Computads and 2 dimensional pasting diagrams

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Introduction

1. This paper is the second installment of a series whose first item is the paper [M]. In [M], a paper was promised, [M4] in the references section there, with the tentative title "A 2-categorical pasting theorem: revisiting John Power's paper [P1] of the same title". The present paper is what [M4] has become.

The introduction of [M] should serve as a general introduction to the present paper as well.

The notions of " ω-category" and "computad" come from the work of Ross Street.

The basic notions of and around " ω-category" and "computad" will not be recalled here. By now, these concepts belong (or should belong ...) to the common knowledge in category theory. For instance, the reader is not far wrong if he/she takes "computad" to mean "free ω-category". However, the ways these concepts are formulated in this paper, and the special notations used when dealing with them, will have to be gleaned from [M], which is intended as a "foundational" paper for these concepts.

In the introductory first two sections, two things are done. First, we recall the necessary background material on computads, mainly by citing definitions and results from [M], but also by introducing new terms and statements which are in [M] only implicitly. The definitions and results cited are relevant or valid in arbitrary dimensions. The results cited from [M] are marked by the symbol [M], and numbered in the style [M](i), [M](ii), ...

Secondly, in sections 1 and 2, we also state some new results. The theorems and propositions in sections 1 and 2 marked in the style 1.1, 2.1, 2.2, ... will be proved only later in the paper. On the other hand, similarly numbered corollaries of the above are proved on the spot.

There is one constraint observed in sections 1 and 2: only such new results are stated which have straightforward *conjectured* higher-dimensional generalizations, although the results themselves are claimed and stated only for dimension 2, and occasionally 3.

In section 2, among others, an analog of John Power's theorem, 3.3 Theorem in [P1], "Every labelling of a pasting scheme has a unique composite", is stated (2.12 Theorem).

In the second part, from section 3 on, the concepts and results of a new "geometric theory" of computads, presently established only in dimension 2, are presented. After the purely combinatorial and elementary section 3, concerning what we call "planar arrangements", the first of two forms of the main result of the paper, Theorem 4.2, is formulated in Section 4. With the exception of those in section 9, all results of the paper, including the ones stated in sections earlier than the fourth, are essentially (that is, modulo the basics in [M]) corollaries of the main result 4.2.

Theorem 4.2 is a reconstruction of the "geometry" of a 2-dimensional pasting diagram (2-pd), valid for the class of 2-pd's called *anchored* (for the definition, see below; the terminology has been suggested by Andre Joyal). The geometry in question is given by *postulation* in [P1];

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here the "geometry" is computed from the algebraic expression of the pd.

2. There are two concerns in the paper, one explicit, the other somewhat implicit: the interest in general laws on the one hand, and computational procedures on the other.

The "geometry" of a pasting diagram is what we display when we *draw* the diagram. This is *the primary* aspect of the subject: it is what *we are given*, informally of course, when we start the investigation (witness the first few sentences of John Power's paper). It is a compelling idea to follow the hunch that there are general laws and procedures behind the drawing of categorical diagrams.

The theorems of the first four sections state the *laws*, proved for a small beginning range of cases and conjectured for others, of pasting diagrams. The *computational procedures* of the subject are shown only later; nevertheless, they are the first motivation for the paper.

For example, 2.2 Theorem, part (c), says that there is a so-called *planar arrangement* of the occurrences of the indeterminate 2-cells of a 2-dimensional pasting diagram (pd), under a mild, but important, restriction on the pd itself. This is our way of stating that a 2-pd *can be drawn in the plane*. But in fact, the complete point is not just that this "drawing" *exists*, but also that it can be *computed*. Namely, given a symbolic representation for the 2-pd, in the form that we call a molecule -- which is just a somewhat normalized syntactical term in the language of operations for the notion of 2-category -- we can effectively and "naturally" calculate said planar arrangement.

This concern for calculation explains a certain repetitiveness in the paper. The calculation just alluded to leads naturally to a *tree*, depending on the given molecule, that represents the steps in the calculation. The given molecule stands for a 2-pd that can be defined by numerous other molecules -- in fact, these latter molecules will all appear at one stage or another in the construction of the tree attached to the given molecule. The trees induced by these variant molecules are all different from one another, but they are all, essentially, spanning trees of a certain *graph* which is an invariant object attached to the 2-pd itself.

The graph is mentioned early on; 2.4 Corollary is a result, in the "anchored 2D" case, that gives an abstract description of it. On the other hand, the trees appear only in section 5. They are used to prove all the results stated in the earlier sections. The trees would have been easy to avoid altogether, by somewhat reformulating the proofs, if we had been only interested in the abstract/invariant laws without the calculations. As things are now, in the preparatory stages of dealing with the computational trees, we are compelled to state variants of a number of constructs that had been mentioned in the context of the graphs.

Computads and pasting diagrams serve as the basic carriers of the syntax of higher dimensional categories, weak and strict, as explained, for instance, in the introduction of [M]. This explains the interest in computational aspects of computads: following the lead of Gottlob Frege and David Hilbert, we adhere to the doctrine that all aspects of pure syntax have to be calculable and/or decidable.

3. I will now comment on the two main new concepts of the paper, that of *planar pasting prescheme*, related to dimension 2, and the more general *pasting prescheme* relevant in arbitrary dimensions.

"Pℓanar Pasting PreSchemes", PℓPPS's for short, are introduced in section 4. John Power has "pasting schemes" in both [P1] and [P2]. PℓPPS's are different from Power's concept for dimension 2 in [P1] (and of course, different from that in [P2] too), but serve in similar roles. The prefix "pre" is there because the term "planar pasting scheme", PℓPS, is reserved for a PℓPPS which is "complete" in a suitable sense. PℓPPS's have unique composites, by design (that is, the proof that they do is more direct than in Power's case). The composite is a general 2-cell, also called 2-pasting diagram (2-pd), in (the underlying 2-category of) the underlying computad.

The main result, 4.2 Theorem, says, in essence, that every 2-pd satisfying a smallish but essential restriction ("anchored") *has* a complete PℓPPS displaying it (the composite of the PℓPPS is the given 2-pd). The uniqueness of the displaying PℓPPS is essentially obvious; but it is returned to in section 10.

Notice the opposite natures of the general outline here and of that in [P1]. In [P1], the 2D diagrams are *defined* as those given by a pasting scheme, and the work to be done is in showing that they make sense as 2-categorical composites. Here, the 2D diagrams are given in advance algebraically as 2-categorical composites of indeterminate cells in a computad ("free 2-category"); the work is to show that there are pasting schemes in the new sense that display them.

The general notion of "Pasting PreScheme", PPS, is introduced in section 9. It is formulated in arbitrary dimensions. The main result of the paper concerning this concept is that any PPS has a unique composite (9.3 Proposition). There is no analog in the paper of the hard work done for the planar pasting schemes, the construction of them for a large class of 2-pd's; this analog is planned for the future installments of the series.

Although it is not true that every 2D pasting prescheme is planar, the truth is not far from saying that. The main result of the paper, expressed in terms of the general notion of pasting prescheme, is that any PPS of an anchored 2-pd has a planar extension, and (therefore, by 4.2 Theorem) there is a unique pasting scheme (complete pasting prescheme) displaying any given anchored 2-pd, which is in fact planar (see 10.4 Theorem).

§1 Types, shapes and occurrences

Pasting diagrams

Let us codify the concept of "pasting diagram". A *pasting diagram* (*Pd* for short) is a pair $(\boldsymbol{X}, \Gamma)$ where $\boldsymbol{X}$ is a computad, Γ is a cell of the ω-category $\boldsymbol{X}$, $\Gamma\in\|\boldsymbol{X}\|$, and

$$\boldsymbol{X}=\mathrm{Supp}_{\boldsymbol{X}}(\Gamma)\ . \tag{1}$$

The idea is that $\boldsymbol{X}$ is the diagram itself, which pastes (composes) into the composite Γ. So, in fact, the expression "pasted diagram" would be more suitable. Fortunately, "Pd" is neutral with respect to the two readings.

The capitalized version "Pd" is used for the concept that contains it's own "context" as $(\boldsymbol{X}, \Gamma)$ contains a reference to $\boldsymbol{X}$. A "pd" uncapitalized is an element of $\|\boldsymbol{X}\|$, with $\boldsymbol{X}$ given in a larger context.

The equality (1) means that *all* the indeterminates in $\boldsymbol{X}$ are *used* in writing Γ. This way of saying the matter is a correct definition if the cells in a computad are taken to be equivalence classes of terms formed from the indeterminates as "variables" (see [Pe], or [M] where Jacques Penon's [Pe] definition of computad ("polygraph" in French) via terms is re-stated). On the other hand, one can define, for any computad $\boldsymbol{X}$ and any $\Gamma\in\|\boldsymbol{X}\|$, $\mathrm{Supp}_{\boldsymbol{X}}(\boldsymbol{X})$, a subcomputad of $\boldsymbol{X}$, whose indeterminates are the ones "used" in Γ, in a purely algebraic manner too; see [M].

The datum Γ, the composite itself, is not a superfluous item. With $\boldsymbol{X}^*$ generated by the single 0-cell X, and the single 1-cell $f:X\to X$, we have infinitely many Γ for which $(\boldsymbol{X}^*, \Gamma)$ is a Pd: all the composites f^m (m=1, 2, 3, ...). The reader will be right if he/she thinks that we should be interested in when a computad $\boldsymbol{X}$ *has a unique composite*, meaning there is a unique Γ for which $(\boldsymbol{X}, \Gamma)$ is a Pd.

The notations $(\boldsymbol{X}, \Gamma)$, $(\boldsymbol{Y}, \Lambda)$ will always mean Pd's in the sense just codified. We also write $\underline{\Gamma}$ for $(\boldsymbol{X}, \Gamma)$, $\underline{\Lambda}$ for $(\boldsymbol{Y}, \Lambda)$.

Let's define the *category of pasting diagrams*, Pd, to have objects the Pd's, and arrows $(\boldsymbol{X}, \Gamma)\xrightarrow{f}(\boldsymbol{Y}, \Lambda)$ those $\boldsymbol{X}\xrightarrow{f}\boldsymbol{Y}$ in Comp for which $f(\Gamma)=\Lambda$. Pd has a forgetful functor $\mathrm{Pd}\longrightarrow\mathrm{Comp}$. (Comp is the category of all (small) computads: see [M]).

The *dimension* of the Pd $(\boldsymbol{X}, \Gamma)$ is the dimension of Γ (as a cell of the ω-category $\boldsymbol{X}$). We have $\dim(\boldsymbol{X},\Gamma)=\max\{\dim(x):x\in|\boldsymbol{X}|\}$.

Pd_n is the full subcategory of Pd whose objects are the Pd's of dimension n. The notation $\mathrm{Pd}_{\leq n}$ is analogous.

An *Indeterminate* (Indet for short) will be a Pd $\underline{x}=(\boldsymbol{X}, x)$ where $x\in|\boldsymbol{X}|$, that is, x is an indeterminate (=free generator; see [M]) in $\boldsymbol{X}$. Indet is the full subcategory of Pd consisting of the Indets.

If $(\boldsymbol{X}, x)$, $(\boldsymbol{X}, y)$ are Indets with the same underlying computad $\boldsymbol{X}$, then they are the same: $x=y$: this is obvious, since x is the unique maximal-dimensional indet in $\boldsymbol{X}$. On the other hand, two different Pd's may have the same underlying computad.

Indet is not only a full subcategory of Pd, but it is a sieve in Pd : if $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)\longrightarrow(\boldsymbol{Y}, y)$, and $(\boldsymbol{Y}, y)$ is an indeterminate, then $\underline{\Gamma}$ is itself an Indet; $f(\Gamma)=y$ implies that $\Gamma\in|\boldsymbol{X}|$: this was proved in [M].

$\mathrm{Indet}_{\mathrm{m/1}}$ is defined as the full subcategory of Indet whose objects $\underline{x}=(\boldsymbol{X}, x)$ are the many-to-one Indets, that is, are such that $\mathrm{c}x$ is an indet too. $\mathrm{Indet}_{\mathrm{m/1}}$ is also a sieve in Pd.

The Indets play the central role among the Pd's; in fact, in a sense, every Pd can be "replaced" by an Indeterminate, albeit by one of one-higher dimension. For the Pd $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$, consider the many-to-one Indet $\underline{\tilde{\Gamma}}=(\boldsymbol{X}[x][y], y)$ defined, in two steps, by first adjoining to $\boldsymbol{X}$ the *new* indet x of the dimension of Γ with the specification $\mathrm{d}x=\mathrm{d}\Gamma$, $\mathrm{c}x=\mathrm{c}\Gamma$, and then adjoining y of one higher dimension with $\mathrm{d}y=\Gamma$, $\mathrm{c}y=x$.

$\tilde{\underline{\Gamma}}$ is, of course, defined up to isomorphism only, although, as usual, we pretend that it is strictly specified.

There is an obvious bijection between $\mathtt{hom}(\underline{\Gamma},\underline{\Lambda})$ and $\mathtt{hom}(\tilde{\underline{\Gamma}},\tilde{\underline{\Lambda}})$. In fact, we have an equivalence of categories

$$(\underline{\Gamma}\mapsto\tilde{\underline{\Gamma}}):\mathtt{Pd}\xrightarrow{\simeq}\mathtt{Indet}_{\mathtt{m/1}}$$

Typing and occurrence

We will call the Pd $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ *separated* if for all $\underline{\Lambda}=(\boldsymbol{Y},\Lambda)\in\mathtt{Pd}$ and all $\underline{\Lambda}\xrightarrow{f}\underline{\Gamma}$ in Pd , f is necessarily an isomorphism.

A *computope* (see [M]) is an Indet $(\boldsymbol{X},x)$ such that for all Indets $(\boldsymbol{Y},y)$ and arrows $(\boldsymbol{Y},y)\xrightarrow{f}(\boldsymbol{X},x)$, f is necessarily an isomorphism.

We say that the computad $\boldsymbol{X}$ is a *computope* if there is a, necessarily unique, computope $(\boldsymbol{X},x)$ with underlying computad $\boldsymbol{X}$.

From the fact that $\mathtt{Indet}$ is a sieve in $\mathtt{Pd}$, we immediately see that an Indet is a computope iff it is a separated Pd, and the Pd $\underline{\Gamma}$ is separated if and only if $\tilde{\underline{\Gamma}}$ is a computope.

The *category of all computopes*, $\mathtt{Ctp}$, is defined as the skeletal full subcategory of $\mathtt{Comp}$ itself, whose objects form a full set of representatives of isomorphism types of all the computopes. (Thus, we allow all computad morphisms $f:A\to B$ for computopes (A,x) and (B,y) , not just the ones in $\mathtt{Indet}$.)

It is an important fact (see [M]) that $\mathtt{Ctp}$ is a *finitary one-way* category. A category $\boldsymbol{D}$ is *finitary* if for all objects X in $\boldsymbol{D}$, the set $\{f\in\mathtt{Arr}(\boldsymbol{D}):c(f)=X\}$ is finite. $\boldsymbol{D}$ is *one-way* if there is no infinite descending chain

$$X_0\xleftarrow{f_0}X_1\xleftarrow{f_1}\ldots X_n\xleftarrow{f_0}X_{n+1}\xleftarrow{\quad}\ldots$$

of non-identity arrows in it. (The finitary-ness of $\mathtt{Ctp}$ is not immediate; the one-way quality is.)

In [M], the following are proved.

Theorem [M] **(i)** For every Indet $\underline{x}$, there is a computope $\underline{y}$ with a morphism $\underline{y}\longrightarrow\underline{x}$.

(ii) For every Indet $\underline{x}$, there are only finitely many non-isomorphic

Indets $\underline{y}$ having an arrow $\underline{y} \to \underline{x}$ to $\underline{x}$.

((i) is 11.(4) in [M]; (ii) is stated in the proof of the same 11.(4) as "the isomorphism types of resolvents of B form a non-empty finite set".)

Corollary For every Pd $\underline{\Gamma}$, there is a separated Pd $\underline{\Lambda}$ with a morphism $\underline{\Lambda} \longrightarrow \underline{\Gamma}$; up to isomorphism, there are only finitely many such $\underline{\Lambda}$.

To get the Corollary, apply the Theorem to $\hat{\underline{\Gamma}}$ as $\underline{x}$.

Referring to the Corollary, $\underline{\Lambda}$ is called a *type* for $\underline{\Gamma}$; a morphism $\underline{\Lambda} \longrightarrow \underline{\Gamma}$ a *specializing morphism* for $\underline{\Gamma}$.

We say that the Pd $\underline{\Gamma}$ is *uniquely typed* if

1) the specializing morphism for $\underline{\Gamma}$ from any type of $\underline{\Gamma}$ to $\underline{\Gamma}$ is unique: for $\underline{\Lambda}$ separated, if $\underline{\Lambda} \underset{g}{\overset{f}{\rightrightarrows}} \underline{\Gamma}$, then $f=g$;
and

2) the type of $\underline{\Gamma}$ is unique up to isomorphism: if $\underline{\Lambda}$, $\underline{\Xi}$ are separated, and $\underline{\Lambda} \longrightarrow \underline{\Gamma} \longleftarrow \underline{\Xi}$, we have $\underline{\Lambda} \cong \underline{\Xi}$.

Note that 1) is equivalent to the seemingly stronger condition

1*) for any $\underline{\Lambda}$, if $\underline{\Lambda} \underset{g}{\overset{f}{\rightrightarrows}} \underline{\Gamma}$, then $f=g$.

The reason is that, given $\underline{\Lambda}$, by the previous Corollary, there are separated $\hat{\underline{\Lambda}}$ and $\hat{\underline{\Lambda}} \xrightarrow{h} \underline{\Lambda}$; by 1), $f \circ h = g \circ h$; but h , as any map of Pd's, is surjective on indeterninates; it follows that $f=g$.

The main motivation for the foregoing notions is the desire to understand the idea of an *occurrence of* a generator $x \in |\boldsymbol{X}|$ in a Pd $(\boldsymbol{X}, \Gamma)$.

In the example $\underline{\Gamma}_m = (\boldsymbol{X}^*, f^m)$ (m=1, 2, 3, ...) above, it is natural to say that the 0-cell X "occurs $m+1$ times", and f "occurs m times", because this way of talking will match the *drawing* of the arrow f^m as the composite of a diagram:

$$X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \dots \xrightarrow{f} X \;. \qquad (2)$$

Let $\underline{\Lambda}_m=(\boldsymbol{Y}^*, \Lambda_m)$ be such that $\boldsymbol{Y}^*$ is generated by the distinct 0-cells X_i ($i=1, \dots, m, m+1$) and the 1-cells $f_i:X_i \to X_{i+1}$. Let $\Lambda_m=f_1 \cdot \dots \cdot f_m$. $\underline{\Lambda}_m$ is separated. The drawing of $\underline{\Lambda}_m$ is

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots \xrightarrow{f_{m-1}} X_m \;.$$

There is a unique map $\underline{\Lambda}_m \longrightarrow \underline{\Gamma}_m$; and, up to isomorphism, $\underline{\Lambda}_m$ is the only separated Pd $\underline{\Lambda}$ with a map $\underline{\Lambda} \to \underline{\Gamma}_m$. These facts allow us to say that *the* ith occurrence of X in (2) *is* X_i, and *the* ith occurrence of f in (2) *is* f_i. We have not only accounted for the number of occurrences of each generator, but have succeeded in defining *what* an occurrence is.

We may conclude that if the Pd $\underline{\Gamma}$ is uniquely typed, by $f:\underline{\Lambda} \longrightarrow \underline{\Gamma}$ say, the notion of an occurrence of any given indet $x\in |\underline{\Gamma}|$, as well as the number of distinct occurrences of x, are clarified: an occurrence of x is any element of the set $f^{-1}(x)$; the number of occurrences of x is the cardinality of the set $f^{-1}(x)$. The fact that the typing $(\underline{\Lambda}, f)$ is defined from $\underline{\Gamma}$ uniquely up to a unique isomorphism tells us that we will have a sound notion of occurrence.

Let us review the low dimensions as to unique typing.

In dimension 0, everything is trivial.

Next, one sees easily that every 1-Pd is uniquely typed.

However, in dimension 2 , it is not difficult to find a Pd that is not uniquely typed. In [M], the following example is given.

We let $\boldsymbol{X}$ be generated by the indets X, u and v, where $\mathrm{dim}(X)=0$, $\mathrm{dim}(u)=\mathrm{dim}(v)=2$, and $1_X \overset{u}{\underset{v}{\rightrightarrows}} 1_X$. We let $\Gamma=u\cdot v$. Since $u\cdot v=v\cdot u$ (Eckmann-Hilton), we have the automorphism $h:(\boldsymbol{X}, \Gamma) \xrightarrow{\cong} (\boldsymbol{X}, \Gamma)$ that flips u and v. Since $(\boldsymbol{X}, \Gamma)$ is separated, $(\boldsymbol{X}, \Gamma)$ is its own type, and 1) fails.

Thorsten Palm showed me an example for which 2) fails -- but, unfortunately, I do not understand it.

On the other hand, every 2-Indet (Indet of dimension 2) is uniquely typed. In fact, if $\underline{x}=(\boldsymbol{X}, x)$ is a 2-Indet, then $(\boldsymbol{Y}, y) \xrightarrow{f} (\boldsymbol{X}, x)$ is a typing for $\underline{x}$ iff, with the definitions $\boldsymbol{Y}_1=\mathrm{Supp}_{\boldsymbol{Y}}(dy)$, $f_1=f\upharpoonright\boldsymbol{Y}_1$, $\underline{dy} = (\boldsymbol{Y}_1, dy)$, etc, we have that

(a) $\mathtt{dd}y\neq\mathtt{cc}y$ unless $\mathtt{d}x$ or $\mathtt{c}x$, hence $\mathtt{d}y$ or $\mathtt{c}y$, is an identity; and

(b) $\underline{\mathtt{d}y}\xrightarrow{f_1}\underline{\mathtt{d}x}$ and $\underline{\mathtt{c}y}\xrightarrow{f_2}\underline{\mathtt{c}x}$ are typings for $\underline{\mathtt{d}x}$ and $\underline{\mathtt{c}x}$, respectively,

and $\boldsymbol{Y}$ is the pushout of $\boldsymbol{Y}_1\xleftarrow{\mathtt{incl}}\boldsymbol{Y}_3\xrightarrow{\mathtt{incl}}\boldsymbol{Y}_2$ where $\boldsymbol{Y}_3=\mathtt{Supp}_{\boldsymbol{Y}}(\{\mathtt{dd}y,\mathtt{cc}y\})$, with f defined compatibly with the pushout diagram.

Since $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ is uniquely typed iff the Indet $\tilde{\underline{\Gamma}}$ is, we have that not all 3-Indets are uniquely typed.

A large class of 2-Pd's, and the corresponding class of 3-Indets, the so-called 2-*anchored* ones, are uniquely typed. We call an indeterminate x *anchored* if x is of dimension ≤ 1, or, if $\mathtt{dim}(x)\geq 2$, $\mathtt{c}x$ is a non-identity cell, $\mathtt{c}x\neq 1_{\mathtt{cc}x}$. A computad $\boldsymbol{X}$ is *anchored* if all indets in $\boldsymbol{X}$ are anchored; a Pd $(\boldsymbol{X},\Gamma)$ is k*-anchored* if all indets of dimension k in $\boldsymbol{X}$ are anchored.

Of course, the dual notion when we disallow identities as domains, rather than codomains, of indeterminates gives rise to similar conclusions. The "Eckmann-Hilton" example above shows that bad effect of allowing indeterminates whose domain and codomain are both identities. In section 4, there will be a (simple) example showing that allowing two indeterminates, one with an identity domain, the other an identity codomain, is also bad. In other words, one has to globally exclude either identity domains, or identity codomains, for indets.

One of the main results of the present paper is

1.1 Theorem All 2-anchored 3-Indets, and as a particular case, all anchored 2-Pd's are uniquely typed.

Shape

The word "shape" instead of that of "type" is appropriate here too.

Let us say that Pd's $\underline{\Gamma}$ and $\underline{\Lambda}$ *have the same shape* if they belong to the same connected component of the category $\mathtt{Pd}$; that is, if there is a zig-zag

$$\underline{\Gamma}=\underline{\Gamma}_0\longrightarrow\underline{\Gamma}_1\longleftarrow\underline{\Gamma}_2\longrightarrow\ldots\longleftarrow\underline{\Gamma}_k=\underline{\Lambda}$$

of morphisms in $\mathtt{Pd}$.

Let me remind the reader of the fact that $\mathtt{Comp}$, the category of computads, is a locally finitely presentable category, in particular, it is both complete and cocomplete; see [M]. In $\mathtt{Comp}$, the colimits are "easy"; but the limits are only inferred from the "aleph-zero accessibility" of $\mathtt{Comp}$ (which is also "easy") plus the existence of the colimits. In particular, $\mathtt{Comp}$ has a terminal object $\boldsymbol{T}$, the terminal computad, but $\mathbf{T}$ is very far from being a trivial object. For more, see (also) [M].

Using a fixed copy of the terminal computad $\boldsymbol{T}$, and the morphism $!_{\boldsymbol{X}}:\boldsymbol{X}\longrightarrow\boldsymbol{T}$, every Pd

$\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$ has a unique morphism $\underline{\Gamma} \xrightarrow{!} \underline{\Sigma}$ to a Pd $\underline{\Sigma}$ where $\underline{\Sigma}=(\boldsymbol{Z}, \Sigma)$ has its underlying computad $\boldsymbol{Z}$ a subcomputad of $\boldsymbol{T}$. Following Ross Street, we call this $\underline{\Sigma}$ the *shape* of $\underline{\Gamma}$. We mean by *a shape*, in general, a Pd whose underlying computad is a subcomputad of $\boldsymbol{T}$.

Note that this fits the previous terminology: the two meanings of "having the same shape" coincide -- and in the zig-zag of the first definition, we may always take $k=2$.

A type of a Pd is also a type of the shape of the Pd.

If two Pd's have the same type (the same separated Pd is a type of both), then they also have the same shape. I do not know if the converse holds.

The concepts of "type" and "shape" are, in a sense, dual to each other. Obviously, the "type" works less smoothly than the "shape". However, this is not simply a drawback of the notion of "type". The non-uniqueness related to "types" is a real difficulty with the idea of occurrence that cannot be ignored.

Concrete presheaf categories of computads

The question whether or not various categories of computads are *presheaf categories*, a question that has been investigated in the literature, is closely related to the question which Pd's are uniquely typed. I introduce this subject with some new terminology.

A class $\boldsymbol{C}$ of computads is said to be *standard* if

1) it is a sieve in $\mathtt{Comp}$: whenever $\boldsymbol{X}\to\boldsymbol{Y}$ is an arrow in $\mathtt{Comp}$, and $\boldsymbol{Y}\in\boldsymbol{C}$, then $\boldsymbol{X}\in\boldsymbol{C}$; and

2) it is closed under coverings in $\mathtt{Comp}$: whenever $(\boldsymbol{X}_i \xrightarrow{f_i} \boldsymbol{Y})_{i\in I}$ is a family of arrows in $\mathtt{Comp}$, $\boldsymbol{X}_i\in\boldsymbol{C}$ for all $i\in I$, and the derived family of the arrows $(|\boldsymbol{X}_i| \xrightarrow{|f_i|} |\boldsymbol{Y}|)_{i\in I}$ on the sets of indeterminates is a surjective family, then $\boldsymbol{Y}\in\boldsymbol{C}$.

There are many important examples of standard classes. The total class is an example. So is the class of *anchored* computads; for the term, see above (it is an easy fact seen in [M] that if $f:\boldsymbol{X}\to\boldsymbol{Y}$ is a morphism of computads, and $a\in\|\boldsymbol{X}\|$ is not an identity, then $f(a)$ is not an identity either). The class of *positive computads,* in which there are no indeterminates with codomain or domain equal to an identity, is a natural standard class; in fact, it seems that the pasting schemes of [P] or [S] are meant to be positive.

An important example is the class of *many-to-one computads*: the class of computads $\boldsymbol{X}$ for which for every $x\in|\boldsymbol{X}|$, $\mathtt{dim}(x)\geq 1$, we have that $\mathtt{c}x$ is an indeterminate itself. (See, e.g., [M] for why many-to-one computads are important.) For any fixed $n\in\mathbb{N}$, the computads of dimension at most n is another example.

Given any class $\boldsymbol{C}$ of computads, the Pd's *associated with* $\boldsymbol{C}$ are those Pd's $(\boldsymbol{X}, \Gamma)$ for which $\boldsymbol{X}\in\boldsymbol{C}$. The class of Pd's associated with $\boldsymbol{C}$ is written as $\mathtt{Pd}(\boldsymbol{C})$. Similarly, we have

$\mathtt{Indet}(\boldsymbol{C})$, the class of Indets associated with $\boldsymbol{C}$.

If $\boldsymbol{C}$ is standard, then each of the classes $\mathtt{Indet}(\boldsymbol{C})$ and $\mathtt{Pd}(\boldsymbol{C})$ uniquely determines $\boldsymbol{C}$. Namely, $\boldsymbol{X}\in\boldsymbol{C}$ iff for all $\Gamma\in\|\boldsymbol{X}\|$, $(\mathtt{Supp}_{\boldsymbol{X}}(\Gamma),\Gamma)\in\mathtt{Pd}(\boldsymbol{X})$ iff for all $x\in|\boldsymbol{X}|$, $(\mathtt{Supp}_{\boldsymbol{X}}(x),x)\in\mathtt{Indet}(\boldsymbol{C})$.

For a class $\boldsymbol{I}$ of Indets, there is a standard class $\boldsymbol{C}$ with $\mathtt{Indet}(\boldsymbol{C})=\boldsymbol{I}$, if and only if the following both hold:

1) whenever $\underline{x}=(\boldsymbol{X},x)\in\boldsymbol{I}$, and $y\in|\boldsymbol{X}|$, then $\underline{y}=(\mathtt{Supp}_{\boldsymbol{X}}(y),y)\in\boldsymbol{I}$;

2) whenever $\underline{x}\in\boldsymbol{I}$, and $\underline{y}\longrightarrow\underline{x}\longrightarrow\underline{z}$ are arrows in $\mathtt{Indet}$, then both $\underline{y}$, $\underline{z}$ belong to $\boldsymbol{I}$.

Note that 2) can be said equivalently in this way: $\boldsymbol{I}$ is *shape-determined*: if two indets have the same shape, and one of them is in $\boldsymbol{I}$, then so is the other.

1) is a natural "reasonability condition": if we "accept" an indeterminate, we should also "accept" all indets involved in it.

A *standard class of Indets* is one that satisfies the last-listed conditions 1) and 2).

We may say that the standard classes of computads, and the standard classes of Pd's are the ones that are selected by the shapes, or equivalently, the types of indets involved in them.

