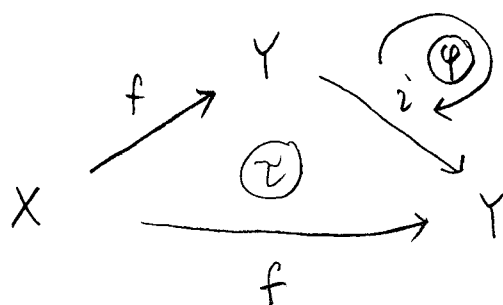
Boundaries

Let us fix a context (of variables) : C
 (over a fixed signature L). Let K be a
 kind in L . With $\overset{\circ}{K}$ the K -niche, we
 call a change-of-variables $\boxed{\beta: \overset{\circ}{K} \rightarrow C}$ a
boundary (maybe, the word is not so good:
 "boundary of something" is the natural use;
 we will have this too, but right now, β is
 a boundary that may be empty; a boundary
 without being a boundary of something).

Let us see some boundaries in the example

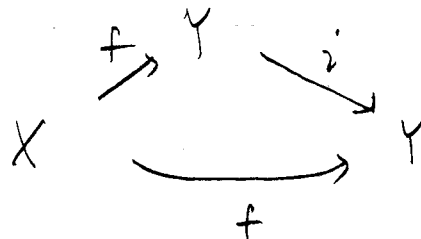
$$C = C_{\text{2nd-unit-law}} \quad [A21]$$

First of all, a more complete picture for C :



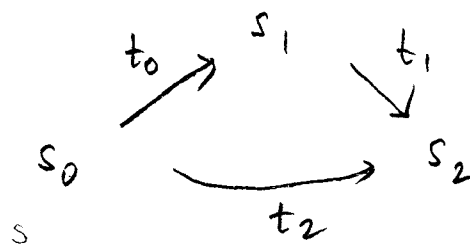
(we have added ε and φ to the picture)

The following is a boundary in C :



This is the map $\beta: |\overset{\circ}{T}| \longrightarrow |C|$ for

which (compare



the picture for $|\overset{\circ}{T}|$) has :

$$\boxed{\beta s_0 = X, \beta s_1 = Y, \beta s_2 = X, \beta t_0 = f, \beta t_1 = i, \beta t_2 = f}$$

Another, "not visible", boundary in C

$$\beta': |\overset{\circ}{A}| \longrightarrow |C|$$

is :

$$\beta' d = X, \quad \beta' c = X.$$

The β above is filled; it is the boundary of ε ;

$\mathcal{P}'$ is empty; there is no arrow $X \dashrightarrow X$

in $\mathcal{C}$ - all according to the following general definition:

given $x \in |C|$, the boundary of x ,

∂x , is given by

$$\boxed{(\partial x)_p \stackrel{\text{def}}{=} p x}$$

In more detail: let $x :: K$; for $p \in |K| = L/K$,

$p: K \rightarrow K_p$, we have $p x$ in $\mathcal{C}$; $p x = (\mathbb{C} p) x$

(reminders!). The fact that ∂x is in fact

a map $\partial x: K \rightarrow \mathcal{C}$ is verified by

$$\begin{array}{ccc} (\partial x)(q p) & \stackrel{?}{=} & q((\partial x)_p) \\ \parallel & & \parallel \\ (q p) x & \stackrel{!}{=} & q(p x) \\ & \checkmark & \end{array}$$

(from A25)

(in the equation) : $\Phi(p x) = p(\Phi x)$

the corresponding things:

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \partial x(q p) & = & q((\partial x)_p) \end{array} \quad \left(\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \Phi & & \Phi \end{array} \right)$$

In the example, the boundary β is the boundary of τ : $\beta = \partial\tau$. The boundary β' is empty: there is ^{no} x such that $\partial x = \beta'$.

In general, any number of x 's may have the same boundary; we could add another τ' to the context with the same boundary β as τ .

The range of a variable in an interpretation

We now come to the main idea of FOLDS: the range of a variable.

Given a variable $x \in K$, and an interpretation,

meaning an L -structure M , and an evaluation $\alpha: D \rightarrow M$, ^(D a subcontext of the fixed C) that evaluates at least the

variables in ∂x (it does not need to evaluate x itself!) ^{(i.e.: $\text{dep}(x) \subseteq D$)} we define the

range of x , $M_\alpha[x]$ as a certain

subset of MK , $M_\alpha[x] \subseteq MK$.

under the given interpretation (M, α) , (A31)

The meanings of the quantifiers $\forall x$,

$\exists x$ will then be specified by

saying:

" $\forall x$ " means "for all $a \in M_\alpha[x]$ "

" $\exists x$ " means "there exists $a \in M_\alpha[x]$ ".

1) Let us start with the following data:

$$\mathcal{L} \xrightarrow{M} \text{Set} ; \quad \mathcal{L}\text{-structure};$$

$$K \in \text{Ob}(\mathcal{L}), \quad a \in MK.$$

We can define the boundary of a ,

$$\partial a: \overset{\circ}{K} \longrightarrow M, \quad \text{exactly as the boundary}$$

$$\partial x: \overset{\circ}{K} \longrightarrow C \quad \text{was defined for a variable } x:$$

$$(\partial a)_u: \overset{\circ}{K}[u] \longrightarrow MU \quad \text{is given by}$$

$$K \xrightarrow{f} u \longmapsto (Mf)a \in MU$$

$$\left(\begin{array}{ccc} \text{note: } MK & \xrightarrow{Mf} & MU \\ & x \longmapsto & (Mf)a \end{array} \right)$$

$$\boxed{(\partial a)f = (Mf)a}$$

('contexts' and 'structures' are, both,
functors $L \rightarrow \text{Set}!$)