A *concrete category* is a pair $(\boldsymbol{A},|-|)$ where $\boldsymbol{A}$ is a category, $|-|$ is a functor $|-|:\boldsymbol{A}\longrightarrow\mathtt{Set}$ to the category of sets. The concrete categories $(\boldsymbol{A},|-|_{\boldsymbol{A}})$, $(\boldsymbol{B},|-|_{\boldsymbol{B}})$ are said to be *equivalent* if there exists an equivalence of categories $E:\boldsymbol{A}\xrightarrow{\simeq}\boldsymbol{B}$ that is compatibly with the underlying-set functors: $|-|_{\boldsymbol{B}}\circ E\cong|-|_{\boldsymbol{A}}$.

Any category of the form $\hat{\boldsymbol{D}}\overset{\mathrm{def}}{=}\mathtt{Set}^{\boldsymbol{D}^{\mathrm{op}}}$, with $\boldsymbol{D}$ a small category, is regarded as a *concrete category* with the underlying-set functor $(F\in\boldsymbol{D}^{\mathrm{op}})\mapsto\coprod_{X\in\mathrm{Ob}(\boldsymbol{D})}F(X)$.

Every subcategory of $\mathtt{Comp}$ is regarded as a *concrete category* with the underlying-set functor defined as $\boldsymbol{X}\mapsto|\boldsymbol{X}|=$the set of indets in $\boldsymbol{X}$.

We say of a concrete category that it is a *concrete presheaf category* if it is equivalent to the concrete category $\hat{\boldsymbol{D}}$ for some small category $\boldsymbol{D}$.

Any class of computads determines a full subcategory of $\mathtt{Comp}$, and thus a concrete category; if the class is standard, we call the resulting concrete category a *standard category of computads*.

The following is stated with a different wording, and proved, in [M].

Proposition [M] (iii) A standard category $\boldsymbol{C}$ of computads is a concrete presheaf category if and only if every Indet in $\mathtt{Indet}(\boldsymbol{C})$ is uniquely typed.

Remarks **1** The phrase " $(\boldsymbol{X}, x)$ is uniquely typed" is meant here in the exact sense stated above, without relativization to the subcategory $\boldsymbol{C}$ -- although such relativization would result in a correct statement too.

2 Modulo Theorem [M] (i), Prop (iii) is elementary category theory, involving the Yoneda functor and the like. On the other hand, I consider the Theorem [M] (i) on the existence of typing, quoted above from [M], to be a real theorem, requiring for its proof more than a superficial look at what it says -- at least until I am shown that I am wrong.

3 Note that we have that $\mathtt{Comp}$ itself is *not* a concrete presheaf category since there are Pd's that are not uniquely typed. In fact, $\mathtt{Comp}$ is not a presheaf category in the usual more general, "non-concrete", sense either: see [M]. Although I do not know, it may be true that a standard category of computads that is a presheaf category is necessarily a concrete presheaf category.

4 The most important example of a standard category of computads which is a concrete presheaf category is the category of many-to-one computads: the class of computads $\boldsymbol{X}$ for which for every $x\in|\boldsymbol{X}|$, $\mathtt{dim}(x)\geq 1$, we have that $\mathtt{c}x$ is an indeterminate itself.

5 Since every 2-Indet is uniquely typed, $\mathtt{Comp}_{\leq 2}$, the category of computads of dimension at most 2 is a concrete presheaf category. This is an old observation of Steve Schanuel's.

In case $(\boldsymbol{A}, |-|)$ is a concrete category which is equivalent to a concrete presheaf category $(\mathtt{Set}^{\boldsymbol{D}^{\mathrm{op}}}, |-|)$, the category $\boldsymbol{D}$ involved is determined *up to isomorphism* by $(\boldsymbol{A}, |-|)$ itself. This contrasts with the "non-concrete" case when the exponent category $\boldsymbol{D}$ is not determined even up to equivalence (although its idempotent-splitting completion is).

When $(\boldsymbol{A}, |-|)$ is equivalent to $(\mathtt{Set}^{\boldsymbol{D}^{\mathrm{op}}}, |-|)$, we call $\boldsymbol{D}$ the *type-category* for $(\boldsymbol{A}, |-|)$.

We would like to call it, rather, the "shape-category"; but it is related to the "types" rather than the "shapes".

In fact, we can identify what $\boldsymbol{D}$ should be even before we know that $(\boldsymbol{A}, |-|)$ is a concrete presheaf category. In particular, we define the *type category* of any standard category $\boldsymbol{C}$ of computads as $\boldsymbol{C}\cap\mathtt{Ctp}$, i.e. the full subcategory of $\boldsymbol{C}$ whose objects consist of exactly one isomorphic copy for each computope that belongs to $\boldsymbol{C}$. The type-category, as a subcategory of $\mathtt{Ctp}$, is always a one-way and finitary category.

(I emphasize again that the notion of computope is an *absolute* notion: whether or not something is a computope is decided in $\mathtt{Comp}$, rather than some subcategory of it -- although if you relativize the definition to the standard subcategory, you still have a correct definition.)

We observe that, if $\boldsymbol{D}$ is the type category of $\boldsymbol{C}$, then we have a canonical functor E and a natural transformation φ as in

$$\begin{tikzcd} \boldsymbol{C} \arrow[rr, "E"] \arrow[dr, "|-|_{\boldsymbol{C}}"'] & & \mathrm{Set}^{\boldsymbol{D}^{\mathrm{op}}} \arrow[dl, "|-|"] \\ & \mathrm{Set} & \end{tikzcd} \qquad \varphi\colon |-|\circ E \longrightarrow |-|_{\boldsymbol{C}}$$

defined by $E(\boldsymbol{X})=\boldsymbol{D}(i(-),\boldsymbol{X})$, where $i:\boldsymbol{D}\to\boldsymbol{C}$ is the inclusion, and

$$\begin{array}{rl} \varphi_{\boldsymbol{X}}(A): & \boldsymbol{D}(A,\boldsymbol{X}) \longrightarrow |\boldsymbol{X}| \qquad (\underline{x}=(A,x) \text{ a computope in } \boldsymbol{C}). \\ & (f:A\to X) \longmapsto f(x) \end{array}$$

We have (exercise!) that $(\boldsymbol{C}, |-|_{\boldsymbol{C}})$ is a concrete presheaf category if an only if E is an equivalence of categories and φ is an isomorphism of functors.

What we have said here about concrete presheaf categories and their type categories is general and simple category theory. On the other hand, the theoretical simplicity of the definition should not mislead one into believing that it is easy to get a concrete, workable description of the type-category, or that it is easy to see whether or not the standard category in question is a concrete presheaf category. For instance, $\mathrm{Comp}_{m/1}$, the category of many-to-one computads is a concrete presheaf category; but the "concrete" description of its type-category, the category of *multitopes*, whose theoretical definition we now have as $\mathrm{Comp}_{m/1}\cap\mathrm{Ctp}$, and the proof that it works as $\boldsymbol{D}$ in the last "exercise", are far from obvious; see [M] and the references there.

We write $\mathrm{Comp}_{\le 3}^{2\text{-anch}}$ for the full subcategory of Comp consisting of the computads of dimension at most 3 all whose 2-indets are anchored. The following is a consequence of 1.1 and (iii).

1.2 Corollary $\mathrm{Comp}_{\le 3}^{2\text{-anch}}$ is a concrete presheaf category.

§2 Factorization and geometry

Background from [M]

Let us fix a dimension n, at least 1, and a computad X, to consider the elements (cells) of X of dimension n, the set of whose is written as $\|X\|_n$.

For $a\in\|X\|$ (all pd's, all cells of the ω-cat X), $\mathrm{supp}_k(a)$ is the set of k-indets "occurring in a" (more precisely, in $|\mathrm{Supp}_X(a)|$).

We say that $\varphi\in\|X\|_n$ is an *atom* (n-atom if n needs to be emphasized) if it is *top*-(dimension)-*indecomposable* in this sense: whenever $\varphi=b\cdot e$ with $b, e\in\|X\|_n$, then either $b=1_{db}$, or $e=1_{ce}$. (As a reminder: $b\cdot e = e\circ_{n-1}b$, since $\mathrm{dim}(b)=\mathrm{dim}(e)=n$.)

It is easy to see (also, see below) that for an n-atom φ, $\mathrm{supp}_n(\varphi)$ is a singleton, $\mathrm{supp}_n(\varphi)=\{u\}$, say. (The converse is far from being true.) We write $\varphi[u]$ for φ to indicate the *nucleus* u of φ.

Let N be a positive integer.

Let us call a tuple $\Phi=(\varphi_1, \varphi_2, \ldots, \varphi_N)$ of n- atoms φ_i such that $\varphi_1\cdot\varphi_2\cdot\ldots\cdot\varphi_N$ is well-defined an *n-dimensional molecule*, or more simply, an *n-molecule*. The *product* $[\![\Phi]\!]$ of the molecule Φ is the pd $\varphi_1\cdot\varphi_2\cdot\ldots\cdot\varphi_N$. A *factorization* of the pd a is any molecule whose product is a.

N is the *length* of the molecule $\Phi=(\varphi_1, \varphi_2, \ldots, \varphi_N)$. The *top-content* of the molecule

$$\Phi = (\varphi_1[u_1], \varphi_2[u_2], \ldots, \varphi_N[u_N]) \quad , \tag{1}$$

denoted $[[\Phi]]$, is the multiset of the nuclei involved. In other words, $[[\Phi]]$ is the function on n-indets whose value at u is

$$[[\Phi]](u) = \#\{i\in\{1, \ldots, N\}:u=u_i\} .$$

Let us write $[\Phi]$ for the total set of all top-dimensional indets in the molecule Φ. In other words, if (1), $[\Phi]=\{u_1, \ldots u_N\}$.

For the sake of completeness, we extend these definitions to include the possibility of length 0 for a molecule. Let $n\geq 1$. A *length*-0 *n-molecule* Φ is given by an $(n-1)$-pd f; we write $\Phi=(f)$. We define the *value* of Φ, $[\![\Phi]\!]=[\![(f)]\!] \overset{\mathrm{def}}{=} \mathrm{id}_f$, the identity n-pd. The top-content $[[\Phi]]$ for $\Phi=(f)$ is the empty multiset; $[\Phi]$ is the empty set.

We define the domain $\mathrm{d}\Phi$ and codomain $c\Phi$ of the molecule Φ by $\mathrm{d}\Phi=\mathrm{d}[\![\Phi]\!]$, $\mathrm{c}\Phi=\mathrm{d}[\![\Phi]\!]$. When Φ has positive length N , and is as in (1), we have $\mathrm{d}\Phi=\mathrm{d}\varphi_1$, $\mathrm{c}\Phi=\mathrm{c}\varphi_N$. When $\Phi=(f)$ of zero length, then $\mathrm{d}\Phi=\mathrm{c}\Phi=f$.

Theorem [M] **(i)** Every pd a of dimension at least 1 can be factored as a product

$$a = \varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_N$$

of atoms φ_i , usually in more than one way. Here, N is a non-negative integer; $N=0$ is allowed.

(If $a=1_{da}$, a is considered to be an empty product of atoms. The empty product is unambiguously defined only when its domain, which is equal to its codomain, is separately specified. When $N\geq 1$, $\mathrm{d}a=\mathrm{d}\varphi_1$, $\mathrm{c}a=\mathrm{c}\varphi_N$.)

Equivalently, every pd a of positive dimension is the value $[\![\Phi]\!]$ of at least one, usually more than one, molecule Φ .

(ii) The length and the content of the factorization of any pd are uniquely determined by the pd: if Φ , Ψ are molecules, $[\![\Phi]\!]=[\![\Psi]\!]$ implies that $[[\Phi]]=[[\Psi]]$. Hence, we can talk about the length $\ell(\underline{\Gamma})$ and the top-content $[[\underline{\Gamma}]]$ of any Pd $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$. Similarly, $[\underline{\Gamma}] \overset{\mathrm{def}}{=} [\Phi]$ for any Φ such that $\Gamma=[\![\Phi]\!]$.

(iii) Every pd has only finitely many distinct factorizations.

(iv) Let $f:\boldsymbol{X}\to\boldsymbol{Y}$ be a morphism of computads, and let $\varphi\in\|\boldsymbol{X}\|$. Then φ is an atom if and only if $f(\varphi)$ is an atom.

(v) If $f:\underline{\Gamma}\to\underline{\Lambda}$, then $\ell(\underline{\Gamma})=\ell(\underline{\Lambda})$ and $[[\Lambda]](v)=\sum_{\substack{u\in[\Gamma]\\ f(u)=v}}[[\Gamma]](u)$ ($v\in[\Lambda]$).

Notation (iv) allows us to see any computad morphism $f:\boldsymbol{X}\to\boldsymbol{Y}$ as acting on the molecules in $\boldsymbol{X}$, and giving rise to molecules in $\boldsymbol{Y}$. For $\Phi=(\varphi_1, \ldots, \varphi_N)$ in $\boldsymbol{X}$, $f(\Phi)$ is defined as the molecule $(f\varphi_1, \ldots, f\varphi_N)$. We can also "contextualize" a molecule, and write $\underline{\Phi}=(\boldsymbol{X}, \Phi)$, with $\boldsymbol{X} = \mathrm{Supp}_{\boldsymbol{X}}(\Phi) = \bigcup_{i=1}^{N}\mathrm{Supp}_{\boldsymbol{X}}(\varphi_i)$, and have arrows of Molecules $f:\underline{\Phi}\to\underline{\Psi}$, all in the obvious senses.

Suppose $\boldsymbol{X}$ is a computad of dimension n (the maximal dimension of an indet in $\boldsymbol{X}$ is n). We write $\bar{\boldsymbol{X}}$ for $\bar{\boldsymbol{X}}=\boldsymbol{X}\upharpoonright(n-1)$, the $(n-1)$-truncation of $\boldsymbol{X}$, another computad. If $f:\boldsymbol{X}\to\boldsymbol{Y}$,

$\bar{f}:\bar{\boldsymbol{X}}\to\bar{\boldsymbol{Y}}$ is the truncation of f to dimensions $\le n-1$. If $f:\underline{\Gamma}\to\underline{\Lambda}$ is a map of Pd's, $\bar{f}$ is the corresponding map of truncated computads.

(vi) Let $f:\underline{\Gamma}\to\underline{\Lambda}$, a morphism of Pd's, and assume that $\bar{f}$ is an isomorphism of computads. Then f induces a surjection on the molecules defining Γ onto the molecules defining Λ. That is, for any molecule Ψ such that $[\![\Psi]\!]=\Lambda$, there is a molecule Φ such that $f(\Phi)=\Psi$ and $[\![\Phi]\!]=\Gamma$.

In [M], "atoms" and correspondingly "molecules", were defined in a more concrete manner than here. According to this definition, an n-atom φ in a computad $\boldsymbol{X}$ is a (well-defined) pd of the form

$$\varphi = b_{n-1}\cdot(b_{n-2}\cdot(\ldots(b_1\cdot u\cdot e_1)\ldots)\cdot e_{n-2})\cdot e_{n-1} \tag{2}$$

where $b_i, e_i\in\boldsymbol{X}_i$, and $u\in|\boldsymbol{X}|_n$ (u is an indeterminate).

(As further reminders: $b\cdot e$ stands for $e\circ_k b$ where $k=\min(\dim(b),\dim(e))-1$; $e\circ_k b$ is $\mathrm{id}_e^{(N)}\circ_k\mathrm{id}_b^{(N)}$ where $N=\max(\dim(b),\dim(e))$; $\mathrm{id}_e^{(m)}=e$ for $m=\dim(e)$, and $\mathrm{id}_e^{(p+1)}=\mathrm{id}_{\mathrm{id}_e^{(p)}}$ for $p\ge m$.)

(I note that when $n\ge3$, the ingredients b_i, e_i in (2) are not determined uniquely by the atom φ itself; an atom can usually be written in more than one way in the form (2).)

Using this definition of "atom", for (i), see (12) Prop in section 8 in [M]; (ii) is contained in section 9 in [M], which contains, more generally, a useful description (see also below) of when two molecules define the same pd.

Once we have (i) and (ii) for atom as in [M], it is obvious that the new (abstract) and the old (concrete) definitions of "atom" and "molecule" coincide.

Part (iii) of the theorem is contained in (the proof of) (4) Theorem in section 11 in [M].

The fact that φ is an atom implies that $f(\varphi)$ is an atom (the "only if" part in (iv)) is immediate under the concrete definition of "atom". The "if" part of (iv) and (v) are now clear on the basis of (i) and (ii).

Part (vi) is fairly clear on the basis of the description (in section 9 of [M]) mentioned above of the relation $\Phi\sim\Psi \overset{\mathrm{def}}{\Longleftrightarrow} [\![\Phi]\!]=[\![\Psi]\!]$. Since this is important, I reproduce the description and give the proof of (vi).

Copying from [M], we define the quaternary relation L on (n-)atoms (in a fixed but arbitrary computad) as follows. For atoms ρ,σ,φ,ψ,

$L(\rho, \sigma, \varphi, \psi) \iff$ there are atoms α and β such that

$$\mathrm{cc}(\alpha)=\mathrm{dd}(\beta)\ ,\ \rho=\alpha\cdot\mathrm{d}\beta\ ,\ \sigma=(\mathrm{c}\alpha)\cdot\beta\ ,\ \varphi=(\mathrm{d}\alpha)\cdot\beta\ ,\ \psi=\alpha\cdot(\mathrm{c}\beta)\ .$$

$L(\rho,\sigma,\varphi,\psi)$ implies that $\rho\cdot\sigma=\varphi\cdot\psi$; in fact, $L(\rho,\sigma,\varphi,\psi)$ says that the equality $\rho\cdot\sigma=\varphi\cdot\psi$ is an instance of the so-called commutative law (see [M]).

We write $E(\rho,\sigma,\varphi,\psi) \iff L(\rho,\sigma,\varphi,\psi)\vee L(\varphi,\psi,\rho,\sigma)$.

For molecules $\Phi=(\varphi_1,\ldots,\varphi_M)$, $\Psi=(\psi_1,\ldots,\psi_N)$ and $k\in\{1,\ldots,N\}$, let's write

$$\mathcal{S}_k(\Phi,\Psi) \overset{\text{def}}{\iff} M=N\ ,\ E(\varphi_k,\varphi_{k+1},\psi_k,\psi_{k+1}) \text{ and } \varphi_i=\psi_i \text{ for all } i\in\{1,\ldots,N\}-(k,k+1)\ ;$$

and

$$\mathcal{S}(\Phi,\Psi) \overset{\text{def}}{\iff} \text{there is } k\in\{1,\ldots,N\} \text{ such that } \mathcal{S}_k(\Phi,\Psi)\ .$$

$\mathcal{S}(\Phi,\Psi)$ says that the molecule Ψ is obtained from Φ by applying an instance of the commutative law to a pair of adjacent atoms in Φ . Since (of course) the relation E on atoms is symmetric, the relation $\mathcal{S}$ on molecules is symmetric too.

$\mathcal{S}^{r/tr}$ is the reflexive and transitive closure of $\mathcal{S}$, an equivalence relation.

Theorem [M] **(vii)** $[\![\Phi]\!]=[\![\Psi]\!]$ iff $\mathcal{S}^{r/tr}(\Phi,\Psi)$.

For (vii), see Section 9 of [M].

Note that (v) is immediate from (vii).

Proof of (vi) We have $f:\mathbf{X}\to\mathbf{Y}$ such that $\bar{f}:\bar{\mathbf{X}}\to\bar{\mathbf{Y}}$ is an isomorphism.

We have the *lemma*: if $f(\hat{\rho})=\rho$, $f(\hat{\sigma})=\sigma$ and $L(\rho,\sigma,\varphi,\psi)$, then there are $\hat{\varphi}$, $\hat{\psi}$ such that $f(\hat{\varphi})=\varphi$, $f(\hat{\psi})=\psi$ and $L(\hat{\rho},\hat{\sigma},\hat{\varphi},\hat{\psi})$.

The lemma shows that the set $\{f(\hat{\Phi}):[\![\hat{\Phi}]\!]=\Gamma\}$ is closed under the relation S : if $E(f(\hat{\Phi}),\Psi)$ then there is $\hat{\Psi}$ such that $f(\hat{\Psi})=\Psi$. The assertion in (vi) then follows by (i) and (vii).

To prove the lemma, we start with α and β witnessing the relation $L(\rho,\sigma,\varphi,\psi)$. Let us denote the inverse image of any cell $a\in\|\bar{\mathbf{Y}}\|$ under the isomorphism $\bar{f}$ by $\hat{a}$. If

$$\alpha = b_{n-1}\cdot(b_{n-2}\cdot(\ldots(b_1\cdot u\cdot e_1)\ldots)\cdot e_{n-2})\cdot e_{n-1}\,,$$

we define

$$\hat{\alpha} = \hat{b}_{n-1} \cdot (\hat{b}_{n-2} \cdot (\ldots(\hat{b}_1 \cdot \hat{u} \cdot \hat{e}_1)\ldots) \cdot \hat{e}_{n-2}) \cdot \hat{e}_{n-1} \ .$$

where we define $\hat{u}$ to be the nucleus of $\hat{\rho}$. Since the nucleus u of α is also the nucleus of ρ, and $f(\hat{\rho})=\rho$, we have $f(\hat{u})=u$. Since $\bar{f}$ is an isomorphism, and α is well-defined, it follows that $\hat{\alpha}$ is well-defined.

Similarly, we define $\hat{\beta}$. Next, from $\hat{\alpha}$ and $\hat{\beta}$, we define the atoms $\hat{\varphi}$ and $\hat{\psi}$ so that $\hat{\alpha}$ and $\hat{\beta}$ will witness the fact that $L(\hat{\rho}, \hat{\sigma}, \hat{\varphi}, \hat{\psi})$, showing the lemma.

This completes the proof (vi).

To discuss the most interesting aspect of factorization, "uniqueness up to the order of top-dimensional indets", we need to take *occurrences* of n-indets in an n-pd, rather than just the indets themselves. To be able to talk about occurrences of the top-dimensional indeterminates, we have to be able to *separate* distinct occurrences of the same indeterminate.

Given a Molecule $\underline{\Phi}=(\boldsymbol{X}, \Phi)$, $\Phi=(\varphi_1[u_1], \ldots, \varphi_N[u_N])$, we can define the $\hat{u}_i$ to be new indeterminates, distinct for distinct i, such that $\hat{u}_i \| u_i$. We put

$\hat{\boldsymbol{X}}=\bar{\boldsymbol{X}}[\hat{u}_i]_{i=1,\ \ldots,\ N}$.

For any fixed i, $\hat{\varphi}_i=\hat{\varphi}_i[\hat{u}_i]$ is the atom in $\hat{\boldsymbol{X}}$ which is "obtained by replacing u_i by $\hat{u}_i$"; we may write $\varphi_i[\hat{u}_i/u_i]$, or even $\hat{\varphi}_i[\hat{u}_i]$, for $\hat{\varphi}_i$. Formally, we have the computad $\bar{\boldsymbol{X}}[u_i]$ (a *single* n-indet, u_i, is being adjoined to $\bar{\boldsymbol{X}}$), and we have the map $g:\bar{\boldsymbol{X}}[u_i] \longrightarrow \hat{\boldsymbol{X}}$ defined to be the identity on $\bar{\boldsymbol{X}}$ and mapping u_i to $\hat{u}_i$; we put $\hat{\varphi}_i \overset{\text{def}}{=} g(\varphi_i)$; by (iv), $\hat{\varphi}_i$ is an atom.

Finally, we let $\hat{\Phi}=(\hat{\varphi}_1[\hat{u}_1], \ldots, \varphi_N[\hat{u}_N])$, and $\hat{\underline{\Phi}}=(\hat{\boldsymbol{X}}, \hat{\Phi})$. We have that $\hat{\underline{\Phi}}$ is *top-(dimensional-)separated*, by the definition that its top-content function is zero-one valued: $[[\hat{\underline{\Phi}}]](u)=1$ for $u\in[\hat{\Phi}]$ (and $[[\hat{\underline{\Phi}}]](u)=0$ otherwise).

We call a Pd *top-separated* if some, equivalently all (see (i) and (ii)), of its representing Molecules are top-separated. We will also use "n-separated" for "top-separated" with n the dimension of the Pd involved, mainly when $n=2$.

We have proved

Proposition [M] **(viii)** For any Pd $\underline{\Gamma}$, there is a top-separated Pd $\hat{\underline{\Gamma}}$, with a map $\gamma:\hat{\underline{\Gamma}}\to\underline{\Gamma}$ such that $\bar{\gamma}$ is the identity map.

We say that a map $\gamma:\hat{\underline{\Gamma}}\to\underline{\Gamma}$ is a *top-separating map for* $\underline{\Gamma}$ if $\hat{\underline{\Gamma}}$ is top-separated and $\bar{\gamma}$ is an isomorphism. (viii) implies that top-separating maps exist for all Pd's.

Proposition [M] **(ix)** If, in the diagram

$$\begin{tikzcd}
\hat{\underline{\Gamma}} \arrow[r, dashed, "\hat{f}"] \arrow[d, "\gamma"'] & \hat{\underline{\Lambda}} \arrow[d, "\lambda"] \\
\underline{\Gamma} \arrow[r, "f"'] & \underline{\Lambda}
\end{tikzcd}$$

of maps of Pd's, first without $\hat{f}$, $\hat{\underline{\Gamma}}$ is top-separated and $\bar{\lambda}$ is an isomorphism (in particular, if γ and λ are top-separating), then $\hat{f}$ exists making the diagram commute.

Proof Let $\hat{\underline{\Gamma}}=(\hat{\boldsymbol{X}},\hat{\Gamma})$, $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$, $\hat{\underline{\Lambda}}=(\hat{\boldsymbol{Y}},\hat{\Lambda})$, $\underline{\Gamma}=(\boldsymbol{Y},\Gamma)$.

Let $\hat{\Phi}$ represent $\hat{\Gamma}$. Let $\Phi=\gamma(\hat{\Phi})$ and $\Psi=f(\Phi)$. Φ represents Γ, Ψ represent Λ. Using (vi), choose $\hat{\Psi}$ representing $\hat{\Lambda}$ such that $\lambda(\hat{\Psi})=\Psi$. Writing $\hat{\Phi}=(\hat{\varphi}_i[\hat{u}_i])$, $\Phi=(\varphi_i[u_i])$, $\Psi=(\psi_i[v_i])$, $\hat{\Psi}=(\hat{\psi}_i[\hat{v}_i])$, we now have, without the top horizontals, the following three diagrams:

$$\begin{tikzcd}
\bar{\hat{\boldsymbol{X}}} \arrow[r, dashed, "\bar{\hat{f}}"] \arrow[d, "\bar{\gamma}"'] \arrow[dr, phantom, "\circ"] & \bar{\hat{\boldsymbol{Y}}} \arrow[d, "\cong"', "\bar{\lambda}"] \\
\bar{\boldsymbol{X}} \arrow[r, "\bar{f}"'] & \bar{\boldsymbol{Y}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{\boldsymbol{X}} \arrow[r, dashed, "\hat{f}"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circ"] & \hat{\boldsymbol{Y}} \arrow[d, "\cong"', "\lambda"] \\
\boldsymbol{X} \arrow[r, "f"'] & \boldsymbol{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\hat{u}_i} \arrow[r, dashed, mapsto, "\hat{f}"] \arrow[d, "\gamma"'] & \underline{\hat{v}_i} \arrow[d, "\lambda"] \\
u_i \arrow[r, mapsto, "f"'] & v_i
\end{tikzcd}$$

The first is filled in by its top horizontal uniquely, by $\bar{\lambda}$ being an isomorphism. Next, we note that stipulating that $\hat{f}$ map the top-dimensional indets as shown in the third diagram is consistent, for the reasons that, first, the $\hat{u}_i$ are distinct for distinct i ($\hat{\underline{\Gamma}}$ is top-separated), and secondly, writing ∂ for either $\mathtt{d}$ or $\mathtt{c}$, we have

$\bar{\lambda}\bar{\hat{f}}(\partial\hat{u}_i)=\bar{f}\bar{\gamma}(\partial\hat{u}_i)=\partial v_i=\lambda(\partial\hat{v}_i)$, from which $\bar{\hat{f}}(\partial\hat{u}_i)=\partial\hat{v}_i$ follows since $\bar{\lambda}$ is an isomorphism.

Having defined $\hat{f}:\hat{\boldsymbol{X}}\to\hat{\boldsymbol{Y}}$, we see that the diagram

$$\begin{array}{ccc}\hat{\boldsymbol{X}} & \xrightarrow{\hat{f}} & \hat{\boldsymbol{Y}} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \boldsymbol{X} & \xrightarrow[f]{} & \boldsymbol{Y}\end{array}$$

commutes since it does when the top dimension is removed, and it does on the top-dimensional indets. We still need to see that $\hat{f}$ is a map of Pd's: $\hat{f}:\underline{\hat{\Phi}}\to\underline{\hat{\Psi}}$; that is, $\hat{f}(\hat{\Phi})=\hat{\Psi}$, i.e., $\hat{f}(\hat{\varphi}_i)=\hat{\psi}_i$ for all i .

This may look obvious; but here is a proof.

Let's fix i and abbreviate $\rho=\hat{f}(\hat{\varphi}_i)$, $\sigma=\hat{\psi}_i$, to show that $\rho=\sigma$.

ρ and σ share the same nucleus, namely $u=\hat{v}_i$. Writing $\boldsymbol{Z}$ for $\hat{\boldsymbol{Y}}$, $\lambda:\boldsymbol{Z}\to\boldsymbol{Y}$ maps ρ and σ to the same atom, namely $\pi=\psi_i$. The *subcomputad* of $\boldsymbol{Z}$ generated by $\bar{\boldsymbol{Z}}$ and the single indet u is $\bar{\boldsymbol{Z}}[u]$, that is, it is obtained by freely adjoining u to $\bar{\boldsymbol{Z}}$ (such trivial-sounding facts, ones that are in need of proof for the pedant as I am, are shown in [M]). Similarly, for $v=\lambda(u)$, $\bar{\boldsymbol{Y}}[v]$ is the subcomputad of $\boldsymbol{Y}$ generated by $\bar{\boldsymbol{Y}}$ and v .

Both ρ and σ are in $\bar{\boldsymbol{Z}}[u]$, π is in $\bar{\boldsymbol{Y}}[v]$. The map $\lambda:\boldsymbol{Z}\to\boldsymbol{Y}$ restricts to $\mu:\bar{\boldsymbol{Z}}[u]\longrightarrow\bar{\boldsymbol{Y}}[v]$, and μ maps both ρ and σ to π . μ is an isomorphism since $\bar{\mu}=\bar{\lambda}$ is an isomorphism and thus μ^{-1} can be defined by stipulating that $\overline{\mu^{-1}}=\overline{\lambda^{-1}}$ and $\mu^{-1}(v)=u$. Since the isomorphism μ maps ρ and σ to the same element π , $\rho=\sigma$ as desired.

This completes the proof of (ix).

Inspired by (ix), we call the domain of a top-separating map for $\underline{\Gamma}$ the *top-type* of $\underline{\Gamma}$. We write $\underline{\hat{\Gamma}}$ for the top-type of $\underline{\Gamma}$.

Corollary [M] **(x)** The top-type and the top-separating map are unique up to isomorphism: if $\underline{\Lambda}\xrightarrow{f}\underline{\Gamma}\xleftarrow{g}\underline{\Xi}$ are both top-separating maps for $\underline{\Gamma}$, then there is an isomorphism $\underline{\Lambda}\xrightarrow[\cong]{h}\underline{\Xi}$ such that $g\circ h=f$.

Proof Immediate from (ix).

Let us note that we cannot, in general, say "up to *unique* isomorphism" in (x). In §1, we saw the example of a 2-Pd $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$, with $|\boldsymbol{X}|=\{X, u, v\}$, which had a non-trivial automorphism h that exchanged u and v. Defining $\dot{\boldsymbol{X}}$ to be generated by X and u alone, and $\dot{\Gamma}=u\cdot u$, $\dot{\underline{\Gamma}}=(\dot{\boldsymbol{X}},\dot{\Gamma})$, we have a unique map $\gamma:\underline{\Gamma}\to\dot{\underline{\Gamma}}$, the one for which $\gamma(X)=X$, $\gamma(u)=\gamma(v)=u$. γ is a top-separator; however, $\gamma\circ h=\gamma\circ\mathrm{id}_{\boldsymbol{X}}=\gamma$.

Let $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ be any Pd. We say that it has (*an essentially*) *unique factorization* if for any molecules

$$\Phi=(\varphi_1[u_1],\ ...,\ \varphi_N[u_N]),\quad \Psi=(\psi_1[v_1],\ ...,\ \psi_N[v_N]),\quad [\![\Phi]\!]=[\![\Psi]\!]=\Gamma$$
$$\text{and } u_i=v_i \text{ for } i=1,\ ...,N \text{ imply that } \Phi=\Psi.$$

Thus, "unique factorization" is "uniqueness up to the order of the top-dimensional indets".

It quickly becomes obvious that it is reasonable to expect this to hold for a top-separated Pd $\underline{\Gamma}$ only.

There is a stronger form of unique factorization that also subsumes a cancellation law, and which, as it happens, I can show to hold whenever I am able to show ordinary unique factorization. However, I don't know if ordinary unique factorization in fact implies strong unique factorization.

We say of the top-separated n-pd Γ that it has *strong unique factorization* if every time

$$\Lambda_1\cdot\Lambda_2\cdot\ ....\ \cdot\Lambda_m\ =\ \Gamma_1\cdot\Gamma_2\cdot\ ....\ \cdot\Gamma_m\ =\ \Gamma\ ,$$

the Λ_i and Γ_i are n-pd's, and

$$[\Lambda_i]=[\Gamma_i]\ \text{ for } i=1,\ ...,m\ ,$$

we must have that $\Lambda_i=\Gamma_i$ for all $i=1,\ ...,m$.

It is clear that the special case $m=2$ implies the general case, and that strong unique factorization for the top-type of $\underline{\Gamma}$ implies *cancellation* for Γ:

$$(\Gamma_1\cdot\Gamma_3=\Gamma_2\cdot\Gamma_3=\Gamma\ \vee\ \Gamma_3\cdot\Gamma_1=\Gamma_3\cdot\Gamma_2=\Gamma)\ \Longrightarrow\ \Gamma_1=\Gamma_2\ .$$

Remarks It may be objected that talking about unique factorization is reasonable only if the (ordinary) commutative law holds. Although full commutativity does not hold, the law we have called "the commutative law" is important, in fact, it is the main mover of the algebra of ω-categories. Under the (restricted) commutative law, when for two atoms $\rho[u]$, $\sigma[v]$ of the same dimension the product $\rho[u]\cdot\sigma[v]$ is well defined, under certain definite circumstances, we can reverse the order of the nuclei u and v, and write $\rho[u]\cdot\sigma[v]$ as $\varphi[v]\cdot\psi[u]$ for suitable atoms $\varphi[v]$, $\psi[u]$. The commutative law is the main basis for the dynamic of the algebra of ω-categories.

In [M], the operations of the laws of ω-category theory were restated in a way that resembles the laws for (commutative) rings: we have unit laws, associative laws, distributive laws and the (restricted) commutative law. It is tempting to consider ω-category theory as a kind of higher dimensional ring theory; the computads play the role of the rings of polynomials.

"Strong unique factorization" would be immediate from ordinary unique factorization if we had the ordinary commutative law available. As things are, this implication is not (yet) seen, but it is interesting that the strong version is provable in those cases when we are able to show ordinary unique factorization.

We write $\mathbf{G}^{\Gamma}$ for the set of representatives of $\underline{\Gamma}$: $\mathbf{G}^{\Gamma}=\{\Phi:[\![\Phi]\!]=\Gamma\}$. Any morphism $f:\underline{\Gamma}\longrightarrow\underline{\Lambda}$ induces a map $\mathbf{G}^{f}:\mathbf{G}^{\Gamma}\to\mathbf{G}^{\Lambda}$. Clearly, $f\mapsto\mathbf{G}^{f}$ is functorial.

Let $\underline{\Gamma}$ be a top-separated Pd of length N, and let $\mathbf{N}_{\underline{\Gamma}}$, or $\mathbf{N}$, denote the set, previously denoted by $[\Gamma]$, of all top-dimensional indets in Γ. For $\Phi\in\mathbf{G}^{\Gamma}$, $\Phi=(\varphi_1[u_1],\ ...,\ \varphi_N[u_N])$, let $<_{\Phi}$ be the (irreflexive total) order of the set $\mathbf{N}$ for which $u_i<u_j\Longleftrightarrow i<j\ \ (i=1,\ ...,\ N)$. We define the (irreflexive) partial order $\prec_{\Gamma}$ on the set $\mathbf{N}$ as the intersection of all $<_{\Phi}$, $\Phi\in\mathbf{G}^{\Gamma}$; thus, $\prec_{\Gamma}$ is the largest partial order on $\mathbf{N}$ that all $<_{\Phi}$, $\Phi\in\mathbf{G}^{\Gamma}$, are compatible with.

The entity $\prec_{\Gamma}$ is perhaps the main object introduced in this paper. It is called the *backbone order* of the Pd Γ.

Note that, for any top-separated $\underline{\Gamma}$, $\underline{\Lambda}$, map $f:\underline{\Gamma}\to\underline{\Lambda}$, $\Phi\in\mathbf{G}^{\Gamma}$, and $u,v\in\mathbf{N}_{\Gamma}$, $u<_{\Phi}v$ iff $fu<_{f\Phi}fv$.

For a partial order $\prec$ on a set $\mathbf{N}$, let's write $\mathbf{G}^{\prec}$ for the set of all total orders $\lll$ on the set $\mathbf{N}$ that are compatible with $\prec$, $\prec\subseteq\lll$.

To say that the mapping

$$\mathbf{o}_{\Gamma}\ \overset{\mathrm{def}}{=}\ (\Phi\mapsto<_{\Phi}):\mathbf{G}^{\Gamma}\to\mathbf{G}^{\prec_{\Gamma}} \tag{3}$$

is 1-1 is to say that $\underline{\Gamma}$ has unique factorization.

Concerning of the functoriality of the mapping (3) under mappings of Pd's, we note the following.

Given an isomorphism of ordered sets, $g:(\mathbf{N}_1, \prec_1) \xrightarrow{\cong} (\mathbf{N}_2, \prec_2)$, we have the induced "direct image" bijection $\mathbf{G}^g:\mathbf{G}^{\prec_1} \xrightarrow{\cong} \mathbf{G}^{\prec_2}$ for which

$$(u, v)\in\ll \iff (gu, gv)\in\mathbf{G}^g(\ll)$$

($\ll\in\mathbf{G}^{\prec_1}$, $u, v\in\mathbf{N}_1$) . On the other hand, a map $f:\underline{\Gamma}\longrightarrow\underline{\Lambda}$ of top-separated Pd's Γ and Λ induces an isomorphism $\prec_f:(\mathbf{N}_\Gamma, \prec_\Gamma) \xrightarrow{\cong} (\mathbf{N}_\Lambda, \prec_\Lambda)$, where $\prec_f(u)=f(u)$ ($u\in\mathbf{N}_\Gamma$) (remember that u is an indet, and f is a map of computads).

Combining these constructions, we have the commutative diagram, induced by a map $f:\underline{\Gamma}\longrightarrow\underline{\Lambda}$ of top-separated Pd's $\underline{\Gamma}$ and $\underline{\Lambda}$:

$$\begin{tikzcd}
\mathbf{G}^{\Gamma} \arrow[r, "\mathbf{o}_\Gamma"] \arrow[d, "\mathbf{G}^f"'] \arrow[dr, phantom, "\circ"] & \mathbf{G}^{\prec_\Gamma} \arrow[d, "\mathbf{G}^{\prec_f}", "\cong"'] \\
\mathbf{G}^{\Lambda} \arrow[r, "\mathbf{o}_\Lambda"'] & \mathbf{G}^{\prec_\Lambda}
\end{tikzcd} \qquad (3.1)$$

New results in dimension 2

The theorems and propositions starting with 2.1 Theorem next will be proved later in the paper.

2.1. Theorem Top-separated anchored 2-Pd's have unique factorization; in fact, strong unique factorization.

In the next section, we will give a (simple) example of a (non-anchored) top-separated 2-Pd which fails to have unique factorization.

For a Pd $\underline{\Gamma}$ of length N having unique factorization, we now have the obvious bound $N!$ on the number of molecules representing it; it is easy to see that this bound is sharp.

We need more notation.

For atoms ρ , σ , we write $\rho\rightarrow\sigma$ for:

$$\rho\rightarrow\sigma \iff \exists\alpha,\beta.\ \mathrm{cc}\alpha=\mathrm{dd}\beta\ \&\ \rho=\alpha\cdot\mathrm{d}\beta\ \&\ \sigma=\mathrm{c}\alpha\cdot\beta\ ; \tag{4}$$

and we write $\varphi\leftarrow\psi$ for:

$$\varphi\leftarrow\psi \iff \exists\alpha,\beta.\ \mathrm{cc}\alpha=\mathrm{dd}\beta\ \&\ \varphi=d\alpha\cdot\beta\ \&\ \psi=\alpha\cdot c\beta\ . \tag{5}$$

Equivalently, using L from above,

$$\rho\rightarrow\sigma \iff \exists\varphi,\psi\ L(\rho,\sigma,\varphi,\psi)\ ,$$
$$\varphi\leftarrow\psi \iff \exists\rho,\sigma\ L(\rho,\sigma,\varphi,\psi)\ .$$

Note that $\rho\rightarrow\sigma$ implies that $\rho\cdot\sigma\downarrow$, since it follows that $\mathrm{c}\rho=\mathrm{c}\alpha\cdot\mathrm{d}\beta=\mathrm{d}\sigma$. Similarly, $\varphi\leftarrow\psi$ implies that $\varphi\cdot\psi\downarrow$. (Note that $\varphi\leftarrow\psi$ is *not* the same as $\psi\rightarrow\varphi$: in both $\rho\rightarrow\sigma$ and $\varphi\leftarrow\psi$, the first term (ρ , respectively, φ) is "above" the second term, using the imagery of vertical compositions $\rho\cdot\sigma$, $\varphi\cdot\psi$ going downward). Finally, note that if $\rho\rightarrow\sigma$, with data α,β as in (4), and we define φ and ψ with the same data as in (5), then φ,ψ are well-defined, and $\varphi\leftarrow\psi$ holds. Of course, the dual statement also holds.

We write $\rho\longleftrightarrow\sigma$ for $\rho\rightarrow\sigma\vee\rho\leftarrow\sigma$.

Of course, we have that

$$L(\rho,\sigma,\varphi,\psi) \Longrightarrow \rho\rightarrow\sigma\ \&\ \varphi\leftarrow\psi\ . \tag{6}$$

It is most important to observe that if $E(\rho,\sigma,\varphi,\psi)$, then the nuclei of ρ and σ "change places: we have

$$\rho[u]\cdot\sigma[v]\ =\ \varphi[v]\cdot\psi[u]\ .$$

Let $\Phi=(\varphi_1[u_1], \ldots \varphi_N[u_N])$ be any top-separated molecule. Let us write $\mathbf{N}$ for the set $[\Phi]=\{u_i : i=1, \ldots, N\}$, and for any $u\in\mathbf{N}$, let us write

$$\varphi^{\Phi}[u] \overset{\text{def}}{=} \varphi_i[u_i]$$

for the i such that $u=u_i$. For $u, v\in\mathbf{N}$, define $u\xrightarrow{\Phi}v$ (" u and v are exchangeable, u left, v right, in Φ ") and $u\xleftarrow{\Phi}v$ by

$$u\xrightarrow{\Phi}v \quad \overset{\text{def}}{\Longleftrightarrow} \quad u<_{\Phi}!v \ \& \ \varphi^{\Phi}[u]\rightarrow\varphi^{\Phi}[v]$$
$$u\xleftarrow{\Phi}v \quad \overset{\text{def}}{\Longleftrightarrow} \quad u<_{\Phi}!v \ \& \ \varphi^{\Phi}[u]\leftarrow\varphi^{\Phi}[v]$$

($u<_{\Phi}!v$ means that v is the immediate successor of u in the order $<_{\Phi}$).

Note that, automatically, for any morphism $f:\underline{\Phi}\rightarrow\underline{\Psi}$, $u\xrightarrow{\Phi}v$ implies $fu\xrightarrow{f\Phi}fv$ and similarly for $\longleftarrow$.

2.2 Theorem Let $\underline{\Gamma}$ be a *top-separated* anchored 2-Pd.

(a) The possible orders of 2-indets in the factorizations of $\underline{\Gamma}$ are precisely those that extend $\prec=\prec_{\Gamma}$. The map (3) is a bijection.

(b) The following two equivalent definitions define $\rightarrow_{\Gamma}$ to be an irreflexive partial order on the set $\mathbf{N}_{\Gamma}$:

$$u\rightarrow_{\Gamma}v \quad \Longleftrightarrow \quad \forall\Phi\in\mathbf{G}^{\Gamma}.\ u<_{\Phi}!v\Longrightarrow u\xrightarrow{\Phi}v$$
$$\Longleftrightarrow \quad \exists\Phi\in\mathbf{G}^{\Gamma}.\ (u<_{\Phi}!v\ \&)\ u\xrightarrow{\Phi}v\ .$$

(c) For $\rightarrow\ =\ \rightarrow_{\Gamma}$ in (b), the pair $(\prec,\rightarrow)$ is a *planar arrangement* on the set $\mathbf{N}$, meaning that for any $u\neq v$ in $\mathbf{N}$, exactly one of the following four alternatives hold:

$$u\prec v\ , \qquad v\prec u\ , \qquad u\rightarrow v\ , \qquad v\rightarrow u\ .$$

2.3 Proposition Any morphism $f:\underline{\Gamma}\longrightarrow\underline{\Lambda}$ of *top-separated* anchored 2-Pd's induces an *isomorphism*

$$[f]:(\mathbf{N}_{\underline{\Gamma}}, \prec_{\underline{\Gamma}}, \rightarrow_{\underline{\Gamma}})\xrightarrow{\ \cong\ }(\mathbf{N}_{\underline{\Lambda}}, \prec_{\underline{\Lambda}}, \rightarrow_{\underline{\Lambda}})$$

of the planar arrangements associated with $\underline{\Gamma}$ and $\underline{\Lambda}$. In other words, for any $u, v\in\mathbf{N}_{\underline{\Gamma}}$,

$u \prec_{\underline{\Gamma}} v \iff fu \prec_{\underline{\Lambda}} fv$, $u \to_{\underline{\Gamma}} v \iff fu \to_{\underline{\Lambda}} fv$.

We consider the set $\mathbf{G}^{\Gamma}$ of variants of $\underline{\Gamma}$ the vertices of a (undirected, loop-free) graph, with edges the pairs (Φ,Ψ) such that $\mathcal{S}(\Phi, \Psi)$. On the other hand, the set $\mathbf{G}^{\prec}$ of total extension of the partial order $\prec$ carries a natural graph-structure: $<_1$ and $<_2$ are connected by an edge, $\mathcal{S}(<_1, <_2)$, if one is obtained by a transposition of two consecutive elements in the other.

It is an easy general fact about total extensions of finite partial orders that $\mathbf{G}^{\prec}$ is a connected graph.

On the other hand, since, for any Θ such that $\Gamma = [\![\Theta]\!]$, we have $\mathbf{G}^{\Gamma} = \{\Phi : \mathcal{S}^{r/tr}(\Theta,\Phi)\}$, it is clear that also $\mathbf{G}^{\Gamma}$ is connected.

2.4 Proposition For top-separated anchored 2-Pd $\underline{\Gamma}$, in (3), we have an isomorphism of graphs.

2.5 Elementary Lemma A finite planar arrangement has no non-trivial automorphism.

(See also section 3 below.) In this, a finite planar arrangement is similar to a finite total linear (1D) order. A planar arrangement is a kind of total order of a portion of the plane.

2.6 Corollary The category of anchored 2-Pd's is a preorder: if $\underline{\Gamma} \overset{f}{\underset{g}{\rightrightarrows}} \underline{\Lambda}$ where $\underline{\Gamma}$, $\underline{\Lambda}$ are anchored 2-Pd's, then $f=g$.

Proof Note that by (viii) and (ix), we may assume without loss of generality that $\underline{\Gamma}$, $\underline{\Lambda}$ are top-separated.

Assume $\underline{\Gamma} \overset{f}{\underset{g}{\rightrightarrows}} \underline{\Lambda}$, $\underline{\Gamma}$, $\underline{\Lambda}$ top-separated. By 2.3 and 2.5, the effects of f and g on the 2-indets in $\underline{\Gamma}$ are the same.

Let $\Phi = (\varphi_1[u_1], \ldots, \varphi_N[u_N]) \in \mathbf{G}^{\Gamma}$. We have that $v_i \overset{\text{def}}{=} f(u_i) = g(u_i)$ $(i=1, \ldots, N)$. Therefore, for $\Psi = f\Phi = (f\varphi_1[v_1], \ldots, f\varphi_N[v_N])$, $\Theta = g\Phi =$

$(g\varphi_1[v_1], ..., g\varphi_N[v_N])$, we have that Ψ and Θ define the same order $\ll = <_\Psi = <_\Theta$ on the top indets, namely $v_1 \ll v_2 \ll ... \ll v_N$. They also represent the same pd, namely Λ. By unique factorization (2.1) for Λ, we have $\Psi=\Theta$.

This means that $\psi_i \overset{\text{def}}{=} f(\varphi_i)=g(\varphi_i)$ for $i=1, ..., N$. With $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$, $\underline{\Lambda}=(\boldsymbol{Y}, \Lambda)$, $\boldsymbol{X}_i=\mathrm{Supp}_{\boldsymbol{X}}(\varphi_i)$, $\boldsymbol{Y}_i=\mathrm{Supp}_{\boldsymbol{Y}}(\psi_i)$, $\underline{\varphi}_i=(\boldsymbol{X}, \varphi_i)$, $\underline{\psi}_i=(\boldsymbol{Y}, \psi_i)$, $f_i = f{\restriction}\boldsymbol{X}_i$, $g_i = g{\restriction}\boldsymbol{X}_i$, we have $\underline{\varphi}_i \underset{g_i}{\overset{f_i}{\rightrightarrows}} \underline{\psi}_i$. The fact that $f_i=g_i$ follows is the "one-atom" special case of the Corollary itself -- but it is something easy to check directly, given that 1-pd's are "obvious".

Since $\bigcup_{i=1}^{N} \boldsymbol{X}_i = \boldsymbol{X}$, and the restrictions of f and g to each $\boldsymbol{X}_i$ are equal, we conclude that $f=g$ as desired.

2.6 is the main ingredient of the proof of 1.1; it ensures condition 1*) in section 1 in "uniquely typed".

2.7 Corollary Let $f:\underline{\Gamma}\longrightarrow\underline{\Lambda}$ be *any* morphism of anchored 2-Pd's. f induces a *bijection* $\mathbf{G}^f:\mathbf{G}^\Gamma \xrightarrow{\cong} \mathbf{G}^\Lambda$ between molecules representing Γ and those representing Λ.

Proof In the case when both $\underline{\Gamma}$ and $\underline{\Lambda}$ are top-separated, the result is immediate from the diagram (3.1) and 2.2(a). Using (ix), we then see that it suffices to prove the assertion for the case when f is a top-separating map; assume it is.

By (vi), $\mathbf{G}^f$ is surjective. Assume that $\mathbf{G}^f(\Phi)=\mathbf{G}^f(\Psi)$, to show that $\Phi=\Psi$. Let $\Phi=(\varphi_1[u_1], ..., \varphi_N[u_N])$, $\Psi=(\psi_1[v_1], ..., \psi_N[v_N])$. Our assumption implies that $fu_i=fv_i$, in particular, $f\mathrm{d}u_i=\mathrm{d}fu_i=\mathrm{d}fv_i=f\mathrm{d}v_i$, and since $\bar{f}$ is an isomorphism, $\mathrm{d}u_i=\mathrm{d}v_i$. Similarly, $\mathrm{c}u_i=\mathrm{c}v_i$. For the underlying computad $\boldsymbol{X}$, $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$, we now have that the mapping h that is the identity on $\bar{\boldsymbol{X}}$, and maps u_i to v_i $(i=1, ..., N)$ is an automorphism of $\boldsymbol{X}$.

We also have $h(\Phi)=\Psi$. To see this, the real issue is that $[\![h(\Phi)]\!]=[\![\Phi]\!]=\Gamma$; once that is known, it is clear that $<_{h(\Phi)}=<_\Psi$, and thus, by 2.1, $h(\Phi)=\Psi$. But, for any molecules Θ and Σ in $\boldsymbol{X}$, $\bar{f}:\bar{\boldsymbol{X}}\to\bar{\boldsymbol{Y}}$ being an isomorphism implies that $\mathcal{S}(\Theta, \Sigma)$ iff $\mathcal{S}(f\Theta, f\Sigma)$, and thus, by the surjectivity of $\mathbf{G}^f$ (see (vi)), $\mathcal{S}^{\mathrm{r/tr}}(\Theta,\Sigma)$ iff $\mathcal{S}^{\mathrm{r/tr}}(f\Theta,f\Sigma)$, and so $[\![\Theta]\!]=[\![\Sigma]\!]$ iff $[\![f\Theta]\!]=[\![f\Sigma]\!]$. Applied to $\Theta=h(\Phi)$ and $\Sigma=\Psi$, since $f(h\Phi)=f\Phi=f\Psi$, this gets us $[\![h(\Phi)]\!]=[\![\Phi]\!]$ as desired.

Having the equality $h(\Phi)=\Psi$, we conclude that h is an automorphism $h:(X,\Gamma)\xrightarrow{\cong}(X,\Gamma)$. This immediately implies that h induces an automorphism of the planar arrangement $(N_\Gamma, \prec_\Gamma, \to_\Gamma)$ of 2-indets in X. But the map of 2-indets induced by h is just $u_i\mapsto v_i$. By 2.5, therefore, this map has to be the identity; $u_i=v_i$ for all $i=1, \ldots, N$. This of course means that $\Phi=\Psi$, which was to be proved.

We call a Molecule $\underline{\Phi}=(X,\Phi)$ *projective* if whenever $\underline{\Phi}\xrightarrow{f}\underline{\Psi}\xleftarrow{g}\underline{\Theta}$, there is $\underline{\Phi}\xrightarrow{h}\underline{\Theta}$ such that $g\circ h=f$. The concept of a Pd being projective is analogous, but different!

2.8 Elementary Lemma For any anchored 2-dimensional Molecule $\underline{\Phi}$, there is a projective Molecule $\hat{\underline{\Phi}}$ with a map $\hat{\underline{\Phi}}\to\underline{\Phi}$.

The proof is indeed elementary, since it depends on understanding 1-Pd's that are easy to understand. See later too.

2.9 Corollary For any anchored 2-Pd $\underline{\Gamma}$, there is a projective 2-Pd $\hat{\underline{\Gamma}}$ with a map $\hat{\underline{\Gamma}}\to\underline{\Gamma}$.

Proof Given $\underline{\Gamma}$, let Φ represent Γ; let $\hat{\underline{\Phi}}$ be a projective Molecule, with $\hat{\underline{\Phi}}\xrightarrow{f}\underline{\Phi}$. Let $\hat{\underline{\Gamma}}$ be the Pd represented by $\hat{\underline{\Phi}}$; then $\hat{\underline{\Gamma}}\xrightarrow{f}\Gamma$. I claim that $\hat{\underline{\Gamma}}$ is projective. Let $\hat{\underline{\Gamma}}\xrightarrow{g}\underline{\Lambda}\xleftarrow{h}\underline{\Xi}$. Let $\Psi=f(\hat{\underline{\Phi}})$. Then Ψ represents Λ. By 2.7, there is Θ representing Ξ such that $h(\Theta)=\Psi$. We now have $\hat{\underline{\Phi}}\xrightarrow{g}\underline{\Psi}\xleftarrow{h}\underline{\Theta}$. Since the Molecule $\hat{\underline{\Phi}}$ is projective, there is $\hat{\underline{\Phi}}\xrightarrow{k}\underline{\Theta}$ such that $h\circ k=g$, which was to be proved.

Recall from section 1 what we mean by a *separated* Pd; of course, this is something more than "top-separated". Recall the notion of "unique typing" also.

2.10 Corollary Anchored 2-Pd's are uniquely typed.

Proof Let $\underline{\Gamma}$ be an anchored 2-Pd. Condition 1), uniqueness of the specializing map, is part of 2.6. To show Condition 2), use 2.9 and fix a projective $\hat{\underline{\Gamma}}$ with $\hat{\underline{\Gamma}}\xrightarrow{f}\underline{\Gamma}$. Let

$\underline{\Gamma} \xleftarrow{g} \underline{\Lambda}$ be any arrow to $\underline{\Gamma}$ from a $\underline{\Lambda}$ separated. Since $\underline{\hat{\Gamma}}$ is projective, there is $\underline{\hat{\Gamma}} \xrightarrow{h} \underline{\Lambda}$ (such that $g \circ h = f$). By the definition of "separated", h is an isomorphism. We have shown that any type $\underline{\Lambda}$ of $\underline{\Gamma}$ must be isomorphic to $\underline{\hat{\Gamma}}$; in particular, any two types of $\underline{\Gamma}$ are isomorphic to each other.

It also follows that we have

2.11 Corollary "Projective" and "separated" for anchored 2-dimensional Pd's are the same property.

The obvious idea of unique composability, obvious in our context when pasting diagrams are defined via computads, was already mentioned in essence at the beginning of section 1.

We say that a computad $\boldsymbol{X}$ is *composable* if there is a pd Γ in $\boldsymbol{X}$ making $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$ into a Pd (that is, $\mathrm{Supp}_{\boldsymbol{X}}(\Gamma)=\boldsymbol{X}$). $\boldsymbol{X}$ is *uniquely composable* if said Γ is unique: there is exactly one Γ such that $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$ is a Pd. We can extend the terminology by saying that a Pd $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$ is *uniquely composed* if $\boldsymbol{X}$ is uniquely composable (Γ is the only pd in $\boldsymbol{X}$ whose support is the whole if $\boldsymbol{X}$.)

The final four assertions are for positive 2-pd's: ones that have domain and codomain both non-identity 1-pd's. Of course, a positive Pd is also anchored: positive = anchored and co-anchored.

Positive pd's are the only ones that [P1] and [P2] deal with.

2.12 Theorem A separated positive 2-Pd is uniquely composed.

Let me point out that "unique composedness" of a Pd in general can fail for at least two reasons. For one thing, if $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$ is a non-separated Pd, it is "bound" to be non-uniquely composed -- although there are uniquely composed positive pd's that are not separated: take 0-cells X, Y, Z, W , 1-indets $f, f':X \to Y$, $g:Y \to Z$, $h, h':Z \to W$, and the single 2-indet $u:f \cdot g \cdot h \to f' \cdot g \cdot h'$; then $\Gamma=u$ is not separated, but it is uniquely composed.

For the simplest example for a non-uniquely composed positive non-separated pd, see the beginning of section 1.

On the other hand, the simplest *non-positive*, anchored, separated 2-Pd fails to be uniquely composed: take the 0-cell X , the 1-indet $f:X \to X$, and the 2-indet $u:\mathrm{id}_X \to f$, to form the computad $\boldsymbol{X}=\langle X, f, u\rangle$. $\underline{u}=(\boldsymbol{X}, u)$ and, for each n , $\Gamma_n=u \cdot (fu) \cdot \ldots \cdot (f^n u)$, all make $(\boldsymbol{X}, \Gamma_n)$ a separated anchored 2-Pd with the same underlying computad $\boldsymbol{X}$.

2.12 is the present paper's version of John Power's theorem of unique composability of 2-pasting schemes in [P].

The proof of 2.12 is by "exhaustion": given a separated, equivalently projective, positive 2-Pd $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$, we can give an account of all the pd's in the computad $\boldsymbol{X}$. This account yields further results, some of which can be conjectured to be true in higher dimensions.

2.13 Proposition Let $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ be a positive 2-Pd; assume it is separated (equivalently, projective). Then every pd in $\boldsymbol{X}$ is separated: for every $\Lambda\in\|\boldsymbol{X}\|$, $\underline{\Lambda}=(\mathrm{Supp}_{\boldsymbol{X}}(\Lambda),\Lambda)$ is a separated Pd.

This is certainly false in higher dimensions. A (simple) example in [P2] of a 3-Indet $\underline{x}=(\boldsymbol{X},x)$ shows that "looping" of 1-cells in a separated positive 3-Indet is possible: $\boldsymbol{X}$ contains 1-indets f and g in the configuration $X\xrightarrow{f}Y\xrightarrow{g}X$; the 1-pd $f\cdot g$ is not separated. However, the weaker version of 2.13 in which one requires that Λ be of the same dimension as Γ may be true for anchored separated Pd's in general.

We also can conclude that, for $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ a separated positive 2-Pd, "all pd's in $\boldsymbol{X}$ are parts of the full composite Γ".

Let $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ be any Pd. Let us call a pd Λ in $\boldsymbol{X}$ a *part of* Γ if Λ belongs to the least class $\mathcal{C}$ of pd's in $\boldsymbol{X}$ such that $\Gamma\in\mathcal{C}$, $\Xi_1\cdot\Xi_2\in\mathcal{C}$ implies $\Xi_1,\Xi_2\in\mathcal{C}$, $1_{\Xi}\in\mathcal{C}$ implies $\Xi\in\mathcal{C}$, and $\Xi\in\mathcal{C}$ implies $\mathrm{d}\Xi,\mathrm{c}\Xi\in\mathcal{C}$.