2) Now, let us have

$$L \xrightarrow{M} \text{Set}$$

$K \in \text{Ob}(L)$, $x :: K$ in C ($x \in C(K)$)

and an evaluation $\alpha: D \rightarrow M$

for a subcontext $D \subseteq C$ such that

$\partial x: K \rightarrow C$ is 'in' D , in the sense that

$$\begin{array}{ccccc} \text{we have} & K & \xrightarrow{\partial x} & D & \xrightarrow{\text{incl.}} & C \\ & & & \searrow \# & \nearrow & \\ & & & \partial x & & \end{array}$$

equivalently, for each $p \in L|K$, px (in the sense of

C) is in D . We can take the composite:

$$\begin{array}{ccc} & K & \\ & \downarrow \partial x & \\ L & \xrightarrow{\quad} & \text{Set} \\ & D & \\ & \downarrow \alpha & \\ & M & \end{array}$$

$$L \xrightarrow[\begin{smallmatrix} \downarrow \alpha(\partial x) \\ M \end{smallmatrix}]{\overset{\circ}{K}} \text{Set}$$

$$\alpha(\partial x) \stackrel{\text{def}}{=} \alpha \circ \partial x;$$

$\alpha(\partial x)$ is the interpretation of the variable-boundary ∂x in M . Given any $a \in MK$, from 1), we have

$$L \xrightarrow[\begin{smallmatrix} \downarrow \partial a \\ M \end{smallmatrix}]{\overset{\circ}{K}} \text{Set}$$

Therefore, it is meaningful to define:

$$M_\alpha[x] \stackrel{\text{def}}{=} \{ a \in MK : \partial a = \alpha(\partial x) \}$$

that is, for $a \in MK$,

$$\boxed{a \in M_\alpha[x] \iff \partial a = \alpha(\partial x).}$$

In words: the range of the variable $x :: K$ under the interpretation (M, α) is the set of ^{all} $a \in MK$ whose boundary is the α -image of the boundary of x .

Example: Returning to our previous example, let $x = z$, $D \subseteq C$ the sub-context that misses z but nothing else in C ,

Let $M: L(=L_{\text{cat}}) \longrightarrow \text{Set}$. For any

$a \in M \underline{I}$, $\partial a: \underline{I}^0 \longrightarrow M$, or rather

$$|\partial a|: |\underline{I}^0| \longrightarrow |M|$$

$$(\equiv \bigcup_{K \in L} MK)$$

is given as $(\partial a) s_0 = (M s_0) a \in M \underline{O}$, ...

$$(\partial a) t_2 = (M t_2) a \in M \underline{A}$$

(see bottom A 31).

Therefore, by box A 31, $\partial x = \beta$ in box A 28,

$M_x[x]$ is the set of all $a \in MK$ such that

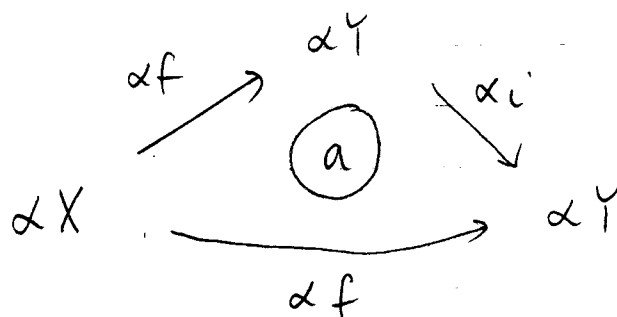
$$\partial a = \alpha(\partial x) = \alpha(\beta), \text{ that is, } , , ,$$

$$(M s_0)(a) = \alpha(X), \quad (M s_1)(a) = (M s_2)(a) = Y,$$

$$(M t_0)(a) = (M t_2)(a) = f, \quad (M t_1)(a) = i$$

that is, a fits into the α -image of the

triangle on top of A28, i.e., into



As expected!

Formulas

Let us fix a context $C: L \rightarrow \text{Set}$. All 'formulas' will use variables from C . We could make C a "large infinite" context as well (any functor $L \rightarrow \text{Set}$, in fact) — but this is not important.

The letters $\underline{X}, \underline{Y}, \dots$ (script $X, Y, \dots$) will

always be subcontexts of C . We will not write $|\underline{X}|$ even when we mean it; we will just write $\underline{X}$. Recursively, we define 'formula' (C -formula),

and 'set $\boxed{\text{var}(\varphi)}$ of free variables of formula φ '.

Auxiliary definition: for context $\underline{X}$

(subcontext of C), let us write:

$$\underline{X}^\uparrow \stackrel{\text{def}}{=} \{y \in |C| : \text{for all } x \in \underline{X}, y \notin \text{dep}(x)\}$$

$\underline{X}^\uparrow$ consists of the variables on which

no variable in $\underline{X}$ depends. When

$y \in \underline{X}^\uparrow$, y may or may not belong to $\underline{X}$

- if it does, it is on the top of $\underline{X}$, in the

sense of the (partial) order of dependence:

- writing $y \prec x$ to mean $y \in \text{dep}(x)$,

$\prec$ is an irreflexive partial order.

$y \in \underline{X}^\uparrow$ iff either y is a $\prec$ -maximal element of $\underline{X}$, or else $y \notin \underline{X}$. Also,

$y \in \underline{X}^\uparrow$ if and only if the set $\underline{X} - \{x\}$ is a context.