2.14 Proposition If $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ is a separated, equivalently projective, positive 2-Pd, then every pd Λ in $\boldsymbol{X}$ is a part of Γ.

2.14 is false for dimensions higher than 2 as the example in [P2] just quoted shows. However, once again, the weaker version of 2.14 in which one requires that Λ be of the same dimension as Γ may be true for positive separated Pd's in general.

We can express the idea of "part" used in 2.14 more "geometrically" as seen in the statement of 2.15 below.

Let $\boldsymbol{X}$ be an n-dimensional computad, Γ a n-pd in $\boldsymbol{X}$, u a particular n-indet in $\boldsymbol{X}$. Let Λ be another n-pd in $\boldsymbol{X}$ such that $\Lambda\|u$ ($\mathrm{d}\Lambda=\mathrm{d}u$, $\mathrm{c}\Lambda=\mathrm{c}u$). Then $\Gamma[\Lambda/u]$, the n-pd obtained by *substituting* Λ for u in Γ, is obtained as

$$\Gamma[\Lambda/u]\overset{\mathrm{DEF}}{=}f(\Gamma),$$

for the map $f:\boldsymbol{X}\to\boldsymbol{X}$ of ω-categories (not necessarily a map of computads!) defined by the stipulation that f is the identity of $|\boldsymbol{X}|-\{u\}$, and $f(u)=\Lambda$. (This is legitimate; $\boldsymbol{X}=\boldsymbol{Y}[u]$

for some $\boldsymbol{Y}$ with $|\boldsymbol{Y}| = |\boldsymbol{X}| - \{u\}$; we can apply the universal property of $\boldsymbol{Y}[u]$; this is why we need u to be top-dimensional. As a matter of fact, one can define meaningful substitution for indeterminates that are not top-dimensional; but this involves suitably replacing higher dimensional indets depending on the one being substituted for).

Let us say that the n-pd Λ is *a part of* the n-pd Γ if, for u a new n-indet parallel to Λ , there is a 2-pd Γ^* in $\boldsymbol{X}[u]$ such that $u\in\mathrm{supp}(\Gamma^*)$ and $\Gamma=\Gamma^*[\Lambda/u]$.

Lemma [M] (xii) Let $\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$ be a separated n-Pd , and suppose that all n-pd's in $\boldsymbol{X}$ are parts of Γ . Then Γ is uniquely composed.

Proof Suppose Λ is an n-pd in $\boldsymbol{X}$ such that $\mathrm{Supp}(\Lambda)=\boldsymbol{X}$. Since Λ is a part of Γ , there are appropriate u and Γ^* as in the definition. We are going to show that, necessarily, Γ^* is equal to u , and thus $\Gamma=\Gamma^*[\Lambda/u] = \Lambda$.

Let us write out Γ^* as the product of an n-molecule $\Phi=(\varphi_1[u_1], \dots, \varphi_N[u_N])$. By assumption, there is $i\in\{1, \dots N\}$ such that $u=u_i$. Let

$$\Gamma_1=\varphi_1\cdot\ldots\cdot\varphi_{i-1}\,, \qquad \varphi=\varphi_i\,, \qquad \Gamma_2=\varphi_{i+1}\cdot\ldots\cdot\varphi_N\,.$$

We **claim** that $N=1$ and $i=1$; that is, $\Gamma^*=\varphi$ itself is an n-atom. Suppose not. Then either Γ_1 or Γ_2 is not an identity n-cell; say Γ_1 is not an identity n-cell. Therefore there is $v\in\mathrm{supp}_n(\Gamma_1)$. We have

$$\Gamma = \Gamma^*[\Lambda/u] = \Gamma_1\cdot\varphi[\Lambda/u]\cdot\Gamma_2\ .$$

Since $\mathrm{Supp}(\Lambda)=\boldsymbol{X}$, we have $v\in\mathrm{supp}(\Lambda)$. Let $\hat{v}$ be a *new* n-indet parallel to v , and let $\hat{\Gamma}_1=\Gamma_1[\hat{v}/v]$. Clearly, $\hat{\Gamma}_1\|\Gamma_1$; thus $\hat{\Gamma} = \hat{\Gamma}_1\cdot\varphi[\Lambda/u]\cdot\Gamma_2$ is well-defined in $\hat{\boldsymbol{X}}=\boldsymbol{X}[\hat{v}]$. Clearly, $\mathrm{Supp}(\hat{\Gamma})=\hat{\boldsymbol{X}}$. We have the n-Pd $\underline{\hat{\Gamma}}=(\hat{\boldsymbol{X}}, \hat{\Gamma})$. Define the map $f:\hat{\boldsymbol{X}}\to\boldsymbol{X}$ that is the identity on $\boldsymbol{X}$ and maps $\hat{v}$ to v . Clearly, f maps $\hat{\Gamma}$ to Γ ; $f:\underline{\hat{\Gamma}}\to\underline{\Gamma}$. However, obviously, f is *not* an isomorphism: $\hat{\boldsymbol{X}}$ has one more indeterminate than $\boldsymbol{X}$. This contradicts the assumption that $\underline{\Gamma}$ is separated. The **claim** is proved.

Let's write out $\Gamma^*=\varphi$ as an n-atom:

$$\varphi = b_{n-1}\cdot(b_{n-2}\cdot\ldots\cdot(b_1\cdot u\cdot e_1)\cdot\ldots\cdot e_{n-2})\cdot e_{n-1}\ .$$

Let us assume that $\Gamma^*=\varphi\neq u$, to arrive at a contradiction. The proof is similar to that of the above claim. There must be some $k\in\{1, \dots, n-1\}$ such that either b_k ore e_k is not an identity. Suppose the first alternative. Let k be the largest integer $\leq n-1$ such that b_k is not an identity. We have

$$\varphi = b_k \cdot (b_{k-1} \cdot \ldots \cdot (b_1 \cdot u \cdot e_1) \cdot \ldots \cdot e_{n-2}) \cdot e_{n-1}$$

and

$$\Gamma = \varphi[\Lambda/u] = b_k \cdot (b_{k-1} \cdot \ldots \cdot (b_1 \cdot \Lambda \cdot e_1) \cdot \ldots \cdot e_{n-2}) \cdot e_{n-1} \ .$$

Let v be a k-indet in b_k ; let $\hat{v}$ be a *new* k-indet parallel to v ; let $\hat{b}_k = b_k[\hat{v}/v]$; we have $\hat{b}_k \| b_k$; let

$$\hat{\Gamma} = \hat{b}_k \cdot (b_{k-1} \cdot \ldots \cdot (b_1 \cdot \Lambda \cdot e_1) \cdot \ldots \cdot e_{n-2}) \cdot e_{n-1} \ ;$$

we have $\hat{\boldsymbol{X}} \overset{\mathrm{DEF}}{=} \boldsymbol{X}[\hat{v}] = \mathrm{Supp}(\hat{\Gamma})$ *because* $v \in \mathrm{supp}(\Lambda)$, and the only indet that *might* have been removed from Γ is v ; it is not removed; and of course, $\hat{v}$ has been added. Let $\underline{\hat{\Gamma}} = (\hat{\boldsymbol{X}}, \hat{\Gamma})$.

Define $f : \hat{\boldsymbol{X}} \to \boldsymbol{X}$ as the identity on $\boldsymbol{X}$, and mapping $\hat{v}$ to v . Clearly, $f(\hat{\Gamma}) = \Gamma$, and $f : \underline{\hat{\Gamma}} \to \underline{\Gamma}$; f is not an isomorphism; contradiction.

The last lemma shows that the next proposition is stronger than 2.12. It is obviously stronger than 2.14.

2.15 Proposition Let $\underline{\Gamma} = (\boldsymbol{X}, \Gamma)$ be a separated positive 2-Pd. Then every 2-pd in $\boldsymbol{X}$ is a part of Γ .

In fact, for every 2-pd Λ in $\boldsymbol{X}$, u new indet parallel to Λ , there is Γ^* in $\boldsymbol{X}[u]$ such that

(a) $\Gamma = \Gamma^*[\Lambda/u]$;

(b) u occurs in Γ^* exactly once;

and

(c) $\mathrm{supp}(\Gamma^*) \cap \mathrm{supp}(\Lambda) = \mathrm{supp}(\mathrm{d}\Lambda) \cup \mathrm{supp}(\mathrm{c}\Lambda)$.

§3 Cuts in partial orders, and planar arrangements

Cuts

Let us fix a finite irreflexive partial order $(\mathbf{N};\prec)$. We read $x\prec y$ as " x is *above* y ", " y is *below* x ". That is, we imagine the order $\prec$ as going "downward". This may cause some linguistic problems such as "minimal" in the sense of $\prec$ means "being on the top", etc.

A *cut* C in $(\mathbf{N};\prec)$, or simply a *cut*, is a pair $C=(U,L)$ such that $U\dot{\cup}L=\mathbf{N}$, and

- **(i)** U is up-closed: $b\prec a\in U\Rightarrow b\in U$;
- **(ii)** L is down-closed: $b\succ a\in L\Rightarrow b\in L$.

Note that if $U\dot{\cup}L=\mathbf{N}$, *then* (i) *iff* (ii). That is, a cut C can be given by a single set U which is up-closed; L is then the complement of U. Of course, the roles of U and L are entirely symmetric. I find that it is better to keep both of "sides" around when we think of a cut.

Let $C=(U,L)$ be a cut. Let μU be the set of all $\prec$-maximal (maximally low) elements of U, νL that of the $\prec$-minimal (maximally high) elements of L :

$$\mu U \overset{\text{def}}{=} \{u\in U:\forall v\in U.\neg(v\succ u)\}, \qquad \nu L \overset{\text{def}}{=} \{\ell\in L:\forall m\in L.\neg(m\prec\ell)\} .$$

$$B \overset{\text{def}}{=} \mu U\,\dot{\cup}\,\nu L$$

is the *border* of the cut C.

The border can also be described as follows:

$$w\in B \iff \forall v[(w\prec v\Longrightarrow v\in L)\ \&\ (w\succ v\Longrightarrow v\in U)] . \tag{1}$$

A *spanning set*, or *span*, X, *of* C is any subset X of the border B which is a maximal antichain in the order $\prec\restriction B$: $X\subseteq B$; and for any x,y in X, $\neg(x\prec y)$ and $\neg(y\prec x)$; and X is maximal among such subsets of B.

3.1 Elementary observation A spanning set X of a cut is a maximal antichain in the order $\prec$ on $\mathbf{N}$: if $X\subseteq Y\subseteq\mathbf{N}$, Y is a $\prec$-antichain, then $X=Y$.

Proof We use the notation associated above with a cut C without comment.

Let X be a span of C. Let $z\in\mathbf{N}$ be arbitrary, to show that there is $x\in X$ such that either $x\succeq z$ or $x\prec z$.

Either $z\in U$, or $z\in L$. Assume, for instance, that $z\in U$; the case $z\in L$ is symmetrically treated. Let u be a $\prec$-maximal (maximally low) element in the set $Y=\{y\in U:y\succeq z\}$ (we have $z\in Y$); in particular, $u\succeq z$. We have $u\in\mu U\subseteq B$ because if $v\succ u$, then $v\in U$ would imply $v\in Y$, contradicting the maximal choice of u; we must have $v\in L$, thus $u\in\mu U$.

Since X is a maximal antichain in B, we have some $x \in X$ such that $x \succeq u$ (case 1), or $x \prec u$ (case 2). In case 1, we have $x \succeq u \succeq z$ and $x \succeq z$ as desired. In case 2, since $x \in B$, $x \prec u$ forces u to be in L; contradiction, since $u \in U$.

Let us write $\mathcal{C}$ for the set of all cuts in $(\mathbf{N}, \prec)$.

In what follows, $C, D, \tilde{C}$ are cuts; $C=(U, L)$, $D=(V, M)$, $\tilde{C}=(\tilde{U}, \tilde{L})$; B is the border of C, $\tilde{B}$ is the border of $\tilde{C}$, E is the border of D.

The cuts form a (reflexive) partial order $(\mathcal{C}, \leq)$ by inclusion of their upper parts: $C \leq D \overset{\mathrm{DEF}}{\iff} U \subseteq V$. $(\mathcal{C}, \leq)$ is in fact a distributive lattice, with an injective homomorphism of lattices $C \mapsto U$ into $(\mathcal{P}(\mathbf{N}), \subseteq)$. Let us write $C<D$ for $C \underset{\neq}{\leq} D$.

The *distance* $\rho(C, D)$ between C and D is defined as the cardinality of the symmetric difference $U \cap M \,\dot{\cup}\, L \cap V$. $\rho(C, D)=0$ iff $C=D$.

Let B be the border of C. Assume $u \in B$. Either $u \in \underline{B}$, or $u \in \bar{B}$. In either case, we can *shift* u over to the other side, and obtain a new cut. Let, e.g., $u \in \underline{B}$. We can form $\tilde{U}=U-\{u\}$, $\tilde{L}=L \cup \{u\}$. Then $\tilde{U}$ is closed upward: if $w \prec v$ & $v \in \tilde{U}$, then $w \in \tilde{U}$: indeed, $w \in U$ is clear, and $w=u$ would imply $u \prec v$, which, together with $v \in U$, would contradict $u \in \underline{B}$.

Thus, $\tilde{C}=(\tilde{U}, \tilde{L})$ is a cut. Also, we have $u \in \overline{\tilde{B}}$: indeed, $w \prec u$ implies $w \in \tilde{U}=U-\{u\}$ since $u \in U$ and U is closed upward. On the other hand, $\tilde{B}$ can very well be different from B.

We say that $\tilde{C}$ is obtained from C by *shifting* x, or simply: $\tilde{C}$ is the x*-shift* of C, and, without referring to x, that $\tilde{C}$ is a *shift* of C, if either $x=u \in B$ and $\tilde{C}$ is obtained from C as described, or $x \in L$, and the dual situation takes place. "Being a shift of" is a symmetric relation.

Suppose $\rho(C, D)>0$; i.e, $U \cap M$ is non-empty, or $L \cap V$ is non-empty. Assume the first alternative; the treatment of the second is a dual affair.

I claim that the set $(\mu U) \cap M$ is non-empty. Let u be a $\prec$-maximal (lowest) element of $U \cap M$. u must be in μU. Indeed, let $u \prec v$. $v \in U$ would imply $v \in U \cap M$ (since M is closed downward), contradicting the extremal property of u.

Let $u \in (\mu U) \cap M$, and consider the cut $\tilde{C}=(\tilde{U}, \tilde{L})$ explained above: $\tilde{U}=U-\{u\}$, $\tilde{L} = L \cup \{u\}$. We have that $\tilde{U} \cap L \subset U \cap L$, and $(U \cap L)-(\tilde{U} \cap L) = \{u\}$. On the other hand, $\tilde{L} \cap V = L \cap V$, since $\tilde{L} \cap V \supseteq L \cap V$, and $u \notin \tilde{L} \cap V$, since $u \in M$. Therefore, the distance $\rho(\tilde{C}, D)$ has gone down by one with respect to $\rho(C, D)$.

In summary, we can characterize the immediate successor (or: Hasse-) relation associated with

$<$ on $\mathcal{C}$, the relation $<!$ ($C<!D \iff C<D\&\neg\exists\tilde{\hat{C}}.C<\tilde{\hat{C}}<D$) as follows: $C<!D$ iff $C<D$ and D is a shift of C. Also, $\rho(C, D)=1$ iff D is a shift of C.

Suppose $\tilde{\hat{C}}$ is the u-shift of C, and let S be any span for C, that is, a maximal $\prec$-antichain in B, such that $u\in S$. I claim that S is a span for $\tilde{\hat{C}}$ too.

E.g., $u\in\underline{B}$, $u\in\overline{\tilde{\hat{B}}}$. What we need is that $S\subseteq\tilde{\hat{B}}$. Let $s\in S$; if $s=u$, we know that $u\in\tilde{\hat{B}}$. Suppose $s\neq u$. But then, since S is an antichain, s is $\prec$-incomparable with $u(\in S)$. Assume $v\prec s$, to show that $v\in\tilde{\hat{U}}$. We have $v\in U$ (since $s\in B$) and $v\neq u$; and this means that $v\in\tilde{\hat{U}}$. The implication $v\succ s \Rightarrow v\in\tilde{\hat{L}}$ is trivial.

Signed spans

Let $C=(U, L)$ be a cut in $(\mathbf{N}, \prec)$.

Given a span S of $C=(U, L)$, let's write $\underline{S} = S\cap\mu U$, $\bar{S} = S\cap\nu L$. The cut C is, clearly, recovered from $(\underline{S}, \bar{S})$ in this way:

$$u\in U \iff u\in\underline{S} \ \dot{\vee}\ \exists s\in S.\ u\prec s \tag{2.1}$$
$$\ell\in L \iff \ell\in\bar{S} \ \dot{\vee}\ \exists s\in S.\ s\prec\ell \tag{2.2}$$

Conversely, let us start with an arbitrary span S in $\mathbf{N}$, that is, an $\prec$-antichain, and an arbitrary partition of it, $S = \underline{S}\dot{\cup}\bar{S}$; we call the data a (*up/down*) *signed span*; we use the symbol $\bar{\underline{S}}$ to denote the signed span as well.

Define U by (2.1) above. It is immediate that U is $\prec$-up-closed: $u\in U \,\&\, x\prec u \Longrightarrow x\in U$. Thus, with $L = \mathbf{N}\text{-}U$, we have a cut $C=(U, L)$. Using the fact that S is a maximal $\prec$-antichain, we immediately see that (2.2) holds. It is now clear that $\underline{S}\subseteq\mu U$ and $\bar{S}\subseteq\nu L$; and thus S is a span of C. Let us write $C[S]$ for the cut just defined.

We say that two signed spans S_1, S_2 are *equivalent*, $S_1\sim S_2$, if $C[S_1]=C[S_2]$. Thus, cuts are in a bijective correspondence with the equivalence classes of $\sim$.

Convex sets

We work in a fixed finite irreflexive partial order $(\mathbf{N}, \prec)$.

A *convex set* (or, in case the reference is needed, a $\prec$-convex set) is a subset P of $\mathbf{N}$ such that $p, q\in P$ and $p\prec x\prec q$ implies $x\in P$.

Let P be any set (subset of $\mathbf{N}$). We derive the following further sets from P:

$$\tilde{P}\uparrow \;=\; \{x\in\mathbf{N}: \exists p\in P. x\preceq p\}$$

$$\tilde{P}\downarrow \;=\; \{x\in\mathbf{N}: \exists p\in P. x\succeq p\}$$

$$P\uparrow \;=\; \tilde{P}\uparrow - P$$

$$P\downarrow \;=\; \tilde{P}\downarrow - P$$

thus

$$\tilde{P}\uparrow = P \dot{\cup} P\uparrow \;,\quad \tilde{P}\downarrow = P\dot{\cup} P\downarrow \;;$$

$$P^* \;=\; \{x\in\mathbf{N}: \forall p\in P. \neg(x\prec\!\!\!\underline{\;\;}\!\!\!\succ p)\}$$

$$P^+ \;=\; P\dot{\cup} P^* \;.$$

By definition,

$$P^* \dot{\cup} (\tilde{P}\uparrow \cup \tilde{P}\downarrow) = \mathbf{N}\;.$$

From now on , we assume that P is convex.

We immediately see that this implies that $P\uparrow$ is up-closed ($x\prec p\in P\uparrow$ imply $x\in P\uparrow$), $P\downarrow$ is down-closed, and $P\uparrow$ and $P\downarrow$ are disjoint. In particular, we have the following partition:

$$P\dot{\cup} P^* \dot{\cup} P\uparrow \dot{\cup} P\downarrow = \mathbf{N}\;; \tag{2.3}$$

that is,

$$P^+ \dot{\cup} P\uparrow \dot{\cup} P\downarrow = \mathbf{N}\;. \tag{2.3'}$$

$P\uparrow$ is up-closed, $P\downarrow$ is downclosed; their complements $\overline{P\uparrow} = \mathbf{N}-P\uparrow$, $\overline{P\downarrow} = \mathbf{N}-P\downarrow$ are downclosed, resp. upclosed, hence all are convex. P^+ is the intersection of $\overline{P\uparrow}$ and $\overline{P\downarrow}$, hence, P^+ is convex.

I **claim** that

$$(P^+)^{\sim}\uparrow = \tilde{P}\uparrow \dot{\cup} P^* \;;$$

The RHS is clearly contained in the LHS. Let $x\in (P^+)^{\sim}\uparrow$, that is, let $p^+\in P^+$ and $x\prec p^+$, to show $x\in$RHS. Suppose $x\notin P^*$, that is, suppose $p\in P$ such that $p\preceq x$ (case 1), or $x\prec p$ (case 2), to show $x\overset{?}{\in}\tilde{P}\uparrow$. In case 1, $p\preceq x\prec p^+$; thus $p\prec p^+$; but $p\in P$, $p^+\in P^+$ and $p\prec p^+$ clearly imply that $p^+\in P$; then, the convexity of P and $p\preceq x\prec p^+$ imply $x\in P$, thus also $x\in\tilde{P}\uparrow$. In case 2, $x\in\tilde{P}\uparrow$ by definition. (**claim** done)

Symmetrically,

$$\tilde{P}^{+}\downarrow \;=\; \tilde{P}\downarrow \dot{\cup}\, P^{*} \quad .$$

Thus, $P^{+}\uparrow = \tilde{P}^{+}\downarrow - P^{+} = \tilde{P}\downarrow \dot{\cup}\, P^{*} - (P \dot{\cup} P^{*}) = P\uparrow$; similarly, $P^{+}\downarrow = P\downarrow$.

From $P^{+} \dot{\cup} P^{+*} \dot{\cup} P^{+}\uparrow \dot{\cup} P^{+}\downarrow = \mathbf{N}$, (2.3) applied to the convex set P^{+} , we get

$$P \dot{\cup} P^{*} \dot{\cup} P^{+*} \dot{\cup} P\uparrow \dot{\cup} P\downarrow = \mathbf{N} ,$$

and since $P \dot{\cup} P^{*} \dot{\cup} P\uparrow \dot{\cup} P\downarrow = \mathbf{N}$, we conclude $P^{+*}=\emptyset$. Therefore, $P^{++}=P^{+}\dot{\cup}P^{+*} = P^{+}$.

Conversely, suppose that $P^{+}=P$; this of course means that $P^{*}=\emptyset$, since $P^{+}=P\dot{\cup}P^{*}$. We have shown

$$P^{+}=P \text{ iff } P^{*}=\emptyset \text{ iff } \exists Q \text{ convex. } P=Q^{+} .$$

Let us call the convex set P *horizontally full* if $P^{+}=P$. (P is *vertically full* if $P=P^{**}$.)

The operation $P \mapsto P^{+}$ on convex sets is *not* monotone, however: if $\mathbf{N}=\{1, 2, 3\}$, $\prec=\{(1, 2), (1, 3)\}$, then $\{1\}^{+}=\{1, 3\}$, and $\{1, 2\}^{+}=\{1, 2\}$.

As a "lemma", let us note that if $V\subseteq P^{*}$ is upward closed in P^{*} , that is, $\forall x.(x\prec v\in V \,\&\, x\in P^{*})\Longrightarrow x\in V$, then $\tilde{P}\uparrow\dot{\cup}V$ is upward closed (absolutely): the only thing to check is that if $w\prec v$ and $v\in V$, then $w\in\tilde{P}\uparrow\dot{\cup}V$; this is true when also $w\in P^{*}$; but if $w\notin P^{*}$, we have $p\in P$ with either $w\precsim p$ (case 1), or $p\prec w$ (case 2); case 1 implies $w\in\tilde{P}\uparrow$, and case 2 is impossible since it gives $p\prec w\prec v$ and $p\prec v$, contradicting $v\in V\subseteq P^{*}$.

Similarly, if $M\subseteq P^{*}$ is downward closed in P^{*} , that is, $\forall x.(x\succ v\in V \,\&\, x\in P^{*})\Longrightarrow x\in V$, then $\tilde{P}\uparrow\dot{\cup}V$ is downward closed (absolutely).

Given two cuts $C_1=(U_1, L_1)$ and $C_2=(U_2, L_2)$ such that $C_1\leq C_2$, that is, $U_1\subseteq U_2$, $L_1\supseteq L_2$, the set $P=L_1\cap U_2$ is obviously convex.

Conversely, given the convex set P , let $D=(V, M)$ be any $\prec$-cut in the set P^{*} : $P^{*}=V\dot{\cup}M$, V closed upward in P^{*} (and M closed upward in P^{*}). From (2.3) and the "lemma" we see that the definitions

$$L_1 \overset{\text{DEF}}{=} P \dot{\cup} P\downarrow \dot{\cup} M = \tilde{P}\downarrow \dot{\cup} M \tag{2.4}$$

$$U_2 \overset{\text{DEF}}{=} P \dot{\cup} P\uparrow \dot{\cup} V = \tilde{P}\uparrow \dot{\cup} V \tag{2.5}$$

give $P=L_1\cap U_2$, with L_1 closed downward, U_2 closed upward, thus defining cuts $C_1=(U_1, L_1)$ and $C_2=(U_2, L_2)$; and $C_1\le C_2$. It is also clear that any $C_1\le C_2$ that give P as $P=L_1\cap U_2$ arises in this way.

Let us summarize. Let us call a pair of cuts C_1 and C_2 a *slicing* (in $(\mathbf{N}, \prec)$) if $C_1\le C_2$; the *slice* of the slicing (C_1, C_2) is the set $\mathrm{P}(C_1, C_2) \overset{\mathrm{DEF}}{=} L_1\cap U_2$. Every slice (of any slicing) is convex; and conversely, every convex set arises as the slice of a slicing. More particularly, the slicings for which a given convex set P is the slice are in a bijective correspondence with the cuts in $(P^*, \prec\restriction P^*)$, according to the formulas (2.4) and (2.5). Therefore, the convex sets that arise as a slice from exactly one slicing are exactly the horizontally full ones.

Planar arrangements

A (finite) *planar arrangement* is a structure $\mathbf{N}=(\mathbf{N}; \prec, \rightarrow)$ with $\mathbf{N}$ a finite set, $\prec, \rightarrow$ binary relations on $\mathbf{N}$, such that:

1) $\prec$ and $\rightarrow$ are irreflexive and transitive (irreflexive partial orders);

2) $\prec \,\dot\cup\, \succ \,\dot\cup\, \rightarrow \,\dot\cup\, \leftarrow \;=\; \mathbf{N}^{2\ne}$;

here, $\succ = \prec^{\mathrm{op}}$, $\leftarrow = \rightarrow^{\mathrm{op}}$, $\mathbf{N}^{2\ne} = \mathbf{N}\times\mathbf{N}-\Delta_{\mathbf{N}}$, $\Delta_{\mathbf{N}} = \{(a, a):a\in\mathbf{N}\}$. We are saying that for any $x\ne y$ in $\mathbf{N}$, *exactly one* of the following four alternatives holds:

$$x\rightarrow y \qquad x\leftarrow y \qquad x\prec y \qquad x\succ y \quad .$$

We read $x\rightarrow y$ as " x is (to the) left of y " , " y is (to the) right of x ". As we said before, we read $x\prec y$ as " x is *above* y " , " y is *below* x "; that is, we imagine the order $\prec$ as going "downward".

In short, $\rightarrow$ is the "left-to-right" (partial) order, $\prec$ is the "(from) up-(to) down" order in the arrangement.

Note the obvious fact that, for a planar arrangement $(\prec, \rightarrow)$, an $\prec$-antichain is the same thing as a $\rightarrow$-chain: a set in which any two elements are $\rightarrow$-comparable. Therefore, a span of $(\mathbf{N}, \prec, \rightarrow)$, respectively a span of a cut C , *always understood as a cut for* $(\mathbf{N}, \prec)$, is a maximal $\rightarrow$-chain in $\mathbf{N}$, respectively, a maximal $\rightarrow$-chain in the border of C .

Further notation

We write $\underset{=}{\rightarrow}$, $\preceq$ for the reflexive versions:

$$x\underset{=}{\rightarrow}y \iff x\rightarrow y \vee x=y ,$$
$$x\preceq y \iff x\prec y \vee x=y .$$

We write $\longleftrightarrow$, $\prec\!\succ$ for the comparability relations:

$$x \longleftrightarrow y \iff x \to y \vee x \leftarrow y\ ,$$
$$x \prec\succ y \iff x \prec y \vee x \succ y\ .$$

There are obvious versions such as $\underset{=}{\leftrightarrow}$, $\prec_{=}\succ$.

All (partial) orders will have finite underlying sets. Thus, we can meaningfully talk about the "Hasse diagram" of an order. We use the notations $\to !$, $\prec !$ in these senses:

$$x \to ! y \iff x \to y \ \& \ \neg\exists z . x \to z \to y\ ,$$
$$x \prec ! y \iff x \prec y \ \& \ \neg\exists z . x \prec z \prec y\ .$$

The notations $<$, $\lll$, possibly with a subscript, will be reserved for *total* (linear, irreflexive) orders.

Intervals

We are in a fixed planar arrangement $(\mathbf{N}; \prec, \to)$.

It is convenient to consider the "2-point compactification" $(\mathbf{N}^{\circ}; \prec, \to)$ of $(\mathbf{N}; \prec, \to)$. Here, with the new symbols $-\infty$, ∞ , we put $\mathbf{N}^{\circ} = \mathbf{N} \dot{\cup} \{-\infty, \infty\}$, and declare that $-\infty \to < \infty$ and $-\infty \to u \to \infty$ for all $u \in \mathbf{N}$. Then $(\mathbf{N}^{\circ}; \prec, \to)$ itself is a planar arrangement -- although this is not very important.

In what follows, we try to adhere to the convention that variables $a, b, \ldots$ range over the extended set $\mathbf{N}^{\circ}$, and $u, v, x, \ell,$ ... range over the original set $\mathbf{N}$.

For any subset S of $\mathbf{N}$, we write S° for the set $S \dot{\cup} \{-\infty, \infty\}$.

Take a and b in $\mathbf{N}^{\circ}$ such that $a \underset{=}{\to} b$, and define

$$[a, b] = [a, b]_{\to} = \{x \in \mathbf{N} : a \underset{=}{\to} x \underset{=}{\to} b\}\ .$$

(Thus, the set $[a, b]$ is a subset of $\mathbf{N}$, although a and b may not be elements of $\mathbf{N}$.)

Similarly,

$$(a, b) = (a, b)_{\to} = \{x : a \to x \to b\}\ ,$$

and we have the obvious versions $(a, b]$, $[a, b)$ too. Of course, $(-\infty, b) = [-\infty, b)$, etc.; and $(-\infty, \infty) = \mathbf{N}$.

We define the subset I of $\mathbf{N}$ to be an *interval with end-points* a *and* b (left end-point a and right end-point b) if $(a, b) \subseteq I \subseteq [a, b]$; here, $a, b \in \mathbf{N}^{\circ}$, and it is assumed that $a \to b$ and $\{a, b\} \neq \{-\infty\}$, $\{a, b\} \neq \{\infty\}$.

In the notion of "interval", we must keep track of the end-points; the same set I could be an interval with two different sets of end-points. For instance, as a set, an interval may be empty;

as an interval in the full sense, it still retains the information of its endpoints. However, we make one exception: we do not allow the empty sets $(-\infty,-\infty)$, (∞,∞) as intervals.

When we denote an interval as, e.g., (a, b) , we automatically mean that the endpoints are taken to be a and b . Thus, an interval is a set I , together with two end-points a and b .

Intervals are $\rightarrow$-*convex sets*: a subset I of $\mathbf{N}$ is an $\rightarrow$-convex iff

$$\forall x\in I.\forall z\in I.\forall y(x\rightarrow y\rightarrow z \Longrightarrow y\in I).$$

An interval is a $\rightarrow$-convex subset of $\mathbf{N}$; but it is also convex with respect to $\prec$; in fact, if I is an interval with end-points a, b , then

$$x, z\in I\ ,\ x\prec y\prec z \Longrightarrow y\in (a, b)\ .$$

To show this, we (easily) exclude each of the possibilities $a\prec y$, $y\prec a$, $y\rightarrow a$ and $y=a$, to conclude $a\rightarrow y$; similarly, we obtain $y\rightarrow b$.

The arrangement $(\mathbf{N},\prec,\rightarrow)$ induces (by restriction) a planar arrangement on any subset of $\mathbf{N}$. For any subset I of $\mathbf{N}$, a *cut in* I , or I-cut, is a cut of the arrangement induced on I . It is obvious that a cut $C=(U, L)$ of the total arrangement induces a cut $C\restriction I = (U\cap I, L\cap I)$ in I , for any subset I of $\mathbf{N}$.

3.2 Proposition Let $C=(U, L)$ be a cut in $\mathbf{N}$, and let B the border of C . Let I be an interval with endpoints a and b , and assume that $a, b\in B^{\circ}$. Then the border of $C\restriction I$ is equal to $B\cap I$.

Proof Denote the border of $C\restriction I$ by $\dot{B}$. It is clear that $B\cap I\subseteq\dot{B}$. To show the converse containment, that is, $\dot{B}\subseteq B$, we verify, for elements w of $\dot{B}$, the RHS of (1).

Let $w\in\dot{B}$ and assume $w\prec v$, to show $v\in L$ (?). Let's compare a and v (in case $a\neq-\infty$). If $a\prec v$, we are done, since $a\in B$. $v\underset{=}{\Longrightarrow} a$ would give $v\underset{=}{\Longrightarrow} a\underset{=}{\Longrightarrow} w$ (since $w\in I$), contradicting $w\prec v$. $v\prec a$ would give $w\prec v\prec a$, contradicting $a\underset{=}{\Longrightarrow} w$. It remains to consider the case that $a\rightarrow v$ (which is OK when $a=-\infty$) . Similarly, we may assume that $v\rightarrow b$. But then, $a\rightarrow v\rightarrow b$, hence, $v\in I$; $v\in L$ follows since $w\in\dot{B}$.

Similarly, we can show, for (1), that $w\succ v \Longrightarrow v\in U$.

A (signed) span in an interval I is, of course, a (signed) span of the planar arrangement induced on I .

3.3 Proposition Suppose that I is an interval, S is a signed span of $\mathbf{N}$, and both endpoints of I belong to S° . Let $C=(U, L)$ be the cut determined by S . The intersection $S\cap I$ inherits a signing from S . Then

(i) $S\cap I$ is a signed span in I .
(ii) The signed span $S\cap I$ in I determines, in the arrangement induced on I , the cut $C\restriction I = (U\cap I,\ L\cap I)$.

Proof Call the end-points of I : a and b .

(i) By Prop 2, $B\cap I$ is the border of $C\restriction I$. I claim $S\cap I$ is a span in $B\cap I$. To see this, suppose that $x\in B\cap I$ and $(S\cap I)\cup\{x\}$ is a $\rightarrow$-chain, wishing to conclude $x\overset{?}{\in}S\cap I$.

Since $x\in I$, we have $a\underset{=}{\rightarrow}x\underset{=}{\rightarrow}b$. For any $y\in S-I$, we have $y\underset{=}{\rightarrow}a$ or $b\underset{=}{\rightarrow}y$; thus, $y\underset{=}{\leftrightarrow}x$: x is $\underset{=}{\rightarrow}$-comparable with any $y\in S-I$. By assumption, x is $\underset{=}{\rightarrow}$-comparable with any $y\in S\cap I$. We conclude that x is $\underset{=}{\rightarrow}$-comparable with any $y\in S$. Since S is a span in B , $x\in S$; $x\in S\cap I$ is true.

(ii) Let (U_I, L_I) be the cut determined by $S\cap I$ in I . It is obvious from the definitions (see (2.1) and (2.2)) that $U_I\subseteq U\cap I$, $L_I\subseteq L\cap I$. But we have $U_I\dot{\cup}L_I = (U\cap I)\dot{\cup}(L\cap I) = I$. It follows that we must have $U_I=U\cap I$, $L_I=U\cap I$.

Intervals I and J are *complementary* if either $I=(-\infty, a)$ and $J=[a.\infty)$, or $I=(-\infty, a]$ and $J=(a.\infty)$, or one of said conditions holds with the roles of I and J reversed.

Given complementary intervals I , J , and given spans S_I , S_J of I , resp. J , we have that $S=S_I\dot{\cup}S_J$ is a span of $\mathbf{N}$.

Indeed, S certainly is a $\rightarrow$-chain in $\mathbf{N}$. Note that a must belong to S : (precisely) one of I and J contains a ; in the first case, $a\in S_I$, in the second, $a\in S_J$; thus, at any rate, $a\in S$.

Assume that $x\notin S$, but $S\cup\{x\}$ is a $\rightarrow$-chain, to reach a contradiction. But then x is $\rightarrow$-comparable to a (since a is in S), and *therefore*, it must belong to either I , or to J ; and either case is a contradiction to the "span" character of the two given spans S_I , S_J . Of course, $S\cap I=S_I$, $S\cap J=S_J$.

We have proved part (i) of

3.4 Proposition Let I , J be complementary intervals, S_I , T_I signed spans in I ; S_J , T_J signed spans in J . Then

(i) $S=S_I\dot{\cup}S_J$ is a signed span in $\mathbf{N}$; $S\cap I = S_I$, $S\cap J = S_J$.

(ii) Let $S=S_I\dot{\cup}S_J$, $T=T_I\dot{\cup}T_J$. Then $S\sim T$ iff $S_I\sim T_I$ & $S_J\sim T_J$.

Proof (ii) Let $C_S=(U_S, L_S)$, $C_T=(U_T, L_T)$ be the cuts determined by S and T in $\mathbf{N}$, respectively.

"Only if": Assume $S\sim T$. From Prop 3.3, it follows that $S\cap I$ and $T\cap I$ determine the same cut in I, namely the cut $(U_S\cap I, L_S\cap I) = (U_T\cap I, L_T\cap I)$. Of course, the same thing goes for J in place of I.

"If": Assume $S_I\sim T_I$ & $S_J\sim T_J$. By Prop 3.3(ii), $U_S\cap(I\cup J) = U_T\cap(I\cup J)$; that is, for every $u\in I\cup J$, C_S and C_T agree on u.

Let C_I be the cut determined in I by S_I as well as T_I; similarly for C_J. Let B_I, B_J be their respective boundaries. For the common endpoint a of I and J, $a\notin\{-\infty, \infty\}$, and exactly one of I and J contains a as an element; call the one K. Then, of course, we have $a\in S_K$ and $a\in B_K$.

For $u\notin I\cup J$, we have that either $u\prec a$ or $u\succ a$. Since $a\in B_K$, in the first case $u\in U_S\cap K$ and $u\in U_T\cap K$, hence, C_S and C_T agree on u; in the second case, $u\in L_S\cap K$ and $u\in L_T\cap K$, and again, C_S and C_T agree on u.

We conclude that $C_S=C_T$ as desired.

3.4' Corollary Let I, J be complementary intervals. Given cuts C on I and D on J, there is a unique cut, denoted $C\cup D$, on $\mathbf{N}$ such that $(C\cup D)\restriction I=C$ and $(C\cup D)\restriction J=D$.

Proof The assertion follows from 3.3, 3.4 and the fact that every cut has at least one signed span determining it.

Local *fattening* **of a planar arrangement**

Let $\mathbf{N}=(N, \prec\rightarrow)$ be a planar arrangement as before.

Let $a, b\in\mathbf{N}$ such that $a\prec!b$ (that is, $a\prec b$ and $\neg\exists c(a\prec c\prec b)$). We describe a new planar arrangement $\mathbf{N}_{a,b}$ obtained by modifying $\mathbf{N}$ by "turning the fact $a\prec!b$ into $a\rightarrow b$", thus "fattening" (extending in the left-to-right direction) the arrangement.

We define

$$\rightarrow_{a,b} \overset{\text{def}}{=} \twoheadrightarrow \overset{\text{def}}{=} (\rightarrow\cup\{(a,b)\})^{\mathrm{tr}} \qquad (\mathrm{tr}: \text{transitive closure})$$

$$\prec_{a,b} \overset{\text{def}}{=} \prec\!\prec \overset{\text{def}}{=} \prec - (\twoheadrightarrow\cup\twoheadleftarrow) \qquad (\twoheadleftarrow = \twoheadrightarrow^{\mathrm{op}})$$

As a general fact, just on the basis that $\rightarrow$ is an irreflexive order, and $a \neq b$ are incomparable in $\rightarrow$, we have that $\twoheadrightarrow$ is an irreflexive order; in fact, the least irreflexive order containing $\rightarrow \,\dot{\cup}\, \{(a, b)\}$. Moreover, with the abbreviation

$$x \overset{*}{\twoheadrightarrow} y \quad \overset{\text{def}}{\Longleftrightarrow} \quad x \xrightarrow[=]{} a \;\&\; b \xrightarrow[=]{} y$$

we have

$$x \twoheadrightarrow y \quad \Longleftrightarrow \quad x \rightarrow y \;\vee\; x \overset{*}{\twoheadrightarrow} y \,.$$

3.5 Proposition $(\mathbf{N};\, \prec\!\prec,\, \twoheadrightarrow)$ so defined is a planar arrangement.

Proof The fact that $(\prec\!\prec,\, \twoheadrightarrow)$ satisfies 2) is clear from the definition.

$\prec\!\prec$ is obviously irreflexive; we have to show that it is transitive.

We have

$$x \prec\!\prec y \iff x \prec y \wedge \neg(x \overset{*}{\twoheadrightarrow} y) \wedge \neg(y \overset{*}{\twoheadrightarrow} x) \,. \qquad (3)$$

Assume

$$x \prec\!\prec y \prec\!\prec z \;,$$

to show $x \prec\!\prec z$ (?). We have $x \prec y \prec z$, thus $x \prec z$. It remains to show that

$$\neg(x \xrightarrow[=]{} a \;\&\; b \xrightarrow[=]{} z) \qquad (?4)$$

and

$$\neg(z \xrightarrow[=]{} a \;\&\; b \xrightarrow[=]{} x) \qquad (?5)$$

Since $x \prec\!\prec y \prec\!\prec z$, we have

either $\neg(x \xrightarrow[=]{} a)$, (6.1)
or $\neg(b \xrightarrow[=]{} y)$ (6.2)

and

either $\neg(y \xrightarrow[=]{} a)$, (7.1)
or $\neg(b \xrightarrow[=]{} z)$ (7.2)

Clearly, (6.1) implies (4).

(6.1) implies (5): $b \xrightarrow[=]{} x \xrightarrow[=]{} a$ implies $b \xrightarrow[=]{} a$, $*$ to $a \prec b$.

Clearly, (7.2) implies (4).

(7.2) implies (5): $b \xrightarrow[=]{} z \xrightarrow[=]{} a$ is false.

Therefore, we may assume that (6.2) and (7.1) hold. This means that

	$b \prec y$	(8.1)
or	$b \succ y$	(8.2)
or	$b \leftarrow y$	(8.3)

and

	$y \prec a$	(9.1)
or	$y \succ a$	(9.2)
or	$y \leftarrow a$	(9.3)

Assume (9.3).

If $x \underset{=}{\longrightarrow} a$, $x \underset{=}{\longrightarrow} a \to y$ * to $x \prec y$: (9.3) $\models$ (4).

If $z \underset{=}{\longrightarrow} a$, $z \underset{=}{\longrightarrow} a \to y$ * to $y \prec z$: (9.3) $\models$ (5).

Assume (8.3).

If $b \underset{=}{\longrightarrow} z$, $y \to b \underset{=}{\longrightarrow} z$ * to $y \prec z$: (8.3) $\models$ (4).

If $b \underset{=}{\longrightarrow} x$, $y \to b \underset{=}{\longrightarrow} x$ * to $x \prec y$; (8.3) $\models$ (5).

Assume (9.1). Since $x \prec y \prec a \prec b$, this excludes both $x \underset{=}{\longrightarrow} a$ and $b \underset{=}{\longrightarrow} x$:

(9.1) $\models$ (4) & (5).

Assume (8.1). Since $a \prec b \prec y \prec z$, this excludes both $b \underset{=}{\longrightarrow} z$ and $z \underset{=}{\longrightarrow} a$:

(8.1) $\models$ (4) & (5).

(9.2) and (8.2) together would mean $a \prec y \prec b$: impossible since we have assumed that $a \prec !\, b$.

This completes the proof of the transitivity of $\prec\prec$, and that of Prop 3.5.

3.5' Proposition For $x, y \in \mathbf{N}^{\circ}$, we have:

(i) $a \twoheadrightarrow !\, b$

(ii) $x \overset{*}{\twoheadrightarrow} y \;\&\; (x \neq a \vee y \neq b) \Longrightarrow \neg(x \twoheadrightarrow !\, y)$.

(iii) $\neg(x \to !\, y) \;\&\; (x \neq a \vee y \neq b) \Longrightarrow \neg(x \twoheadrightarrow !\, y)$.

(iv) $x \twoheadrightarrow !\, y \Longleftrightarrow (x = a \;\&\; y = b) \vee (x \to !\, y \;\&\; \neg(x \overset{*}{\twoheadrightarrow} y))$.

Proof (i) $a \twoheadrightarrow b$ since $a \overset{*}{\twoheadrightarrow} b$. The combinations $a \to z \;\&\; z \overset{*}{\twoheadrightarrow} b$, $a \overset{*}{\twoheadrightarrow} z \;\&\; z \to b$, $a \overset{*}{\twoheadrightarrow} z \;\&\; z \overset{*}{\twoheadrightarrow} b$ are all impossible, and so is $a \to z \;\&\; z \to b$ because $a \to !\, b$. This shows that $a \twoheadrightarrow z \twoheadrightarrow b$ is impossible.

(ii) Immediate.

(iii) If $\neg(x \twoheadrightarrow y)$, we are done. Otherwise, $x \to y$ (case 1) or $x \overset{*}{\twoheadrightarrow} y$ (case 2). In case 1, we have z with $x \to z \to y$, so $x \twoheadrightarrow z \twoheadrightarrow y$, thus $\neg(x \twoheadrightarrow !\, y)$. In case 2, (ii) applies.

(iv) The $\Longrightarrow$ direction is (ii)&(iii). For the other direction: because of (i), we are left with showing $x \to !\, y \;\&\; \neg(x \overset{*}{\twoheadrightarrow} y) \Longrightarrow x \twoheadrightarrow !\, y$. Assume

$$x \to\, !y \ \& \ \neg(x \overset{*}{\to\!\!\to} y) \ \& \ x \to\!\!\to z \to\!\!\to y \,,$$

to derive a contradiction. Since either of $x \to\!\!\to z \overset{*}{\to\!\!\to} y$, $x \overset{*}{\to\!\!\to} z \to\!\!\to y$ implies $x \overset{*}{\to\!\!\to} y$, and $x \overset{*}{\to\!\!\to} z \overset{*}{\to\!\!\to} y$ is impossible, we must have $x \to z \to y$, contradicting $x \to\, !y$.

3.6 Proposition Let $(\prec, \to)$ be a planar arrangement on the set $\mathbf{N}$, now not assumed to be finite. Define, for $x, y \in \mathbf{N}$

$$x <_1 y \overset{\text{def}}{\iff} x \prec y \vee x \to y \tag{10.1}$$

$$x <_2 y \overset{\text{def}}{\iff} x \succ y \vee x \to y \tag{10.2}$$

Then $<_1$, $<_2$ are total (irreflexive) orders on $\mathbf{N}$, and

$$x \prec y \iff x <_1 y \ \& \ y <_2 x \tag{11.1}$$

$$x \to y \iff x <_1 y \ \& \ x <_2 y \,. \tag{11.2}$$

Conversely, if $<_1$, $<_2$ are total orders on $\mathbf{N}$, and we define $\prec$ and $\to$ by (11.1) and (11.2), then $(\prec, \to)$ is a planar arrangement on $\mathbf{N}$, and (10.1), (10.2) hold.

Proof Easy

3.7 Corollary A finite planar arrangement is rigid: it has no non-trivial automorphism.

Proof Any automorphism of the structure $(\mathbf{N}, \prec, \to)$ is also an automorphism of $(\mathbf{N}, <_1)$, by (10.1). But a finite total order has no non-trivial automorphism.

3.8 Elementary Lemma Let $(\mathbf{N}, \prec)$ be any finite irreflexive partial order; let $x, y \in \mathbf{N}$. Then the following are equivalent:

(i) There is a total order $<^*$ on $\mathbf{N}$ extending $\prec$ such that $x <^* !y$.
(ii) $x \prec !y \vee \neg(y \underset{=}{\prec\succ} x)$.

Proof. (i)$\Longrightarrow$(ii) is obvious.

Assume (ii). Let $V_1 = \{v : v \prec x\}$, $V_2 = \{v : v \prec y\} - \{x\}$, $W_1 = \{w : w \succ x\} - \{y\}$, $W_2 = \{w : w \succ y\}$, $V = V_1 \cup V_2$, $W = W_1 \cup W_2$. Let $Z = \mathbf{N} - (V \cup \{x\} \cup \{y\} \cup W)$. Note that

$$z \in Z \iff z \neq x \ \& \ z \neq y \ \& \ \neg(z \prec\succ x) \ \& \ \neg(z \prec\succ y) \,.$$

The four sets $X_1 = V \cup Z$, $X_2 = \{x\}$, $X_3 = \{y\}$, $X_4 = W$ are pairwise disjoint; in fact,

Claim: if $i<j$, then $\neg(s\in X_i \ \& \ t\in X_j \ \& \ t\preceq s)$.

We assume $s\in X_i \ \& \ t\in X_j \ \& \ t\preceq s$, and see that there is a contradiction.

Checking:

Case 1. $(i, j)=(1, 2)$:

Case 1.1: $s\in V_1$: $s\prec x \ \& \ t=x \ \& \ t\preceq s$: *

Case 1.2: $s\in V_2$: $s\prec y \ \& \ s\neq x \ \& \ t=x \ \& \ t\preceq s$: $x\prec s\prec y$: * to (ii)

Case 1.3: $s\in Z$: s is incomparable to x & $t=x \ \& \ t\preceq s$: *

Case 2. $(i, j)=(1, 3)$:

Case 2.1: $s\in V_1$: $s\prec x \ \& \ t=y \ \& \ t\preceq s$: $y\preceq s\prec x$: * to (ii)

Case 2.2: $s\in V_2$: $s\prec y \ \& \ s\neq x \ \& \ t=y \ \& \ t\preceq s$: *

Case 2.3: $s\in Z$: s is incomparable to y & $t=y \ \& \ t\preceq s$: *

Case 3. $(i, j)=(1, 4)$:

Case 3.1.1: $s\in V_1 \ \& \ t\in W_1$: *

Case 3.1.2: $s\in V_1 \ \& \ t\in W_2$: $s\prec x \ \& \ y\prec s$: * to (ii).

Case 3.2.1: $s\in V_2 \ \& \ t\in W_1$: $s\prec y \ \& \ s\neq x \ \& \ x\prec s$: $x\prec s\prec y$: * to (ii).

Case 3.2.2: $s\in V_2 \ \& \ t\in W_2$: $s\prec x \ \& \ y\prec t \ \& \ t\preceq s$: $y\prec t\preceq s\prec x$: * to (ii).

Case 3.3.1: $s\in Z \ \& \ t\in W_1$: *

Case 3.3.2: $s\in Z \ \& \ t\in W_2$: *

Case 4. $(i, j)=(2, 3)$: $s=x \ \& \ t=y \ \& \ t\preceq s$: * to (ii).

The case $(i, j)=(2, 4)$ is similar to $(i, j)=(1, 3)$, and the case $(i, j)=(3, 4)$ is similar to $(i, j)=(1, 2)$.

This checks the Claim.

Let's totally order each of the sets $V\cup Z$, $\{x\}$, $\{y\}$, W compatibly with $\prec$, and let us take the ordered sum of the total orders, itself a total order $<^*$, on the union $V\cup Z \ \dot\cup \ \{x\} \ \dot\cup \ \{y\} \ \dot\cup \ W = \mathbf{N}$. By the Claim, $<^*$ is compatible with $\prec$. It is clear that $x<^*!y$.

This proves the Lemma.

The next series of technical lemmas will be used in section 6. Their placing in this section is justified by their elementary nature.

We are working in an arbitrary fixed planar arrangement $(\mathbf{N}, \prec, \to)$. Throughout, the following additional items are fixed:

$a, b, x, y\in\mathbf{N}$ such that

$$a\prec!b \ , \tag{12.1}$$

$$x\to!y \ , \tag{12.2}$$

$$x\underset{=}{\to}a \tag{12.3}$$

and $\qquad b \xrightarrow[=]{} y\,.$ (12.4)

$\twoheadrightarrow$ is defined as in 3.5.

3.9.1 (0) If $b \neq y$, then $x \prec b$.

(i) $u \in [x, a]_{\rightarrow}$ & $v \in [b, y]_{\rightarrow}$ & $(u, v) \neq (x, y) \Longrightarrow u \prec v$.

(ii) $[x, y]_{\twoheadrightarrow} = [x, a]_{\rightarrow} \,\dot{\cup}\, [b, y]_{\rightarrow}$.

Proof (0): Assume $b \neq y$, to show $x \prec b$.

?: $x \rightarrow b$: $x \rightarrow b \rightarrow y$ (! since $b \neq y$) : * to $x \rightarrow !\, y$;
?: $b \xrightarrow[=]{} x$: $b \xrightarrow[=]{} x \xrightarrow[=]{} a$: * to $a \prec b$;
?: $b \prec x$: $a \prec b \prec x$: * to $x \xrightarrow[=]{} a$.

This proves (0).

(i): Assume LHS of "$\Longrightarrow$". First, we show $u \prec b$:

?: $b \preceq u$: $a \prec b \preceq u$: * to $a \rightarrow u$;

?: $u \rightarrow b$: $x \xrightarrow[=]{} u \rightarrow b$: * to $x \prec b$ if $b \neq y$ by (0); and if $b = y$, then the assumption implies that $u \neq x$, and thus $x \rightarrow u \rightarrow b = y$, * to $x \rightarrow !\, y$;
?: $b \rightarrow u$: $b \rightarrow u \xrightarrow[=]{} a$: * to $a \prec b$.

Next, $u \prec v$:

?: $v \preceq u$: $v \preceq u \prec b$: * to $b \xrightarrow[=]{} v$;

?: $u \rightarrow v$: $x \xrightarrow[=]{} u \rightarrow v \xrightarrow[=]{} y$ & "=" does not hold at both places: * to $x \rightarrow !\, y$;

?: $v \rightarrow u$: $b \rightarrow v \rightarrow u$: * to $u \prec b$ (known from before).

(ii): The fact that the RHS is a disjoint union is contained in part (i).

It is clear that the RHS is contained in the LHS.

Assume z is in the LHS, that is, $x \overset{\geq}{\twoheadrightarrow} z \overset{\geq}{\twoheadrightarrow} y$, to show that z belongs to the RHS.

By definition, we have one of the following four logical possibilities:

$$x \xrightarrow[=]{} z \;\&\; z \xrightarrow[=]{} y$$
$$x \xrightarrow[=]{} z \;\&\; z \overset{*}{\twoheadrightarrow} y$$
$$x \overset{*}{\twoheadrightarrow} z \;\&\; z \xrightarrow[=]{} y$$
$$x \overset{*}{\twoheadrightarrow} z \;\&\; z \overset{*}{\twoheadrightarrow} y\,.$$

The last one is impossible. Since $x \to !y$, the first one is possible only if z is equal to one of x, y, in which case z does belong to the RHS. In the second case $x \underset{=}{\to} z$ & $z \underset{=}{\to} a$, that is, $z \in [x, a]_{\to}$; in the third, $b \underset{=}{\to} z$ & $z \underset{=}{\to} y$, that is, $z \in [b, y]_{\to}$, and we are done.

Define

$$Z \overset{\text{def}}{=} \{z : a \prec z \prec y\}, \tag{13.1}$$

$$Z_! \overset{\text{def}}{=} \{z : a \prec z \prec !y\}, \tag{13.2}$$

For $z \in \mathbf{N}$:

$$W_z \overset{\text{def}}{=} \{w : x \to w \to z\}, \tag{13.3}$$

$$W'_z \overset{\text{def}}{=} \{u : x \underset{=}{\to} u \to z\}. \tag{13.4}$$

3.9.2 For any $z \in Z$, we have $b \to z$ and $x \to z$.

Proof Assume $z \in Z$. Because $a \prec !b$, we must have $y \neq b$, and thus $x \prec b$ by 3.9.1(0).

We show $b \to z$ by excluding the four other possibilities.

? $b = z$: * to $b \notin Z$.
? $z \to b$: $z \to b \to y$: * to $z \prec y$.
? $z \prec b$: $a \prec z \prec b$: * to $a \prec !b$.
? $b \prec z$: $b \prec z \prec y$: * to $b \underset{=}{\to} y$.

Next, we show $x \to z$ in a similar manner (but also using $b \to z$):

? $z \underset{=}{\to} x$: $z \underset{=}{\to} x \to a$: * to $a \prec z$.
? $z \prec x$: $z \prec x \prec b$(!) : * to $b \to z$.
? $x \prec z$: $x \prec z \prec y$: * to $x \to y$.

3.9.3 If $z \in Z$ and $w \in W_z$, then $w \prec y$.

Proof To prove $w \prec y$:
? $w \to y$: $x \to w \to y$: * to $x \to !y$.
? $y \underset{=}{\to} w$: $y \underset{=}{\to} w \to z$: * to $z \prec y$.
? $y \prec w$: $z \prec y \prec w$: * to $w \to z$.

3.9.4 Any $\to$-minimal element of $Z_!$ is an $\to$-minimal element of Z.

Proof Suppose $z \in Z$ is not a $\to$-minimal element of Z. There is $u \in Z$ such that $u \to z$. If $u \in Z_!$, then z is not a $\to$-minimal element of $Z_!$, and we are done. Otherwise, there is v such that $u \prec v \prec y$; we can choose v so that, in addition, $v \prec !y$. Obviously,

$v \in Z_!$. I claim that $v \to z$. Indeed:

? $z \prec v$: $z \prec v \prec y$: * to $z \prec ! y$;
? $v \prec z$: $u \prec v \prec z$: * to $u \to z$;
? $z \xrightarrow[=]{} v$: $u \to z \xrightarrow[=]{} v$: * to $u \prec v$.

But $v \in Z_!$ & $v \to z$ says that z is not a $\to$-minimal element of $Z_!$.

3.9.5 Let z be a $\to$-minimal element of $Z_!$, and $u \in W_z$. Then $u \to a$.

Proof To prove $u \to a$:
? $a \prec u$: $a \prec u \prec y$ by 3.9.3. Hence, $u \in Z$. By $u \in W_z$, $u \to z$. But this contradicts the $\to$-minimality of z in Z , given by 3.9.4. .
? $u \prec a$: $u \prec a \prec z$: * to $u \to z$.
? $a \xrightarrow[=]{} u$: $a \xrightarrow[=]{} u \to z$: * to $a \prec z$.

This shows $u \to a$.

3.9.6 Let z be a $\to$-minimal element of $Z_!$. We have that $u \in W'_z$ implies $u \to a$.

Immediate from 3.9.5.

3.9.7 Let z be a $\to$-minimal element of $Z_!$. Then $(b, z)_\to \subseteq (b, y)_\to$.

Proof Assume $b \to v \to z$, to show $v \to y$.

First, we show $a \prec v$:

?: $a \xrightarrow[=]{} v$: $a \xrightarrow[=]{} v \to z$: * to $z \in Z$;
?: $v \to a$: $b \to v \to a$: * to $a \prec b$;
?: $v \prec a$: $v \prec a \prec b$: * to $b \to v$.

This shows $a \prec v$.

Therefore, $v \prec y$ would imply that $v \in Z$; then by $v \to z$, z is not a $\to$-minimal element of Z , contradicting 3.9.4. Thus, $\neg(v \prec y)$. But also $y \prec v$ would give $z \prec y \prec v$, contradicting $v \to z$, and $y \xrightarrow[=]{} v$ would give $y \xrightarrow[=]{} v \to z$, contradicting $z \prec y$.

We have shown that $v \to y$ as desired.

3.9.8 Let z be a $\to$-minimal element z of $Z_!$. Certainly, $W''_z \overset{\text{def}}{=} W'_z \cap \mu[x, a)_\to \neq \emptyset$ since $x \to z$ (3.9.2), and thus $x \in W''_z$. Since $\mu[x, a)_\to$ is linearly ordered by $\rightarrow$, we can take *the* $\rightarrow$-maximal element u of W''_z . I **claim** that $u\rightarrow !z$.

Proof Suppose that, on the contrary, there is v such that $u\rightarrow v\rightarrow z$. Of course, $v\in W'_z$, and thus $v\rightarrow a$ by 3.9.6. Let $w\in\mu[x, a)_{\rightarrow}$ such that $v\preceq w$. We must have $w\rightarrow z$: any other possibility leads to a contradiction:

? $z\preceq w$: $a\prec z\preceq w$: * to $w\rightarrow a$;

? $w\prec z$: $v\preceq w\prec z$: * to $v\rightarrow z$;

? $z\rightarrow w$: $v\rightarrow z\rightarrow w$: * to $v\preceq w$.

Since $\mu[x, a)_{\rightarrow}$ is linearly ordered by $\rightarrow$, we either have $u\rightarrow w$ or $w\underset{=}{\rightarrow} u$. But the second possibility gives $w\underset{=}{\rightarrow} u\rightarrow v$, contrary to $v\prec w$. Thus, we must have $u\rightarrow w$. We now have $w\in W''_z$ and $u\rightarrow w$, and this contradicts the "maximal" choice of u . We have proved that $u\rightarrow !z$.

3.9.9 Let $D=(V, M)$ be a $\prec$-cut such that $b\in E$=the border of D .

Let's write $\prec_y$ for the set $\{u:u\prec y\}$.

Assume that $a\prec !y$. Then, for any $w\in\prec_y\cap M$, we have $b\rightarrow w$ and $a\rightarrow w$.

Proof: For $b\rightarrow w$:

? $b\preceq w$: $b\preceq w\prec y$: * to $b\rightarrow y$.

? $w\prec b$: since $b\in B$, it would follow that $w\in U$, * to $w\in M$.

? $w\rightarrow b$: $w\rightarrow b\rightarrow y$: * to $w\prec y$.

This proves $b\rightarrow w$.

For $a\rightarrow w$:

? $a\prec w$: $a\prec w\prec y$: * to $a\prec !y$.

? w$\preceq a$: $w\preceq a\prec b$: * to $b\rightarrow w$ (proved earlier).

? $w\rightarrow a$: $b\rightarrow w\rightarrow a$: * to $a\prec b$.

This proves $a\rightarrow w$.

§4 The 2D case

We start by (re-)stating some properties of 2-Pd's.

First, some trivial, but basic, properties of 1-pd's;

4.0. Proposition Let S, S_1, S_2, S_3 denote 1-pd's in the computad $\boldsymbol{X}$, $\hat{S}_1, \hat{S}_2$ 1-pd's in the computad $\boldsymbol{Y}$, $f:\boldsymbol{X}\to\boldsymbol{Y}$ a map of computads.

$$S_1\cdot S_2 = \mathrm{id}_X \Longrightarrow S_1=S_2=\mathrm{id}_X ;$$
$$S_1\cdot S_2 = S_1\cdot S_3 \Longrightarrow S_2=S_3 ;$$
$$S_1\cdot S_3 = S_2\cdot S_3 \Longrightarrow S_1=S_2 ;$$
$$f(S) = \hat{S}_1\cdot\hat{S}_2 \Longrightarrow \exists!(S_1, S_2).[S=S_1\cdot S_2 \ \&\ f(S_1)=\hat{S}_1= \ \&\ f(S_2)=\hat{S}_2] .$$

A 2-atom φ is a 2-pd of the form

$$\varphi = b\cdot u\cdot e \ , \tag{1}$$

where b, e are 1-pd's, and u is a 2-indet. (For a 1-pd b and a 2-pd φ, $b\cdot\varphi = \varphi\circ_0 b$: "whiskering".) The expression (1) is uniquely determined for each φ (a special circumstance for dimension 2). Thus, 2-atoms are the same as *well-defined* expressions of the form (1).

u is the *nucleus* of the atom (1). To indicate that the nucleus of the atom φ is u, we write $\varphi[u]$ for φ.

In what follows, $\alpha,\beta,\rho,\sigma,\varphi,\psi$ will denote 2-atoms.

As we know from §2, a 2-*molecule* Φ is a finite tuple $\Phi:=:(\varphi_1, \ldots, \varphi_N)$ of 2-atoms such that the composite

$$[\![\Phi]\!] \overset{\mathrm{def}}{=} \varphi_1\cdot\ldots\cdot\varphi_N$$

("vertical" composite) is well-defined. N is called the *length* of Φ.

Every 2-pd is of the form $[\![\Phi]\!]$ for some, usually several different, molecules Φ. The main concern of the paper is with the question when two 2-molecules define the same 2-pd's: what is the "concrete" condition on 2-molecules Φ and Ψ for $[\![\Phi]\!]=[\![\Psi]\!]$?

We use the notation of §2.

4.1 Proposition Let $\boldsymbol{X}, \boldsymbol{Y}$ be computads, $f:\boldsymbol{X}\to\boldsymbol{Y}$ a map of computads. We assume that, for any 2-indets u and v in $\boldsymbol{X}$ or in $\boldsymbol{Y}$ (whether or not $u=v$), we have it is *not* the case that $\exists X.\mathrm{c}u=\mathrm{d}v=\mathrm{id}_X$; (2)

this holds in particular if the computads are 2-anchored, or 2-co anchored. ρ,σ,φ,ψ are any atoms in $\boldsymbol{X}$, Φ, Ψ, ... are any molecules in $\boldsymbol{X}$.

(i) Writing the atoms ρ, σ as $\rho = b\cdot u\cdot e$, $\sigma = \hat{b}\cdot v\cdot\hat{e}$

we have

$$\rho\to\sigma \iff \exists S.\ e = S\cdot \mathrm{d}v\cdot\hat{e}\ \&\ \hat{b} = b\cdot\mathrm{c}u\cdot S\ ;$$

here, the 1-pd S is uniquely determined.

Moreover, if $\rho\to\sigma$, then

$$\mathrm{c}\rho = \mathrm{d}\sigma = b\cdot\mathrm{c}u\cdot S\cdot\mathrm{d}v\cdot\hat{e}\ .$$

(ii) $L(\rho, \sigma, \varphi, \psi)$ if and only if, for suitable 1-pd's b, $\hat{e}$ and S, we have

$$\begin{aligned}\rho &= b\cdot u\cdot S\cdot\mathrm{d}v\cdot\hat{e}\\ \sigma &= b\cdot\mathrm{c}u\cdot S\cdot v\cdot\hat{e}\\ \varphi &= b\cdot\mathrm{d}u\cdot S\cdot v\cdot\hat{e}\\ \psi &= b\cdot u\cdot S\cdot\mathrm{c}v\cdot\hat{e}\end{aligned}$$

(" φ comes from σ by replacing $\mathrm{c}u$ by $\mathrm{d}u$; ψ from ρ by replacing $\mathrm{d}v$ by $\mathrm{c}v$ "). If $L(\rho, \sigma, \varphi, \psi)$, the data $b, \hat{e}, S$ are uniquely determined.

(iii) $\rho\to\sigma$ and $\rho\leftarrow\sigma$ cannot happen at the same time.

(iv) If $\rho\to\sigma$, then the pair (φ, ψ) for which $L(\rho, \sigma, \varphi, \psi)$ holds is uniquely determined. As a consequence, in a 2-anchored computad, there is a bijection between pairs (ρ, σ) of 2-atoms such that $\rho\to\sigma$ and pairs (φ, ψ) such that $\varphi\leftarrow\psi$; the bijection is given by the relation $L(\rho, \sigma, \varphi, \psi)$.

(v) If $\rho\longleftrightarrow\sigma$, then there is a unique pair (φ, ψ) such that $E(\rho, \sigma, \varphi, \psi)$. (Recall that $E(\rho, \sigma, \varphi, \psi)\iff L(\rho, \sigma, \varphi, \psi)\& L(\varphi, \psi, \rho, \sigma)$.)

(vi) $\rho\to\sigma\Longrightarrow f(\rho)\to f(\sigma)$; $\rho\cdot\sigma\!\downarrow\ \&\ f(\rho)\to f(\sigma) \Longrightarrow \rho\to\sigma$.

(vii) $\mathcal{S}_k(\Phi, \Psi_1)\ \&\ \mathcal{S}_k(\Phi, \Psi_2) \Longrightarrow \Psi_1=\Psi_2$.

Proof Using 4.0, the proofs are easy.

Example Condition (2) is necessary. Let the 0-indet X, distinct 1-indets f, g, and distinct 2-indets u, v be as follows:

$$f, g{:}X\to X\ ,\quad u{:}f\to\mathrm{id}_X\ , v{:}\mathrm{id}_X\to g\ .$$

Let $\rho=u$, $\sigma=v$. Then $\rho\to\sigma$ holds with $\alpha=u$ and $\beta=v$ (see (4) in §2); but also $\rho\leftarrow\sigma$ with $\alpha=v$ and $\beta=u$ in (5) in §2. Thus, 4.1(iii) is false now. We have $L(\rho,\sigma,f\cdot v,u\cdot g)$ and $L(v\cdot f,g\cdot u,\rho,\sigma)$ and $v\cdot f\neq f\cdot v$, $u\cdot g\neq g\cdot u$. Therefore, 4.1(v) is false too.

Planar pasting preschemes

Let $\boldsymbol{X}$ be a computad. A *planar pasting prescheme* on $\boldsymbol{X}$ is six-tuple $(\mathbf{N},\prec,\to,\mathbf{M},\mathbf{P},\vec{S})$ where

$\mathbf{N}$ is a finite set of 2-indets in $\boldsymbol{X}$,
$(\prec,\to)$ is a planar arrangement on the set $\mathbf{N}$,
$\mathbf{M},\mathbf{P}$ are 0-cells in $\boldsymbol{X}$,

and

$$\vec{S}=\langle (S^x_y)^C : x\to y,\ C \text{ a cut in } (x,y)_\to\rangle$$

is a family of 1-pd's $(S^x_y)^C$ in $\boldsymbol{X}$, one for each pair (x,y) of elements of $\mathbf{N}^\circ$ such that $x\to y$, and for a cut C for $(\prec,\to)$ restricted to $(x,y)_\to$,

these data are required to satisfy conditions 1) and 2) below. To formulate them, we extend the notation $(S^x_y)^C$ to any x,y and cut C for the whole arrangement $(\mathbf{N},\prec,\to)$ such that x and y are on the border of C, $x,y\in B[C]$, by the definition

$$(S^x_y)^C \overset{\text{DEF}}{=} (S^x_y)^{C\restriction(x,y)_\to} .$$

The data are required to satisfy the identities:

1) $$\mathrm{d}(S^x_y)^C=\mathrm{cc}x\ ,\ \mathrm{c}(S^x_y)^C=\mathrm{dd}y \qquad (x\to y,\ x,y\in B[C]\cup\{-\infty,\infty\})$$

(we have made here the convention that $\mathrm{cc}(-\infty)$ equals $\mathbf{M}$, and $\mathrm{dd}(\infty)=\mathbf{P}$ (note that $-\infty$, ∞ are not 2-indets, and $\mathrm{c}(-\infty)$, $\mathrm{d}(\infty)$ are not defined at all));

and

2)(*compositionality*)

$$(S^a_b)^C=(S^a_x)^C\cdot\partial^C x\cdot(S^x_b)^C$$
$(a\to x\to b\ \&\ a,x,b\in B[C]\cup\{-\infty,\infty\})$

(here: with $C=(U,L)$, $\partial^C x \overset{\text{def}}{=} \mathrm{c}x$ if $x\in U$, $\partial^C x \overset{\text{def}}{=} \mathrm{d}x$ if $x\in L$).

Remarks

1 For $x\to!y$, let us write S^x_y for $(S^x_y)^D$, where D is the unique cut in the empty interval $(x, y)_{\rightarrow}$. It is clear that the subsystem of the s^x_y for $x \rightarrow ! y$ determines the whole system $\vec{s}$, by the formula

$$(s^a_b)^C = s^a_{x_1} \cdot \partial^C x_1 \cdot s^{x_1}_{x_2} \cdot \partial^C x_2 \cdot \ldots \cdot \partial^C x_m \cdot s^{x_m}_b$$

where $C=(U, L)$ is a cut in $(a, b)_{\rightarrow}$, $x_1 \rightarrow ! x_2 \rightarrow ! \ldots \rightarrow ! x_m$ is a span of C, and $\partial^C x = \mathrm{c}x$ for $x \in U$, $\partial^C x = \mathrm{d}x$ for $x \in L$.

2 We can put the definition of a planar pasting prescheme in the form of a functor. First we define two categories $\boldsymbol{C}_1 = \boldsymbol{C}_1[\mathbf{N}, \prec, \rightarrow]$ and $\boldsymbol{C}_2 = [\mathbf{N}, \prec, \rightarrow]$, with the same objects. The objects in both will be the distinct symbols $-\infty$, ∞, and all "signed" 2-indets $u=(x, \varepsilon)$, where x is a 2-indet in $\mathbf{N}$, and ε is either "up" or "down"; $(x, \text{"}up\text{"})$ is written as $\underline{x}$, $(x, \text{"}down\text{"})$ as $\overline{x}$; $x = |u|$; $|-\infty| = -\infty$, $|\infty| = \infty$.

An arrow $u \rightarrow v$ in $\boldsymbol{C}_1$ exists only if $|u| \underset{=}{\longrightarrow} |v|$. If $|u| = |v|$, then an arrow exists only if $u=v$, and it is the unique identity arrow. If $|u| \rightarrow |v|$, an arrow $u \rightarrow v$ is a cut in the interval $(u, v)_{\rightarrow}$. The composition of arrows makes use of the signing: given non-identity arrows $u \rightarrow v \rightarrow w$ ($|u| = x$, $|v| = y$, $|w| = z$) in the form of cuts $u \xrightarrow{C} v \xrightarrow{D} w$, with $C=(U, L)$, we put $\hat{C} = (U \cup \{y\}, L)$ if $v = \underline{y}$, $\hat{C} = (U, L \cup \{y\})$ if $v = \overline{y}$. Obviously, $\hat{C}$ is a cut in $(x, y]_{\rightarrow}$, and y is on the border of $\hat{C}$. By 3.4', we have the cut $\hat{C} \cup D$ in $(x, z)_{\rightarrow}$, which is an arrow $D \circ C \overset{\text{def}}{=} \hat{C} \cup D : u \longrightarrow v$ in $\boldsymbol{C}_1$. Obviously, we have a category in this way; in fact, $\boldsymbol{C}_1$ is a finite 1-way category.

As before, we introduce the conventions $\mathrm{cc}(-\infty) = \mathbf{M}$, $\mathrm{dd}(\infty) = \mathbf{P}$.

When, for $|u| = x$, $|v| = y$, we have $x \neq y$, an arrow $f : u \rightarrow v$ ($|u| = x$, $|v| = y$) in $\boldsymbol{C}_2$ is a 1-arrow $f : \mathrm{cc}x \longrightarrow \mathrm{dd}y$ in the given computad $\boldsymbol{X}$ (underlying $\underline{\Gamma}$); the only other arrows are identities. Given non-identity arrows $u \xrightarrow{f} v \xrightarrow{g} w$, the composite $g \square f : u \rightarrow w$ is $g \square f = g \circ \partial v \circ f$, where $\partial v = dv$ if $v = \underline{y}$, and $\partial v = cv$ if $v = \overline{y}$.

A planar pasting prescheme is the same thing as a functor $\vec{s} : \boldsymbol{C}_1 \rightarrow \boldsymbol{C}_2$ which is the identity on objects.

The main result

We now fix a 2-separated anchored 2-Pd $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$. We use the terminology and notation of §2. We write $\mathbf{N}$, $\prec$, etc. for $\mathbf{N}_{\underline{\Gamma}}$, $\prec_{\underline{\Gamma}}$, etc. Recall especially the definition of $\prec$ as the intersection for all $\Phi\in\mathbf{G}^{\Gamma}$ of the total orders $<_{\Phi}$ on $\mathbf{N}$.

For $\Phi\in\mathbf{G}^{\Gamma}$, we use the notation

$$\varphi^{\Phi}[x] = (S_x^{-\infty})^{\Phi}\cdot x\cdot(S_{\infty}^{x})^{\Phi} \ . \tag{3}$$

that is, in the expression $b\cdot x\cdot e$ for the atom $\varphi^{\Phi}[x]$, with b, e suitable 1-pd's, we write $(S_x^{-\infty})^{\Phi}$ for b , and $(S_{\infty}^{x})^{\Phi}$ for e .

We repeat the definition of the relation $\rightarrow_{\Gamma}$, abbreviated $\rightarrow$, as in the second statement in 2.3(c). That is, for $x, y\in\mathbf{N}$:

$$x\rightarrow_{\Gamma}y \overset{\text{def}}{\Longleftrightarrow} \exists\Phi\in\mathbf{G}^{\Gamma}.\,x<_{\Phi}!\,y\ \&\ \varphi^{\Phi}[x]\rightarrow\varphi^{\Phi}[y] \tag{4}$$

For any $\Phi\in\mathbf{G}^{\Gamma}$, and any $x\in\mathbf{N}$, we define two cuts of the order $\prec$, $C_1=(U_1, L_1)$ and $C_2=(U_2, L_2)$, C_1 denoted also as $\underline{x}\Phi$, C_2 as $\overline{x}\Phi$, as follows. $y\in U_1\Longleftrightarrow y\leqq_{\Phi}x$ (and $y\in L_1\Longleftrightarrow x<_{\Phi}y$) ; $y\in U_2\Longleftrightarrow y<_{\Phi}x$ (and $y\in L_2\Longleftrightarrow x\leqq_{\Phi}y$) . It is clear that C_1 , C_2 are cuts, and, with B_1 , B_2 their respective borders, we have $x\in\underline{B}_1$, $x\in\overline{B}_2$; for instance, $x\prec z$ implies $x<_{\Phi}z$, that is, $z\in L_1$.

For the statements 4.2, 4.3 and 4.4 that follow, $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ is a 2-separated anchored 2-Pd; $\prec=\prec_{\Gamma}$, $\rightarrow=\rightarrow_{\Gamma}$.

4.2 Main Theorem Let $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ be a 2-separated anchored 2-Pd; $\mathbf{N}=\mathbf{N}_{\Gamma}$, $\prec=\prec_{\Gamma}$, $\rightarrow=\rightarrow_{\Gamma}$.

(i) $(\mathbf{N},\prec,\rightarrow)$ is a planar arrangement.

(ii) There exists a planar pasting prescheme $(\mathbf{N},\prec,\rightarrow,\mathrm{dd}\Gamma,\mathrm{cc}\Gamma,\vec{S})$ with the part $(\mathbf{N},\prec,\rightarrow)$ given in (i), and such that for any $\Phi\in\mathbf{G}^{\Gamma}$ and any $x\in\mathbf{N}$, we have

$$(S_x^{-\infty})^{\Phi} = (S_x^{-\infty})^{\underline{x}\Phi} = (S_x^{-\infty})^{\overline{x}\Phi}$$

$$(S^{x}_{\infty})^{\Phi} \;=\; (S^{x}_{\infty})^{\frac{x}{\ }\Phi} \;=\; (S^{x}_{\infty})^{\frac{\ }{x}\Phi} \quad .$$

Note that the expressions $(S^{-\infty}_{x})^{C}$, $(S^{x}_{\infty})^{C}$ in (ii) are well-defined for either $C = \frac{x}{\ }\Phi$ or $C = \frac{\ }{x}\Phi$ since x is on the border of C . Note also that the cuts $\frac{x}{\ }\Phi$, $\frac{\ }{x}\Phi$ differ on the particular element x *only* (x being "up" in $\frac{x}{\ }\Phi$, "down" in $\frac{\ }{x}\Phi$), and therefore automatically $(S^{-\infty}_{x})^{\frac{x}{\ }\Phi} = (S^{-\infty}_{x})^{\frac{\ }{x}\Phi}$ and $(S^{x}_{\infty})^{\frac{x}{\ }\Phi} = (S^{x}_{\infty})^{\frac{\ }{x}\Phi}$.

On the other hand, the expressions $(S^{-\infty}_{a})^{\Phi}$, $(S^{a}_{\infty})^{\Phi}$ are to be recalled from (3).

The proof of 4.2 is the main task of the rest of the paper.

For any $\Phi\in\mathbf{G}^{\Gamma}$, and any (x, y) such that $x<_{\Phi}!y$, let us write $x\rightarrow_{\Phi}y$ for $\varphi^{\Phi}[x]\longrightarrow\varphi^{\Phi}[y]$, and $x\leftarrow_{\Phi}y$ for $\varphi^{\Phi}[x]\longleftarrow\varphi^{\Phi}[y]$ as in section 2 .

4.3 Corollary Let $\underline{\Gamma}$ be as in 4.2.

(i) We have, with the notation of 4.2,

$$\varphi^{\Phi}[x] \;=\; (S^{-\infty}_{x})^{\frac{x}{\ }\Phi}\cdot x\cdot(S^{x}_{\infty})^{\frac{x}{\ }\Phi} = (S^{-\infty}_{x})^{\frac{\ }{x}\Phi}\cdot x\cdot(S^{x}_{\infty})^{\frac{\ }{x}\Phi} \quad ,$$

$$\mathrm{c}(\varphi^{\Phi}[x]) = (S^{-\infty}_{\infty})^{\frac{x}{\ }\Phi} \quad , \qquad\qquad \mathrm{d}(\varphi^{\Phi}[x]) = (S^{-\infty}_{\infty})^{\frac{\ }{x}\Phi} \quad .$$

(i)* Let $\Phi,\Psi\in\mathbf{G}^{\Gamma}$; let $x\in\mathbf{N}$. Assume $\forall y\in\mathbf{N}.(y<_{\Phi}x \iff y<_{\Psi}x)$. Then the atoms $\varphi^{\Phi}[x]$, $\psi^{\Psi}[x]$ are equal.

(ii) The mapping $\mathbf{o}_{\Gamma}=(\Phi\mapsto<_{\Phi}):\mathbf{G}^{\Gamma}\longrightarrow\mathbf{G}^{\prec}$ is one-to-one: Γ has unique factorization.

(ii)* Γ has strong unique factorization (see section 2).

(iii) For any $\Phi\in\mathbf{G}^{\Gamma}$, and any (x, y) such that $x<_{\Phi}!y$,

$$x \to_\Phi y \quad \text{iff} \quad x \longrightarrow y \ ,$$
$$x \leftarrow_\Phi y \quad \text{iff} \quad x \longleftarrow y \ ,$$
$$\neg(x \longleftrightarrow_\Phi y) \ \text{ iff } \ x \prec y \quad \text{iff} \quad x \prec ! y \ .$$

As a consequence, if $x, y \in \mathbf{N}$, $\Phi, \Psi \in \mathbf{G}^\Gamma$ and $x <_\Phi ! y$, $x <_\Psi ! y$ both hold, then $x \to_\Phi y$ iff $x \to_\Psi y$, and $x \leftarrow_\Phi y$ iff $x \leftarrow_\Psi y$.

(iv) The mapping $\mathbf{o}_\Gamma = (\Phi \mapsto <_\Phi) : \mathbf{G}^\Gamma \longrightarrow \mathbf{G}^\prec$ is a surjection.

(v) For any $x, y \in \mathbf{N}$,

$$x \prec ! y \ \text{ iff } \ \exists \Phi \in \mathbf{G}^\Gamma . x <_\Phi ! y \ \& \ \neg(x \longleftrightarrow_\Phi y) \ .$$

(vi) The mapping $\mathbf{o}_\Gamma = (\Phi \mapsto <_\Phi) : \mathbf{G}^\Gamma \longrightarrow \mathbf{G}^\prec$ is an isomorphism of the graphs $\mathbf{G}^\Gamma$ and $\mathbf{G}^\prec$.

(vii) Any morphism $f : \underline{\Gamma} \longrightarrow \underline{\Lambda}$ of *top-separated* anchored 2-Pd's induces an *isomorphism*

$$[f] : (\mathbf{N}_{\underline{\Gamma}}, \prec_{\underline{\Gamma}}, \to_{\underline{\Gamma}}) \xrightarrow{\ \cong\ } (\mathbf{N}_{\underline{\Lambda}}, \prec_{\underline{\Lambda}}, \to_{\underline{\Lambda}})$$

of the planar arrangements associated with $\underline{\Gamma}$ and $\underline{\Lambda}$. In other words, for any $u, v \in \mathbf{N}_{\underline{\Gamma}}$,
$u \prec_{\underline{\Gamma}} v \iff fu \prec_{\underline{\Lambda}} fv$, $u \to_{\underline{\Gamma}} v \iff fu \to_{\underline{\Lambda}} fv$.

(viii) The planar pasting prescheme for Γ with the properties in 4.2(ii) is uniquely determined.

Proof of 4.3 from 4.2 (i): immediate from 4.2(ii) : by 4.2(ii), the ingredients $(S_x^{-\infty})^\Phi$, $(S_\infty^x)^\Phi$ of $\varphi^\Phi[x]$ (see (3)) are determined by what the cut $\overline{x}\Phi$ is.

(ii): immediate from (i).

(ii)*: see section 10.

(iii): Let $\Phi \in \mathbf{G}^\Gamma$, $x <_\Phi ! y$, and assume $x \to y$, to show $x \to_\Phi y$. Let $C = \underline{x}\Phi = \overline{y}\Phi$. As we said above, $x \in \underline{B}$. $y \in \bar{B}$ for the border B of C . By 2), $(S_\infty^x)^C = (S_y^x)^C \cdot \mathrm{d}y \cdot (S_\infty^y)^C$. On the other hand, $(S_\infty^x)^C = (S_\infty^x)^\Phi$, $(S_\infty^y)^C = (S_\infty^y)^\Phi$ by 4.2 (ii). Thus, with $S \overset{\text{def}}{=} (S_y^x)^C$, we have $(S_\infty^x)^\Phi = S\cdot \mathrm{d}y\cdot (S_\infty^y)^\Phi$. Similarly, we show that $(S_y^{-\infty})^\Phi = (S_x^{-\infty})^\Phi\cdot \mathrm{c}x\cdot S$. We have verified the condition in 4.1(i) for $\varphi^\Phi[x]\longrightarrow\varphi^\Phi[y]$; we have $x\rightarrow_\Phi y$.

The second item in (iii), of course, proved in the same way.

For the third item: the first "iff" is a consequence of the first two items and the fact that $(\mathbf{N}, \prec, \rightarrow)$ is a planar arrangement. Since $\prec\subseteq<_\Phi$, and

$$x<_\Phi !y \iff x<_\Phi y \;\&\; \neg\exists z.x<_\Phi z<_\Phi y \;,$$
$$x\prec !y \iff x\prec y \;\&\; \neg\exists z.x\prec z\prec y \;,$$

$x\prec y$ & $x<_\Phi !y$ implies $x\prec !y$.

Before turning to the rest of the assertions, we introduce some more notation. First,

$$(x, y):\mathcal{S}(\Phi, \Psi) \overset{\mathrm{def}}{\iff} x\rightarrow_\Phi y \;\&\; L(\varphi^\Phi[x], \varphi^\Phi[y], \varphi^\Psi[y], \varphi^\Psi[x]) \;\&$$
$$\forall u\in\mathbf{N}\text{-}\{x, y\}.\varphi^\Phi[u]=\varphi^\Psi[u] \;.$$

This should be compared to the definition of the relation $\mathcal{S}(\Phi, \Psi)$ early in section 2. Clearly, we have $\mathcal{S}(\Phi, \Psi)$ iff $\exists x, y\in\mathbf{N}.(x, y):\mathcal{S}(\Phi, \Psi)$.

Note that

$$x\rightarrow_\Phi y \iff \exists\Psi.(x, y):\mathcal{S}(\Phi, \Psi) \tag{5.1}$$

and

$$x\leftarrow_\Phi y \iff \exists\Psi.(y, x):\mathcal{S}(\Phi, \Psi) : \tag{5.2}$$

indeed, if $x\rightarrow_\Phi y$, that is, $L(\varphi^\Phi[x], \varphi^\Phi[y], \rho, \sigma)$ for some ρ,σ , we can define Ψ by $\varphi^\Psi[u]=\varphi^\Phi[u]$ for $u\in\mathbf{N}\text{-}\{x, y\}$, $\varphi^\Psi[x]=\sigma$ and $\varphi^\Psi[y]=\rho$; Ψ is well-defined and $(x, y):\mathcal{S}(\Phi, \Psi)$.

On the other hand, for total orders $<^*$, $<^{**}$ of the set $\mathbf{N}$, we write

$$(x, y):\mathcal{S}(<^*, <^{**}) \overset{\mathrm{def}}{\iff} (x, y)\in <^* \;\&\; <^{**}=<^*\text{-}\{(x, y)\}\cup\{\{y, x\}\} \;.$$

We have that $\mathcal{S}(<^*, <^{**})$ iff $\exists x, y\in\mathbf{N}.(x, y):\mathcal{S}(<^*, <^{**})$; here, $\mathcal{S}$ is the graph-relation for $\mathbf{G}^\prec$ and $<^*, <^{**}\in\mathbf{G}^\prec$.

(iv): Since $\mathbf{G}^\prec$ is a connected graph (an elementary fact, true for any partial order $\prec$), and $\mathbf{G}^\Gamma$ is non-empty, it suffices to show that, assuming $\Phi\in\mathbf{G}^\Gamma$, $<^*\in\mathbf{G}^\prec$ and $\mathcal{S}(<_\Phi, <^*)$ ($<_\Phi$

and $<^*$ are connected by an edge in the graph $\mathbf{G}^{\prec}$), there is $\Psi\in\mathbf{G}^{\Gamma}$ such that $<^*=<_\Psi$. $\mathcal{S}(<_\Phi, <^*)$ means that there are $x, y\in\mathbf{N}$ such that $(x, y):\mathcal{S}(<_\Phi, <^*)$. But then, since $\prec\subseteq<^*$ and $(y, x)\in<^*$, we must have that $\neg(x\prec y)$. Looking at (iii), we see that either $x\longrightarrow_\Phi y$ or $y\longrightarrow_\Phi x$. If, e.g., $x\longrightarrow_\Phi y$, then $\varphi^\Phi[x]\to\varphi^\Phi[y]$. Thus, we can define $\Psi\in\mathbf{G}^\Gamma$ by stipulating $\varphi^\Psi[x]=\varphi^\Phi[y]$, $\varphi^\Psi[y]=\varphi^\Phi[x]$ and $\varphi^\Psi[u]=\varphi^\Phi[u]$ for $u\in\mathbf{N}-\{x, y\}$, and get $(x, y):\mathcal{S}(\Phi, \Psi)$; in particular, $<_\Psi=<^*$. This proves (iv).

(v): The "if" part is contained in (iii). Assume $x\prec!y$, to show $\exists\Phi\in\mathbf{G}^\Gamma.x<_\Phi!y\ \&\ \neg(x\longleftrightarrow_\Phi y)$. Apply 3.8 Elementary Lemma, to obtain a total order $<^*$ on $\mathbf{N}$ extending $\prec$ such that $x<^*!y$. By the present (iv), there is $\Phi\in\mathbf{G}^\Gamma$ such that $<_\Phi=<^*$. We have $x<_\Phi!y$. Since $x\prec y$, it follows from (iii) that we have $\neg(x\longleftrightarrow_\Phi y)$.

(vi): The fact that the mapping in question is a bijection is (ii) and (iv). The obvious fact that the graph-relation is preserved by the map $(\Phi\mapsto<_\Phi)$ was noted earlier. To see that it is reflected, assume that $\mathcal{S}(<_\Phi, <_\Psi)$, say $(x, y):\mathcal{S}(<_\Phi, <_\Psi)$; hence, in particular, $(x, y)\in<_\Phi!$, $(y, x)\in<_\Psi!$. Thus we cannot have either $x\prec y$ or $y\prec x$, since $x<_\Phi y$ and $y<_\Psi x$, and both $<_\Phi$ and $<_\Psi$ are compatible with $\prec$. Therefore, by 4.2 (i), either $x\to y$ or $y\leftarrow x$, which, by (iii) and $x<_\Phi!y$, implies that $x\to_\Phi y$ or $y\leftarrow_\Phi x$. By (5.1) and (5.2), there is $\hat{\Psi}$ such that $(x, y):\mathcal{S}(\Phi, \hat{\Psi})$ if $x\to_\Phi y$, and $(y, x):\mathcal{S}(\Phi, \hat{\Psi})$ if $y\leftarrow_\Phi x$. But then clearly, $<_{\hat{\Psi}}=<_\Psi$, which, by (ii), says that $\Psi=\hat{\Psi}$, and $\mathcal{S}(\Phi, \Psi)$, as was to be shown.

(vii): This is consequence of 4.1(vi) (especially the second part), and the preceding parts 4.3 (ii), (iii), (iv) and (v).

(viii): see section 10.

For the record: 2.1=4.3(ii), 2.2(a)=4.3(iv), 2.2(b) is part of 4.3(iii), 2.2(c)=4.2(i), 2.3=4.3(vii), 2.4=4.2(vi).

§5 The tree of variants of a molecule

The proof of the main theorem 4.2 is by an induction. We recursively construct the items of the structure mentioned in 4.2. Starting with a molecule Θ , the recursion builds a tree, which tree, when completed, becomes a spanning tree for the graph $\mathbf{G}^{[\![\Theta]\!]}$ used for 4.2. It is therefore no surprise that the formulation of the statement proved by induction repeats several aspects of the statement of 4.2.

Let $\underline{\Theta}=(\boldsymbol{X},\Theta)$ be an anchored 2-separated (2-)Molecule; $\Theta=(\theta_1[u_1], \ldots, \theta_N[u_N])$. We write $\mathbf{N}=\{u_1, \ldots, u_N\}$. We will use a, b, u, v, w, x, y, z ... to denote elements of $\mathbf{N}$. We write $x<y$ to mean that $x=u_i$, $u=u_j$ and $i<j$; we have the *natural order* $<$ of the indets in $\mathbf{N}$.

Recall the relations $\rho\rightarrow\sigma$, $\rho\leftarrow\sigma$, $\rho\longleftrightarrow\sigma$ for atoms ρ,σ .

For variants Φ , Ψ of Θ , we write $\mathcal{S}_x(\Phi, \Psi)$ (" Φ is switchable to Ψ at x $(\in\mathbf{N})$ ") if x is not the last element in the order $<_\Phi$, and for the y for which $x<_\Phi!y$, we have

$$\varphi^\Phi[x]\longleftrightarrow\varphi^\Phi[y] ,$$
$$\varphi^\Psi[x]=\varphi^\Phi[y] ,\quad \varphi^\Psi[y]=\varphi^\Phi[x]$$

and

$$\varphi^\Psi[u]=\varphi^\Phi[u] \text{ for } u\in\mathbf{N}-\{x, y\} .$$

This is the same as $\mathcal{S}_k(\Phi, \Psi)$ used in section 2 , where $k=h_{<_\Phi}(x)$ with

$h_{<_\Phi}:(\mathbf{N}, <_\Phi)\xrightarrow{\cong}(\{1, \ldots, N\}, <)$ the unique isomorphism as shown. To avoid any misunderstanding, we *discontinue* the use of the notation $\mathcal{S}_k(\Phi, \Psi)$ with k an integer.

Note that if $x<_\Phi!y$ and $\varphi^\Phi[x]\longleftrightarrow\varphi^\Phi[y]$, then there is a unique Ψ such that $\mathcal{S}_x(\Phi, \Psi)$.

For total orders $\ll_1$, $\ll_2$ of $\mathbf{N}$, and $x, y\in\mathbf{N}$, we write $\mathcal{S}_x(\ll_1, \ll_2)$ for

$$x<y \;\&\; x\ll_1!y \;\&\; \ll_2=\ll_1-\{(x, y)\}\cup\{(y, x)\}$$

(since y is uniquely determined by x if it exists, it does not have to be mentioned in the notation).

We define $\mathbf{T}=\mathbf{T}[\Theta]$, the *tree of variants of* Θ . $\mathbf{T}$ is a labelled tree. At each node $t\in\mathbf{T}$, we will have labels as follows:

$\Phi^{t} = (\varphi^{t}_{1}, \ldots, \varphi^{t}_{N})$, a molecule of length N, a variant of Θ (that is, Φ^{t} and Θ define the same 2-pd).

$<_{t}$, a total order of the set $\mathbf{N}$;

$\rightarrow_{t}$, $\leftarrow_{t}$: relations on $\mathbf{N}$ (set of ordered pairs of elements $\mathbf{N}$)

We define the tree $\mathbf{T}$ by recursion: we define the root; and having defined a node t with its labels, we define what the (immediate) successors (or "children") of t and their labels are.

The root $\mathbf{r}$ has labels:

$\Phi^{\mathbf{r}} = \Theta$;

$<_{\mathbf{r}} = <$, the natural order of the 2-indets in Θ.

$\rightarrow_{\mathbf{r}}$, $\leftarrow_{\mathbf{r}}$ will be explained as a special case of the general definition below.

Suppose we have defined t, Φ^{t}, and $<_{t}$. First, we explain what $\rightarrow = \rightarrow_{t}$, $\leftarrow = \leftarrow_{t}$ are.

Let us write $\varphi^{t}[x]$ for $\varphi^{\Phi^{t}}[x]$ (the component φ^{t}_{k} whose nucleus is x). We define

$$x \rightarrow_{t} y \overset{\text{def}}{\Longleftrightarrow} x<y \;\&\; x<_{t}!\,y \;\&\; \varphi^{t}[x] \longrightarrow \varphi^{t}[y] \;, \tag{1}$$

$$x \leftarrow_{t} y \overset{\text{def}}{\Longleftrightarrow} x<y \;\&\; x<_{t}!\,y \;\&\; \varphi^{t}[x] \longleftarrow \varphi^{t}[y] \;. \tag{2}$$

Note that for both $x \rightarrow_{t} y$, $x \leftarrow_{t} y$, we have as prerequisite that $x<y$ holds. In other words, the construction of the tree proceeds in steps each of which is the act of passing an indeterminate x ahead past its successor y in the correct order ($x<_{t}!\,y$) (if and) *only if* x precedes y in the natural order and the atoms in question are exchangeable ($\varphi^{t}[x] \longleftrightarrow \varphi^{t}[y]$).

For an arbitrary total order $\ll$ of the set $\mathbf{N}$, the *inversion number* of $\ll$. $\mathrm{inv}(\ll)$, is the number of pairs (x, y) such that $x \ll y$ but $x>y$. Let *the level* $\ell(t)$ of the node t be the number for which $\ell(\mathbf{r})=0$, and $\ell(\hat{t})=\ell(t)+1$ for a successor $\hat{t}$ of t. The previous paragraph says that $\ell(t)=\mathrm{inv}(<_{t})$ in general.

To continue the construction of $\mathbf{T}$: the successors of t in the tree are, by definition, in a bijective correspondence with the elements of the set $\longleftrightarrow_{t} \overset{\text{def}}{=} \rightarrow_{t} \dot{\cup} \leftarrow_{t}$. We write $t[x, y]$ for the successor $\hat{t}$ of t corresponding to the pair $(x, y) \in \longleftrightarrow_{t}$. Then $\Phi^{\hat{t}}$ is the (unique) molecule for which $\mathcal{S}_X(\Phi^t, \Phi^{\hat{t}})$, and $<_{\hat{t}}$ is the order for which $\mathcal{S}_X(<_t, <_{\hat{t}})$.

Note that

$<_t = <_{\Phi^t}$ implies that $<_{\hat{t}} = <_{\Phi^{\hat{t}}}$;

and therefore, by a trivial induction, $<_t = <_{\Phi^t}$ for all t .

This completes the definition of the labelled tree $\mathbf{T}$.

We will use the notation

$$\varphi^t[x] = (s_X^{-\infty})^t \cdot x \cdot (s_\infty^X)^t \quad . \tag{3}$$

that is, in the expression $b \cdot x \cdot e$ for the atom $\varphi^t[x]$, with b, e suitable 1-pd's, we write $(s_X^{-\infty})^t$ for b , and $(s_\infty^X)^t$ for e .

In what follows, *a tree of variants of* Θ , or simply: a *tree,* is any *subtree* of $\mathbf{T}$; that is, any subset T of (the set of nodes of) $\mathbf{T}$ which contains the root $\mathbf{r}$, and for which, if $\hat{t} \in T$, and $\hat{t}$ is a successor of t in $\mathbf{T}$, then $t \in T$. The labelling of a tree is inherited from $\mathbf{T}$.

Of course, $\mathbf{T}$ itself is a tree; and so is $\{\mathbf{r}\}$, consisting of the root only.

Let T be a tree. We define

$$\rightarrow_T^0 \stackrel{\text{def}}{=} \bigcup_{t \in T} \{(x, y) : (x, y) \in \rightarrow_t \ \& \ t[x, y] \in T\} \ ,$$

$$\leftarrow_T^0 \stackrel{\text{def}}{=} \bigcup_{t \in T} \{(x, y) : (x, y) \in \leftarrow_t \ \& \ t[x, y] \in T\}$$

(Note that $\rightarrow_{\{\mathbf{r}\}}^0 = \leftarrow_{\{\mathbf{r}\}}^0 = \emptyset$.)

$$\rightarrow_T \stackrel{\text{def}}{=} (\rightarrow_T^0 \cup (\leftarrow_T^0)^{\text{op}})^{\text{tr}}$$

(tr is "transitive closure")

$$\longleftrightarrow_T \stackrel{\text{def}}{=} \rightarrow_T \cup (\rightarrow_T)^{\text{op}}$$

$$\prec_T \stackrel{\text{def}}{=} <\text{-}\longleftrightarrow_T$$

Next, we state an equivalent versionof the concept of planar pasting presceme of the last section, in a form that is most suitable for the procedures of this and the next two sections.

T is *annotated* if the conditions and data (that are soon seen to be uniquely determined if they exist) listed in (i)-(iv) are present.

(i) $(\mathbf{N}, \prec_T, \rightarrow_T)$ is a planar arrangement.

Notation In what follows, we write $\prec$ for $\prec_T$, and $\rightarrow$ for $\rightarrow_T$.

(ii) With $\mathbf{N}^\circ = \mathbf{N}\dot{\cup}\{-\infty, \infty\}$ and $-\infty\rightarrow\infty$, $-\infty\rightarrow x$, $y\rightarrow\infty$ for all $x, y\in\mathbf{N}$, for every $x, y\in\mathbf{N}^\circ$ such that $x\rightarrow !y$, we have a specified 1-pd s^x_y such that $\mathrm{d}(s^x_y)=\mathrm{cc}(x)$, $\mathrm{c}(s^x_y)=\mathrm{dd}(y)$. (Conventions: $\mathrm{dd}(-\infty)=\mathrm{dd}(\llbracket\Theta\rrbracket)$, $\mathrm{cc}(-\infty)=\mathrm{cc}(\llbracket\Theta\rrbracket)$.)

We call the s^x_y the *basic* 1-pd's.

Notation For any open interval $(a, b)_\rightarrow$ of $(\mathbf{N}, \prec, \rightarrow)$, ($a=-\infty$, $b=\infty$ are allowed), and for any signed span

$$\xi :=: a=x_0\rightarrow !x_1\rightarrow !x_2\rightarrow !\ldots\rightarrow !x_{n-1}\rightarrow !x_n=b$$

in $(a, b)_\rightarrow$ with appropriate signs for each x_i, we define

$$(s^a_b)^\xi \stackrel{\text{def}}{=} s^a_{x_1}\cdot\partial^\xi x_1\cdot s^{x_1}_{x_2}\cdot\partial^\xi x_1\cdot\ldots\cdot\partial^\xi x_{n-1}\cdot s^{x_{n-1}}_b \quad .$$

Here, $\partial^\xi x_i = \mathrm{c}x_i$ if x_i is signed $\underline{x_i}$, $\partial^\xi x_i = \mathrm{d}x_i$ if x_i is signed $\overline{x_i}$ in ξ.

$(s^a_b)^\xi$ is well-defined by (ii).

We have the following obvious *(de)composition equation*: whenever a, b, x are in the signed span ξ, then

$$(s^a_b)^\xi = (s^a_x)^\xi\cdot\partial^\xi x\cdot(s^x_b)^\xi \quad . \tag{4.0}$$

(iii) For any two signed spans ξ and ζ of the same open interval $I=(a,b)_{\rightarrow}$, if $\xi\sim\zeta$ (they define the same cut in I), we have

$$(S^a_b)^\xi = (S^a_b)^\zeta \ .$$

Notation For a cut C of $I=(a,b)_{\rightarrow}$, we write $(S^a_b)^C \stackrel{\text{def}}{=} (S^a_b)^\xi$ for some (any) signed span ξ defining C; by (iii), $(S^a_b)^C$ is well-defined.

Let $C=(U,L)$ be a cut of $I=(a,b)_{\rightarrow}$, and x a proper element of the border B of C (in particular, $x\in I$). Let $\partial^C x = \mathrm{c}x$ if $x\in U$, $\partial^C x = \mathrm{d}x$ if $x\in L$. As a consequence of (iii), we have the all-important *decomposition equation*

$$(S^a_b)^C = (S^a_x)^C \cdot \partial^C x \cdot (S^x_b)^C \ . \tag{4}$$

Indeed, it suffices to take any signed span ξ of B containing x (such exists!), and apply (4.0).

Condition (iii) and the relation (4) allow us to ignore "signed spans", and use exclusively the notation $(S^x_y)^C$ relatively to a cut C. Without talking about signed spans and the like as we did in (iii), we could have formulated (iii) by saying that we have an operation

$$\langle a\rightarrow b\ ,\ C\rangle \ \longmapsto\ (S^a_b)^C$$

satisfying (4).

Notation For any $t\in T$, and any $x\in\mathbf{N}$, we define two cuts of the order $\prec$, $C_1=(U_1,L_1)$ and $C_2=(U_2,L_2)$, C_1 denoted also as $\underline{{}^x}t$, C_2 as $\overline{{}_x}t$, as follows. $y\in U_1 \Longleftrightarrow y\leqq_t x$ (and $y\in L_1 \Longleftrightarrow x<_t y$); $y\in U_2 \Longleftrightarrow y<_t x$ (and $y\in L_2 \Longleftrightarrow x\leqq_t y$). It is clear that C_1, C_2 are cuts, and, with B_1, B_2 their respective borders, we have $x\in\underline{B}_1$, $x\in\bar{B}_2$.

(iv) For any $t\in T$ and any $x\in\mathbf{N}$, we have, for both $C=\underline{{}^x}t$ and $C=\overline{{}_x}t$,

$$(S^{-\infty}_x)^t = (S^{-\infty}_x)^C \ ,$$
$$(S^x_\infty)^t = (S^x_\infty)^C \ .$$

(Here, we have related the notation introduced in (3) above, and the one introduced after (iii).)

Theorem 5.1 Every tree of variants of Θ is annotated. In particular, the full tree $\mathbf{T}[\Theta]$ is annotated.

Proof of 4.2 from 5.1

We apply 5.1 to the full tree $T=\mathbf{T}[\Theta]$.

Let $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ be a 2-separated anchored 2-Pd. Let Θ be any molecule such that $[\![\Theta]\!]=\Gamma$. We use the notation developed above, and Theorem 5.1, in relation to Θ . Note, especially, that we have the *natural order* $<$ on $\mathbf{N}$, the set of 2-indets in Θ .

It clearly follows from 5.1 that, for $s, t\in\mathbf{T}[\Theta]$, $<_s=<_t$ implies $\Phi^s=\Phi^t$.

Let $\rightarrow$ denote $\rightarrow_{\mathbf{T}}$, $\prec$ denote $\prec_{\mathbf{T}}$.

By induction on $\ell(t)$, we see that for every $t\in\mathbf{T}$, $<_t$ is compatible with $\prec$.

The same argument that gave 4.3(iii) from 4.2 gives, using 5.1, that, for any $t\in\mathbf{T}$, $x<y$ & $x<_t!y$ & $x\rightarrow y$ implies (and, of course, is equivalent to) $x\rightarrow_t y$, and $x<y$ & $x<_t!y$ & $y\rightarrow x$ implies $x\leftarrow_t y$ (**Claim 1**).

I claim (**Claim 2**) that for every total order $\ll$ of $\mathbf{N}$ compatible with $\prec$ there is $t\in\mathbf{T}$ such that $<_t=\ll$. The proof is by induction on $\mathrm{inv}(\ll)$. When $\mathrm{inv}(\ll)=0$, $\ll=<$, and $<_{\mathbf{r}}=\ll$. Assume that $\mathrm{inv}(\ll)>0$. Let $(x, y)\in\mathbf{N}\times\mathbf{N}$ be such that $x\ll!y$ and $x>y$; there must be such a pair, because if $x\ll!y$ always implies $x<y$, then clearly $\ll=<$ follows. But then, we must have that x and y are incomparable in $\prec$: $x\prec y$ would imply $x<y$ since $<=<_{\mathbf{r}}$ is compatible with $\prec$; and $y\prec x$ would imply $y\ll x$. Therefore, $\tilde{\ll}=\ll-\{(x, y)\}\cup\{(y, x)\}$ is a total order of $\mathbf{N}$, still compatible with $\prec$. Clearly, $\mathrm{inv}(\tilde{\ll})=\mathrm{inv}(\ll)-1$.

By the induction hypothesis, there is $\tilde{t}\in\mathbf{T}$ such that $\ll=<_{\tilde{t}}$. Since $\neg(x\prec\succ y)$, by (i), we have $x\longleftrightarrow y$; since also $x<y$ & $x<_{\tilde{t}}!y$, by Claim 1 we conclude that $x\longleftrightarrow_{\tilde{t}}y$. Therefore, by the definition of the tree $\mathbf{T}$, there is $t\in\mathbf{T}$, successor of $\tilde{t}$, such that $\mathcal{S}_x(<_{\tilde{t}}, <_t)$; clearly, $<_t=\ll$.

Next, I claim (**Claim 3**) that for every variant Φ of Θ (every Φ such that $[\![\Phi]\!]=\Gamma$), there is $t\in\mathbf{T}$ such that $\Phi^t=\Phi$, and, as a consequence, also $<_t=<_\Phi$. We prove this by showing that the set $\{\Phi^t:t\in\mathbf{T}\}$ is closed under the "switching" relation $\mathcal{S}$; recall that $[\![\Phi]\!]=\Gamma$ iff $(\Theta,\Phi)\in\mathcal{S}^{r/tr}$; and, of course, $\Theta\in\{\Phi^t:t\in\mathbf{T}\}$.

Suppose $t\in\mathbf{T}$, Ψ a molecule, and $(\Phi^t, \Psi)\in\mathcal{S}$ (in particular, Ψ is a variant of Θ), to show that there is $\tilde{t}\in\mathbf{T}$ such that $\Psi=\Phi^{\tilde{t}}$. The fact that $(\Phi^{t}, \Psi)\in\mathcal{S}$ means that there is $x\in\mathbf{N}$ such that $\mathcal{S}_{x}(\Phi^{t}, \Psi)$. Let y be the element for which $x<_{\Phi^{t}}!y$, that is, $x<_{t}!y$. There are two cases: $x<y$ (case 1) or $y<x$ (case 2). In case 1, the definition of the tree $\mathbf{T}$ immediately entails that, for a suitable successor $\hat{t}$ of t , $\Psi=\Phi^{\hat{t}}$.

Suppose case 2. Then x and y are incomparable in $\prec$: $x\prec y$ would entail $x<y$, and $y\prec x$ would entail $y<_{t}x$. Therefore, $\ll=<_{t}-\{(x, y)\}\cup\{(y, x)\}$ is also compatible with $\prec$. By Claim 2, there is $\tilde{t}\in\mathbf{T}$ such that $\ll=<_{\tilde{t}}$. Consider $\Phi^{\tilde{t}}$. Since $y<x \,\&\, y<_{\tilde{t}}!x \,\&\, \neg(y\prec\succ x)$, we have (by Claim 1) $y<x \,\&\, y<_{\tilde{t}}!x \,\&\, y\longleftrightarrow_{\tilde{t}}x$; thus, there is $\hat{t}$, successor of $\tilde{t}$, such that $\mathcal{S}_{y}(\Phi^{\tilde{t}}, \Phi^{\hat{t}})$ and $\mathcal{S}_{y}(<_{\tilde{t}}, <_{\hat{t}})$. From the latter, it follows that $<_{\hat{t}}=<_{t}$. Therefore, by (iii), $\Phi^{\tilde{t}}=\Phi^{t}$. $\mathcal{S}_{y}(\Phi^{\tilde{t}}, \Phi^{\hat{t}})$ gives $\mathcal{S}_{x}(\Phi^{\hat{t}}, \Phi^{\tilde{t}})$. Since also $\mathcal{S}_{x}(\Phi^{t}, \Psi)$, it follows that $\Psi=\Phi^{\tilde{t}}$.

This completes the proof of Claim 3.

By claim 2, the set $\{<_{t}: t\in\mathbf{T}\}$ equals the set of all total extensions on $\mathbf{N}$ of $\prec$, hence, in particular, $\bigcap\{<_{t}: t\in\mathbf{T}\}=\prec$. By claim 3, the same set $\{<_{t}: t\in\mathbf{T}\}$ equals $\{<_{\Phi}:\Phi\in\mathbf{G}^{\Gamma}\}$. Therefore,

$$\prec_{\mathbf{T}} = \prec = \bigcap\{<_{t}: t\in\mathbf{T}\} = \{<_{\Phi}:\Phi\in\mathbf{G}^{\Gamma}\} = \prec_{\Gamma} ,$$

the last equality being the definition of $\prec_{\Gamma}$.

Next, we go back to the definition of $\rightarrow_{\Gamma}$ in section 4 (see (7) there). I claim (**Claim 4**) that

$$\rightarrow_{\Gamma} = \bigcup_{t\in\mathbf{T}}(\rightarrow_{t}\cup(\leftarrow_{t})^{\mathrm{op}}) . \qquad (5)$$

This is virtually obvious from Claim 2. In particular, that the right-hand side of (5) is contained in the left=hand side is clear.

For the converse, suppose $x\rightarrow_{\Gamma}y$, that is, we have Φ , a variant of Θ , such that $x<_{\Phi}!y \,\&\, \varphi^{\Phi}[x]\rightarrow\varphi^{\Phi}[y]$. Either $x<y$ (case 1), or $y<x$ (case 2). In case 1, take $t\in\mathbf{T}$ such that $\Phi=\Phi^{t}$, and (x, y) is clearly in the right-hand side of (5) . In case 2 , we can pass to the unique Ψ such that $\mathcal{S}_{x}(\Phi, \Psi)$; we will have $y<x \,\&\, y<_{\Psi}!x \,\&\, \varphi^{\Psi}[y]\leftarrow\varphi^{\Psi}[x]$;

now, take $t\in\mathbf{T}$ such that $\Psi=\Phi^{t}$.

I claim (**Claim 5**) that $\to\overset{\mathrm{DEF}}{=}\to_{\mathbf{T}}=\bigcup_{t\in\mathbf{T}}(\to_{t}\cup(\leftarrow_{t})^{\mathrm{op}})$ (which means, of course, that $\bigcup_{t\in\mathbf{T}}(\to_{t}\cup(\leftarrow_{t})^{\mathrm{op}})$ is already transitive). Assume that $x\to y$. Either $x<y$ (case 1), or $y<x$ (case 2). In any case, x and y are incomparable in $\prec$ since $x\to y$ and we have (i). In case 1, find, by 3.8, a total order $\ll$ of $\mathbf{N}$ compatible with $\prec$ for which $x\ll!y$; by Claim 2, let $t\in\mathbf{T}$ be such that $\ll=<_{t}$. since $x<y$ & $x<_{t}!y$, by Claim 1, we have $x\to_{t}y$ as desired. In case 2, we proceed by exchanging the roles of x and y .

Claims 4 and 5 say that $\to_{\mathbf{T}}=\to_{\Gamma}$, with the latter as defined in section 4.

4.2(i) is now the same as 5.1(i).

It is clear that clauses (ii), (iii) and (iv) of the definition of "annotated" 5.1 add up to what is contained in 4.2(ii).

Proof of 5.1 modulo 7.1 Lemma from section 7.

The proof of Theorem 5.1 is an induction on the number of nodes of the tree T .

For the smallest tree $T=\{\mathbf{r}\}$, $\to=\to_{T}$ is empty (on $\mathbf{N}$, not on $\mathbf{N}^{\circ}$); $\prec_{T}=<$, the natural order. For $x\in\mathbf{N}$, we write $S_{x}^{-\infty}$ for $(S_{x}^{-\infty})^{\mathbf{r}}$, S_{∞}^{x} for $(S_{\infty}^{x})^{\mathbf{r}}$.
$\varphi^{\mathbf{r}}[x]=\varphi^{\Theta}[x]=S_{x}^{-\infty}\cdot x\cdot S_{\infty}^{x}$; we have $-\infty\to!x$, $x\to!\infty$, and these are all the $\to!$ relations; the *basic* 1-pd's are the $S_{x}^{-\infty}$, S_{∞}^{x} $(x\in\mathbf{N})$; the cuts of the planar arrangement $(\prec,\to)$ are the cuts $\frac{x}{\ }\mathbf{r}$ and $\frac{\ }{x}\mathbf{r}$, the $N+1$ cuts of the total order $<$ (if $x<!y$, then $\frac{x}{\ }\mathbf{r}=\frac{\ }{y}\mathbf{r}$). It is clear that the tree $\{\mathbf{r}\}$ is annotated.

For the induction step, we assume that we have an annotated tree T , $t\in T$, and $a,b\in\mathbf{N}$ such that $a<b$ & $a<_{t}!b$ & $\varphi^{t}[a]\longleftrightarrow\varphi^{t}[b]$, but the corresponding successor $\hat{t}=t[a,b]$ of t in the full tree $\mathbf{T}[\Theta]$ is not in T (yet). We consider the subtree $\hat{T}=T\dot{\cup}\{\hat{t}\}$ of $\mathbf{T}[\Theta]$, and we prove that $\hat{T}$ is annotated, that is, we have the data and conditions for $\hat{T}$ as set out in clauses (i), (ii), (iii) and (iv) above.

We write $\prec$, $\to$ for $\prec_{T}$, $\to_{T}$, respectively, with the *given* tree T .

A remark is, perhaps, in place here. The assumption $\varphi^{t}[a]\longleftrightarrow\varphi^{t}[b]$, which is the disjunction $\varphi^t[a]\longrightarrow\varphi^t[b] \vee \varphi^t[b]\longleftarrow\varphi^t[a]$, is not directly related to the notations $x\to y$, $x\leftarrow y$ for nuclei $x, y\in\mathbf{N}$. The latter refer to the induction hypothesis on the "locally" given tree T, giving a meaning to the relations $\to = \to_T$, $\leftarrow = \leftarrow_T$ on $\mathbf{N}$. On the other hand, $\varphi^t[a]\longrightarrow\varphi^t[b]$ is an instance of the notation $\rho\longrightarrow\sigma$ for atoms ρ,σ in general, and it is an "absolute" concept, the truth or falsity of $\rho\longrightarrow\sigma$ depending on the atoms ρ and σ solely.

Henceforth, we assume $\varphi^t[a]\longrightarrow\varphi^t[b]$; the other case $\varphi^t[a]\longleftarrow\varphi^t[b]$ is, of course, symmetric.

The fact $\varphi^t[a]\longrightarrow\varphi^t[b]$, via the notations $\varphi^t[a] = (S_a^{-\infty})^t\cdot a\cdot(S_\infty^a)^t$, $\varphi^t[b] = (S_b^{-\infty})^t\cdot b\cdot(S_\infty^b)^t$, and via 4.1(i), translate into the two equations

$$(S_\infty^a)^t = S_b^a\cdot \mathrm{d}b\cdot(S_\infty^b)^t \qquad\qquad (S_b^{-\infty})^t = (S_a^{-\infty})^t\cdot \mathrm{c}a\cdot S_b^a ,$$

with a new 1-pd that we have denoted by S_b^a. If it so happens that $a\to !b$ holds (with reference to the given planar arrangement $(\prec, \to)$) then S_b^a had a meaning prior to our new notation, as one of the basic 1-pd's of the system given with T; as we will immediately see, the two meanings must in fact coincide.

Note that $\mathrm{d}(S_b^a)=\mathrm{cc}a$, $\mathrm{c}(S_b^a)=\mathrm{dd}b$.

Let us denote the cut ${}^a_{-}t = {}_{\overline{b}}t$ for $(\mathbf{N}, \prec, \to)$ by D. With B the border of D, $a\in\underline{B}$, $b\in\overline{B}$.

Using (iv) for the old system for T, we have that

$$(S_\infty^a)^t = (S_\infty^a)^D,\ (S_\infty^b)^t = (S_\infty^b)^D,\ (S_b^{-\infty})^t = (S_b^{-\infty})^D,\ (S_a^{-\infty})^t = (S_a^{-\infty})^D,$$

thus,

$$(S_\infty^a)^D = S_b^a\cdot \mathrm{d}b\cdot(S_\infty^b)^D \qquad\qquad (S_b^{-\infty})^D = (S_a^{-\infty})^D\cdot \mathrm{c}a\cdot S_b^a \tag{6}$$

If $a\to !b$, and thus S_b^a is given as a basic 1-dp for T, then, clearly, $(S_b^a)^D$ is well defined, it equals S_b^a, and by (iii),

$$(S_\infty^a)^D = S_b^a\cdot \mathrm{d}b\cdot(S_\infty^b)^D \qquad\qquad (S_b^{-\infty})^D = (S_a^{-\infty})^D\cdot \mathrm{c}a\cdot S_b^a ;$$

comparing these with (6), we see that, indeed, now the two meanings of S^a_b coincide.

There are two cases as to the position of a and b in the planar arrangement given with T; in **Case 1** $a \longleftrightarrow b$, in **Case 2** $a \prec b$; note that $b \prec a$ is excluded by $a <_t b$.

We are going to define the planar arrangement (clause (i)) and the system of 1-pd's (clause (ii)) for the tree $\hat{T}$ separately for the two cases. Next, we look at the invariance property (iii) for the new system: in Case 1, this will be immediate; in Case 2, it will take us two more sections, §6 and §7, to prove. Finally, we complete the proof of clause (iv) for the new tree $\hat{T}$ in essentially the same way for the two cases.

As a reminder: we have assumed that $\varphi^t[a] \to \varphi^t[b]$.

Case 1: $a \longleftrightarrow b$.

We note, first of all, that we must have $a \to b$; $a \leftarrow b$ is impossible. The reason is that $a \leftarrow b$ would entail $\varphi^t[a] \leftarrow \varphi^t[b]$: one sees this by the same argument that was used in the proof of 4.3(iii) and Claim 1 above. Remember that $\varphi^t[a] \to \varphi^t[b]$ and $\varphi^t[a] \leftarrow \varphi^t[b]$ cannot hold at the same time.

Looking at the definitions above, we see that $\to^0_{\hat{T}} = \to^0_T \dot{\cup} \{(a, b)\}$, $\leftarrow^0_{\hat{T}} = \leftarrow^0_T$. Because, however, $(a, b) \in \to_T$ already, $\to_{\hat{T}}$ remains the same as $\to_T$. Thus, the planar arrangement $(\to, \prec)$ remains the same for $\hat{T}$ as it was for T.

The system of basic 1-pd's

$$\langle S^x_y \rangle_{x,\, y \in \mathbf{N}^\circ,\, x \to !y}$$

is, by definition, identical to the old one as given by (ii) and (iii) for T. We have clauses (ii) and (iii) established in Case 1.

Case 2: $a \prec b$.

From the fact that $a <_t !b$, it is immediate that we must have $a \prec !b$.

We clearly have that $\to_{\hat{T}} = (\to \dot{\cup} \{(a, b)\})^{tr}$.

We apply 3.5 Proposition, and its accompanying notation. Thus, $(\mathbf{N}, \to_{\hat{T}}, \prec_{\hat{T}})$ is the same as $(\mathbf{N}, \to\!\!\to, \prec\!\!\prec)$; it is a planar arrangement by 3.5(i).

By 3.5'(i), $a \to\!\!\to ! b$.

Recall from (6) the definition of s^a_b ; now, s^a_b is "new"; it did not appear as a basic 1-pd in the system for T .

Recall 3.5'. We define the system of basic 1-pd's for $(\to\!\!\to, \prec\!\!\prec)$ as

$$\langle s^x_y \rangle_{x,\, y \in \mathbf{N}^\circ \& x \to\!\!\to ! y \& (x \neq a \vee y \neq b)} \cup \langle s^a_b \rangle \quad . \tag{7}$$

(the s^x_y are given in the induction hypothesis; we restrict the given system, and augment it by a single new element).

The proof that clause (iii) holds for $\hat{T}$ and the new system (7) is hard work: it will occupy sections 6 and 7. At this point, we assume that (iii) has been shown (this is 7.1 Lemma in section 7).

We turn to the (easy) proof of clause (iv) for the new systems in both cases simultaneously. For simplicity of notation, we write $(\prec\!\!\prec, \to\!\!\to)$ for the new arrangement in both cases, not just in case 2 ; in case 1, $(\prec\!\!\prec, \to\!\!\to)$ is the same as $(\prec, \to)$.

An "old" cut C , a cut of $(\prec, \to)$ is automatically a cut for $(\prec\!\!\prec, \to\!\!\to)$. Moreover, the border of C in the sense of $(\prec, \to)$ is contained in the border of C in the sense of $(\prec\!\!\prec, \to\!\!\to)$, as it is easy to see (see also section 7). Thus, if $[\![C \restriction (x, y)_{\to}]\!]$ is well-defined, so is $[\![C \restriction (x, y)_{\to\!\!\to}]\!]$. By 7.1 Lemma (b), the values $[\![C \restriction (x, y)_{\to}]\!]$ and $[\![C \restriction (x, y)_{\to\!\!\to}]\!]$ are in fact the same; we denote it $(s^x_y)^C$, unambiguously for the two arrangements.

It follows that clause (iv) for elements of T is inherited from T to $\hat{T}$. It remains to show clause (iv) for $\hat{t}$.

The construction gives $\Phi^{\hat{t}}$ as the molecule for which $\mathcal{S}_a(\Phi^t, \Phi^{\hat{t}})$.

Thus, in the first place, for $x \neq a, b$, $\varphi^{\hat{t}}[x]$ is the same as $\varphi^t[x]$, $(s^{-\infty}_x)^{\hat{t}}$ the same as $(s^{-\infty}_x)^t$, $(s^x_\infty)^{\hat{t}}$ the same as $(s^x_\infty)^t$; moreover, clearly, $\mathtt{if}\ x \neq a, b$, the cuts $\frac{x}{\ }\hat{t}$ and $\frac{\ }{x}\hat{t}$ ($x \in \mathbf{N}$) are the same as $\frac{x}{\ }t$ and $\frac{\ }{x}t$, respectively. Therefore, when $x \neq a, b$, the equalities in (iv) for $\hat{t}$ are inherited from t .

We abbreviate the cut ${}^{a}_{-}t = {}_{\overline{b}}t$ as D (this was done before), and ${}_{\overline{a}}\hat{t} = {}^{b}_{-}\hat{t}$ as $\hat{D}$. D is a cut for $(\prec,\rightarrow)$, hence for $(\prec\!\prec,\rightarrow\!\!\!\rightarrow)$ as well; $\hat{D}$ is a cut for $(\prec\!\prec,\rightarrow\!\!\!\rightarrow)$ only. D and $\hat{D}$ differ only in the positions of a and b, which are the reversed in $\hat{D}$ with respect to D.

It remains to show that

$$(S_a^{-\infty})^{\hat{t}} = (S_a^{-\infty})^{\hat{D}}, \qquad (S_\infty^{a})^{\hat{t}} = (S_\infty^{a})^{\hat{D}}, \tag{8.1?}$$

$$(S_b^{-\infty})^{\hat{t}} = (S_b^{-\infty})^{\hat{D}}, \qquad (S_\infty^{b})^{\hat{t}} = (S_\infty^{b})^{\hat{D}}. \tag{8.2?}$$

Let's abbreviate $\rho{=}\varphi^{t}[a]$, $\sigma{=}\varphi^{t}[b]$, $\varphi{=}\varphi^{\hat{t}}[b]$, $\psi{=}\varphi^{\hat{t}}[a]$.

To connect up with 4.2(i) and (ii), we use (iv) and write (b in 4.2 was ch*an*ged to $\tilde{b}$ to avoid the clash with the present b)

$$e \overset{\text{def}}{=} (S_\infty^{a})^{t} = (S_\infty^{a})^{D}, \qquad \tilde{b} \overset{\text{def}}{=} (S_a^{-\infty})^{t} = (S_a^{-\infty})^{D}, \tag{9.1}$$

$$\hat{e} \overset{\text{def}}{=} (S_\infty^{b})^{t} = (S_\infty^{b})^{D}, \qquad \hat{b} \overset{\text{def}}{=} (S_b^{-\infty})^{t} = (S_b^{-\infty})^{D}. \tag{9.2}$$

$$S \overset{\text{def}}{=} (S_b^{a})^{D}. \tag{9.3}$$

The choice of the notation makes the following hold:

$$\rho = \tilde{b}\cdot u\cdot e, \qquad \sigma = \hat{b}.v\cdot\hat{e}. \tag{10}$$

By (6),

$$e = S\cdot \mathrm{d}b\cdot\hat{e}, \qquad \hat{b} = \tilde{b}\cdot \mathrm{c}a\cdot S. \tag{11}$$

As we said, we have $\mathcal{S}_a(\Phi^{t}, \Phi^{\hat{t}})$, thus, in particular, $L(\rho, \sigma, \varphi, \psi)$. By 4.3(ii), (10), (11),

$$\varphi = \tilde{b}\cdot \mathrm{d}a\cdot S\cdot b\cdot\hat{e}, \qquad \psi = \tilde{b}\cdot a\cdot S\cdot \mathrm{c}b\cdot\hat{e}$$

This means that

$$(S_a^{-\infty})^{\hat{t}} {=} \tilde{b}, \ (S_\infty^{a})^{\hat{t}} {=} S\cdot \mathrm{c}b\cdot\hat{e}, \ (S_b^{-\infty})^{\hat{t}} {=} b\cdot \mathrm{d}a\cdot S, \ (S_\infty^{b})^{\hat{t}} {=} \hat{e}. \tag{12}$$

Since $\hat{D}$ and D agree on the interval $(-\infty, a)_{\rightarrow} = (-\infty, a)_{\rightarrow\!\!\!\rightarrow}$, we have

$(S_a^{-\infty})^{\hat{D}}=(S_a^{-\infty})^D=b$. By (12), we have the first of the four desired equalities in (8.1?).

Since $a \rightarrow\!\!\rightarrow b$, b is in $U[\hat{D}]$, and b is on the border of $\hat{D}$, by compositionality, $(S_\infty^a)^{\hat{D}}=(S_v^a)^{\hat{D}} \cdot \mathrm{c}b \cdot (S_\infty^b)^{\hat{D}}$. On $(a, b)_{\rightarrow\!\!\rightarrow}$, C and D agree; $(S_b^a)^{\hat{D}}=(S_b^a)^D=S$. Also, C and D agree on $(b, \infty)_{\rightarrow}$ (since $a \notin (b, \infty)_{\rightarrow}$), thus $(S_\infty^b)^{\hat{D}}=(S_\infty^b)^D=\hat{e}$. We have assembled all that is needed to conclude, also using (12), that $(S_\infty^a)^D=S \cdot cb \cdot \hat{e}=(S_\infty^a)^{\hat{t}}$, the second of the equalities in (8.1?).

The rest of the questioned equalities are dealt with "symmetrically".

§6 Continuation of the proof of 5.1 Theorem

We continue the following notational conventions: $C, \tilde{C}, D$ denote cuts; $C=(U, L)$, $\tilde{C}=(\tilde{U}, \tilde{L})$, $D=(V, M)$; $B, \tilde{B}, E$ are the respective borders of $C, \tilde{C}, D$.

In this, and the next section, we have a more restricted context than in section 5.

We assume a planar pasting prescheme $(\mathbf{N}, \prec, \rightarrow, \mathbf{M}, \mathbf{P}, \vec{S})$ as in section 4. In addition, we assume to have the following data:

a pair of elements a, b of $\mathbf{N}$ such that $a\prec!b$;

a particular cut D of $(\prec, \rightarrow)$ such that $a\in\underline{E}$, $b\in\overline{E}$;

a particular 1-pd S_b^a;

and decompositions

$$(S_b^{-\infty})^D = (S_a^{-\infty})^D \cdot \mathrm{c}a \cdot S_b^a \ , \tag{0.1}$$
$$(S_\infty^a)^D = S_b^a \cdot \mathrm{d}b \cdot (S_\infty^b)^D \tag{0.2}$$

((0.1) and (0.2) are equivalent, on the basis of the rest of the conditions; see also below.)

(In Case 2 of the induction step of the proof of 5.1 Theorem these data and conditions were present; note, however, that we do not need the 2-pd, the tree and the molecules now.)

The issue is adding the pair (a, b) to $\rightarrow$, and the datum S_b^a to the given system, to get a new planar arrangement $(\prec\prec, \rightarrow\!\!\!\rightarrow)$ and a new compositional system on it. The new arrangement $(\prec\prec, \rightarrow\!\!\!\rightarrow)$ was defined in 3.5.

6.1 Lemma Suppose that $C=(U, L)$ is an (arbitrary) cut for $(\prec, \rightarrow)$ with border B such that $a\in B$, $b\in B$. (Since $a\prec b$, obviously we must have $a\in\underline{B}$, $b\in\bar{B}$). We have

$$(S_\infty^a)^C = S_b^a \cdot \mathrm{d}b \cdot (S_\infty^b)^C \tag{1.1}$$

and, equivalently,

$$(S_b^{-\infty})^C = (S_a^{-\infty})^C \cdot \mathrm{c}a \cdot S_b^a \ . \tag{1.2}$$

and, as a consequence,

$$(S_\infty^{-\infty})^C = (S_a^{-\infty})^C \cdot \mathrm{c}a \cdot S_b^a \cdot \mathrm{d}b \cdot (S_\infty^b)^C \tag{1.3}$$

Proof Note that (1.1) is *not* a special case of the decomposition equation 5.(4), since

s_b^a is not a basic 1-pd for the given arrangement $(\rightarrow, \prec)$.

First, let us show that the two assertions (1.1), (1.2) are indeed equivalent.

Assume (1.1), to show (1.2) (the converse is obviously symmetric). We have

$$(s_\infty^{-\infty})^C = (s_a^{-\infty})^C \cdot \mathrm{c}a \cdot (s_\infty^a)^C = (s_b^{-\infty})^C \cdot \mathrm{d}b \cdot (s_\infty^b)^C \ .$$

Substituting for $(s_\infty^a)^C$ from (1.1):

$$(s_a^{-\infty})^C \cdot \mathrm{c}a \cdot s_b^a \cdot \mathrm{d}b \cdot (s_\infty^b)^C = (s_b^{-\infty})^C \cdot \mathrm{d}b \cdot (s_\infty^b)^C \ .$$

From this, by cancellation (4.0), we obtain (1.2).

The assertion (1.1) will be proved by induction on the distance $\rho(C, D) \overset{\mathrm{DEF}}{=} \#((U \cap M) \dot{\cup} (L \cap V))$ between C and D , When $\rho(C, D)=0$, $C=D$, and (1.1) is (0.1).

Suppose $\rho(C, D)>0$; i.e, $U \cap M$ is non-empty, or $L \cap V$ is non-empty. Assume the first alternative; the treatment of the second is a dual affair.

In section 3, we saw that there is $u \in \underline{B} \cap M$. As in section 3, we consider the cut $\tilde{C}$ which is the u-shift of C . We have that $\rho(\tilde{C}, D)=\rho(C, D)-1$.

Since $a \in U \cap V$, $b \in L \cap M$, we have $u \neq a$, $u \neq b$.

We have that $u \in \overline{\tilde{B}}$ by what we know about "shifting". Furthermore, $a \in \underline{\tilde{B}}$ and $b \in \overline{\tilde{B}}$. The first fact is clear, since $\tilde{U}$ has become smaller than U , but a stayed in $\tilde{U}$. For the second fact: since $u \in M$, and $b \in \bar{E}$, $u \prec b$ is not possible. For $v \in L$, $v \prec b$ is not possible since $b \in \bar{B}$. Thus, for $v \in \tilde{L} = L \cup \{u\}$, $v \prec b$ is not possible; which means that $b \in \overline{\tilde{B}}$.

From the facts that a and b are on the borders of both cuts C and $\tilde{C}$, it follows that for any u that positioned differently for C and $\tilde{C}$ (that is, u is in the set $U \cap \tilde{L} \cup L \cap \tilde{U}$) must satisfy

$$(a \rightarrow u \ \& \ b \rightarrow u) \vee (a \leftarrow u \ \& \ b \leftarrow u) \ . \qquad (2)$$

This is because, firstly, $a \prec u$ would force $u \in U \cap \tilde{U}$, and $a \succ u$ would force $u \in L \cap \tilde{L}$; therefore, we must have $a \longleftrightarrow u$ and $b \longleftrightarrow u$. Secondly, a pair of opposite relations such as $a \rightarrow u$ and $u \rightarrow b$ is clearly impossible.

We see that the induction hypothesis applies to $\tilde{\hat{C}}$; we have

$$(S^a_\infty)^{\tilde{\hat{C}}} = S^a_b \cdot \mathrm{d}b \cdot (S^b_\infty)^{\tilde{\hat{C}}} \quad . \tag{3}$$

Assume the first alternative in (2) :

$$a \to u \ \& \ b \to u \quad ;$$

of course, the other alternative is treated similarly (although the "dual" form (1.2) would be the one to directly tackle).

Remembering that $u \in \underline{B}$ and $u \in \overline{\tilde{\hat{B}}}$, by *decomposition*, we have:

$$(S^a_\infty)^C = (S^a_u)^C \cdot \mathrm{c}u \cdot (S^u_\infty)^C \quad , \tag{3.1}$$

$$(S^b_\infty)^C = (S^b_u)^C \cdot \mathrm{c}u \cdot (S^u_\infty)^C \quad . \tag{3.2}$$

and

$$(S^a_\infty)^{\tilde{\hat{C}}} = (S^a_u)^{\tilde{\hat{C}}} \cdot \mathrm{d}u \cdot (S^u_\infty)^{\tilde{\hat{C}}} \quad , \tag{3.3}$$

$$(S^b_\infty)^{\tilde{\hat{C}}} = (S^b_u)^{\tilde{\hat{C}}} \cdot \mathrm{d}u \cdot (S^u_\infty)^{\tilde{\hat{C}}} \quad . \tag{3.4}$$

Substituting (3.3) and (3.4) into (3), we get

$$(S^a_u)^{\tilde{\hat{C}}} \cdot \mathrm{d}u \cdot (S^u_\infty)^{\tilde{\hat{C}}} = S^a_b \cdot \mathrm{d}b \cdot (S^b_u)^{\tilde{\hat{C}}} \cdot \mathrm{d}u \cdot (S^u_\infty)^{\tilde{\hat{C}}} \quad .$$

By cancellation,

$$(S^a_u)^{\tilde{\hat{C}}} \ = \ S^a_b \cdot \mathrm{d}b \cdot (S^b_u)^{\tilde{\hat{C}}} \quad .$$

Clearly, the cuts C , $\tilde{\hat{C}}$ restrict to the same cuts in either of the intervals $(a, u)_{\rightarrow}$, $(b, u)_{\rightarrow}$; hence, $(S^a_u)^{\tilde{\hat{C}}} = (S^a_u)^C$, $(S^b_u)^{\tilde{\hat{C}}} = (S^b_u)^C$. Therefore

$$(S^a_u)^C \ = \ S^a_b \cdot \mathrm{d}b \cdot (S^b_u)^C \quad .$$

Multiply by the factor $\mathrm{c}u \cdot (S^u_\infty)^{\tilde{\hat{C}}}$ ($\mathrm{d}u$ has been switched to $\mathrm{c}u$)

$$(S^a_u)^C \cdot \mathrm{c}u \cdot (S^u_\infty)^C \ = \ S^a_b \cdot \mathrm{d}b \cdot (S^b_u)^C \cdot \mathrm{c}u \cdot (S^u_\infty)^C \quad .$$

By (3.1), (3.2), this means that

$$(S^a_\infty)^C = S^a_b \cdot \mathrm{d}b \cdot (S^b_\infty)^C \quad .$$

This completes the proof of Lemma 6.1.

Let u and v be arbitrary elements of $\mathbf{N}$, and assume that $u \underset{=}{\longrightarrow} v$. Consider the cut $C_{u,\,v}=(U,\,L)$ in the (open) interval $(u,\,v)_{\rightarrow}$ for which $L=\emptyset$, $U=(u,\,v)_{\rightarrow}$. It's border is $B=\mu(u,\,v)_{\rightarrow}$, the set of $\prec$-maximal (lowest) elements of $(u,\,v)_{\rightarrow}$. It has a unique signed span, the set of whose elements is B itself, all signed "down". Let us write $\underline{S}^u_v$ for the value $[\![C_{u,\,v}]\!]$; that is, $\underline{S}^u_v \; = \; (S^u_v)^{C_{u,\,v}}$.

Entirely analogously, we define $\bar{S}^u_v \; = \; (S^u_v)^{C^{u,\,v}}$; $C^{u,\,v}=(\emptyset,\,(u,\,v)_{\rightarrow})$.

Assume now, in addition to what we have on a, b, D and S^a_b, that we have elements $x,\,y \in \mathbf{N}$ such that

$$x \rightarrow !y\,, \quad x \underset{=}{\longrightarrow} a \;\text{ and }\; b \underset{=}{\longrightarrow} y\,. \tag{4}$$

The pairs $(x,\,y)$ satisfying these conditions are the ones for which we have a basic 2-pd S^x_y in the arrangement $(\prec,\,\rightarrow)$, but which no longer appear as pairs $x \rightarrow\!\!\!\rightarrow !y$. Therefore, S^x_y is no longer a basic 1-pd for $(\prec\!\prec,\,\rightarrow\!\!\!\rightarrow)$, and it must be expressed in terms of the new set of basic 1-pd's. Lemma 6.2 below does this job.

Note that this is the context which the group of lemmas starting with 3.9.1 in section 3 applies to. In particular, by 3.9.1(i), we have that $x \prec b$ and $a \prec y$, and since a and b are on the border E of D, we have $x \in V$ and $y \in M$.

6.2 Lemma Assuming (4), we have

$$S^x_y = \underline{S}^x_a \cdot ca \cdot S^a_b \cdot db \cdot \bar{S}^b_y \quad .$$

Proof Assume $u,\,v \in \mathbf{N}$, $u \preceq v$. Let us define $\delta_{\prec}(u,\,v)$ as the *maximal* integer n for which there is a $\prec$-chain $u=u_0 \prec u_1 \prec ... \prec u_n = v$ connecting u and v. (When $\neg(u \preceq v)$, $\delta_{\prec}(u,\,v) = \infty$.) Note that $\delta_{\prec}(u,\,v)=0$ iff $u=v$, $\delta_{\prec}(u,\,v)=1$ iff $u \rightarrow !v$, and $u \rightarrow v \underset{=}{\longrightarrow} w$ implies that $\delta_{\prec}(u,\,w) < \delta_{\prec}(v,\,w)$.

The proof of the lemma is by induction on

$$\delta_{\prec}(x, b) + \delta_{\prec}(a, y) = \delta_{x, y} \ .$$

More precisely, the induction statement $P(n)$ is: *for every pair* (x, y) such that (4) holds and $\delta_{x, y}=n$, we have the identity in the lemma.

The more detailed plan is as follows:

Basis: $\delta_{x, y} = 2$ (the minimal value!); that is,

$$\delta_{\prec}(x, b)=1 \text{ and } \delta_{\prec}(a, y)=1$$

Within the Basis, we distinguish the cases:

$\succ_x \subseteq M \ \& \ \prec_y \subseteq V$	**Case B1.1**
$\succ_x \subseteq M \ \& \ \neg(\prec_y \subseteq V)$	**Case B1.2**
$\neg(\succ_x \subseteq M) \ \& \ \prec_y \subseteq V$	**Case B2.1**
$\neg(\succ_x \subseteq M) \ \& \ \neg(\prec_y \subseteq V)$	**Case B2.2**

(I have used $\succ_x \overset{\text{DEF}}{=} \{z : z \succ x\}$, etc). The cases B1.2 and B2.1 are dual to each other; it suffices to look at one of them only.

Induction step: $\delta_{x, y} > 2$.

Case I1: $\delta_{\prec}(a, y) > 1$

Case I2: $\delta_{\prec}(x, b) > 1$.

Of course, cases I1 and I2 are dual to each other.

When we deal with case I1, we will make a reduction of the pair (x, y) to a pair (u, z) such that

$$\delta_{\prec}(a, z) < \delta_{\prec}(a, y) \ .$$

and

$$\delta_{\prec}(u, b) \le \delta_{\prec}(x, b) \ ;$$

in particular, $\delta_{u, z} < \delta_{x, y}$.

(**end of plan**)

Case B1.1:

This hypothesis means (since $x \in V$, $y \in M$) that x and y are on the border of D: $x \in \underline{E}$, $y \in \bar{E}$. Remember that $a \in \underline{E}$, $b \in \bar{E}$.

We have, by decomposition,

$$(S^x_\infty)^D = (S^x_a)^D \cdot \mathrm{c}a \cdot (S^a_\infty)^D = S^x_y \cdot \mathrm{d}y \cdot (S^y_\infty)^D ; \tag{5}$$

by $x \to !y$, we have $(S^x_y)^D = S^x_y$, the latter a basic 1-pd for the given prescheme.

By (0.2) and by decomposition of $(S^b_\infty)^D$, we get

$$(S^a_\infty)^D = S^a_b \cdot \mathrm{d}b \cdot (S^b_y)^D \cdot \mathrm{d}y \cdot (S^y_\infty)^D . \tag{6}$$

We substitute the value of $(S^a_\infty)^D$ in (6) into (5):

$$(S^x_a)^D \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot (S^b_y)^D \cdot \mathrm{d}y \cdot (S^y_\infty)^D = S^x_y \cdot \mathrm{d}y \cdot (S^y_\infty)^D .$$

By cancellation,

$$(S^x_a)^D \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot (S^b_y)^D = S^x_y .$$

Finally, we note that the cut D, when restricted to $(x, a)_\to$, has its upper set equal to the whole of $(x, a)_\to$ (and its lower set is empty). This is because, by 3.9.1, for every u in $(x, a)_\to$, one has $u \prec b$. Therefore, $(S^x_a)^D$ is the same as what we wrote as $\underline{S}^x_a$. Similarly for $(S^b_y)^D$.

This completes the proof in Case B1.1.

We skip to

Case B2.2:

Let

$$U = (V \cup \prec_y) - \succ_x$$
$$L = (M \cup \succ_x) - \prec_y .$$

One notes that $\prec_y \cap \succ_x = \emptyset$, and as a consequence, $C \overset{\text{def}}{=} (U, L)$ is a cut for $\prec$. By the case hypothesis, we have some $v \in \succ_x \cap V$, and $w \in \prec_y \cap M$. Clearly, we may, and do, choose v and w "extremally" so that, in addition, also $w \prec! y$ and $x \prec! v$.

6.2.1 Sublemma [B denotes the border of C] We have

$$x \in \underline{B}\,,\ v \in \overline{B}\,,\ a \in \underline{B}\,,\ b \in \overline{B}\,,\ w \in \underline{B} \text{ and } y \in \overline{B}\ .$$

Proof of 6.2.1 Assume $a \prec s$ to conclude $s \in L$. $s \in M$ holds since $a \in E$. $s \prec y$ would entail $a \prec s \prec y$, contradicting $a \prec! y$ (Basis assumption). Also, $a \in V$, and $a \notin \succ_x$; thus $a \in U$. This shows $a \in \underline{B}$.

If $x \prec s$, then $s \in L$ by $\succ_x \subseteq L$. $x \in V$ and $x \notin \succ_x$; thus $x \in U$. This shows $x \in \underline{B}$.

Since $v \in \succ_x$ by choice, and $\succ_x \cap \prec_y = \emptyset$, we have $v \in L$. Assume $s \prec v$. Since $x \prec! v$, we must have $s \notin \succ_x$. But since $v \in V$, also $s \in V$, $s \notin M$. This says that $s \notin L$, $s \in U$. This shows that $v \in \overline{B}$.

The other three similar claims are dual statements. (**End of proof of 6.2.1**)

By 3.9.9, we have $a \to w$, $b \to w$. Dually, also $v \to a$, $v \to b$.

6.2.1 is used to ensure that the border of the cut C contains various elements so that decomposition can be applied at those elements as dividing points.

The conditions for Lemma 6.1 are fulfilled to conclude (1.3), and as a result, that

$$(s^v_w)^C = (s^v_a)^C \cdot \mathrm{c}a \cdot s^a_b \cdot \mathrm{d}b \cdot (s^b_w)^C\ .$$

On the other hand, by decomposition,

$$(s^{-\infty}_{\infty})^C = (s^{-\infty}_x)^C \cdot \mathrm{c}x \cdot s^x_y \cdot \mathrm{d}y \cdot (s^y_{\infty})^C = (s^{-\infty}_v)^C \cdot \mathrm{d}v \cdot (s^v_w)^C \cdot \mathrm{c}w \cdot (s^w_{\infty})^C\ ;$$

thus, by substitution,

$$(s^{-\infty}_x)^C \cdot \mathrm{c}x \cdot s^x_y \cdot \mathrm{d}y \cdot (s^y_{\infty})^C = (s^{-\infty}_v)^C \cdot \mathrm{d}v \cdot (s^v_a)^C \cdot \mathrm{c}a \cdot s^a_b \cdot \mathrm{d}b \cdot (s^b_w)^C \cdot \mathrm{c}w \cdot (s^w_{\infty})^C\ .$$

By decompositions of $C \restriction (-\infty, a)$ and of $C \restriction (b, \infty)$:

$$(s^{-\infty}_a)^C = (s^{-\infty}_v)^C \cdot \mathrm{d}v \cdot (s^v_a)^C = (s^{-\infty}_x)^C \cdot \mathrm{c}x \cdot (s^x_a)^C\ ,$$

$$(s^b_{\infty})^C = (s^b_w)^C \cdot \mathrm{c}w \cdot (s^w_{\infty})^C = (s^b_y)^C \cdot \mathrm{d}y \cdot (s^y_{\infty})^C$$

and thus,

$$(S^{-\infty}_x)^C \cdot \mathrm{c}x \cdot S^x_y \cdot \mathrm{d}y \cdot (S^y_\infty)^C = (S^{-\infty}_x)^C \cdot \mathrm{c}x \cdot (S^x_a)^C \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot (S^b_y)^C \cdot \mathrm{d}y \cdot (S^y_\infty)^C .$$

By cancellation on both sides, we obtain

$$S^x_y = (S^x_a)^C \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot (S^b_y)^C .$$

We have seen before that $(x, a)_{\rightarrow} \subseteq V$. It is obvious that $(x, a)_{\rightarrow} \cap \succ_x = \emptyset$. Therefore, $(x, a)_{\rightarrow} \subseteq U$. Thus, $(S^x_a)^C = \underline{S}^x_a$. Dually, $(S^b_y)^C = \bar{S}^b_y$. We conclude the desired equality

$$S^x_y = \underline{S}^x_a \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot \bar{S}^b_y .$$

This concludes case B2.2.

The (similar) cases B2.1 and B1.2 are left to the reader.

Case I1: We now assume $\delta_{\prec}(a, y) > 1$. It follows that the set $Z_! \overset{\text{def}}{=} \{z : a \prec z \prec ! y\}$ is non-empty; let us choose and fix a $\rightarrow$-minimal element z of $Z_!$. Note that $\delta_{\prec}(a, z) < \delta_{\prec}(a, y)$.

By 3.9.2, we have $x \rightarrow z$ and $b \rightarrow z$. Thus, in particular, for $W'_z \overset{\text{def}}{=} \{u : x \underset{=}{\rightarrow} u \rightarrow z\}$, and $W''_z \overset{\text{def}}{=} W'_z \cap \mu[x, a)_{\rightarrow}$, we have $x \in W''_z$, and W''_z is non-empty. ($\mu[x, a)_{\rightarrow}$ is the set of $\prec$-maximal elements of the set $[x, a)_{\rightarrow}$, and x is clearly $\prec$-maximal in $[x, a)_{\rightarrow}$.) Besides, W''_z is linearly ordered by $\rightarrow$. Let us choose u to be *the* $\rightarrow$-maximal element of W''_z. By 3.9.8, we have $u \rightarrow ! z$.

Let us summarize. We have x and y such that $x \rightarrow ! y$, $x \underset{=}{\rightarrow} a$ and $b \underset{=}{\rightarrow} y$. In addition, we have u and z such that $u \in \mu[x, a)_{\rightarrow}$, $u \rightarrow ! z$, $u \prec b$ (by 3.9.1), $b \rightarrow z$, $a \prec z \prec ! y$, and z is an $\rightarrow$-minimal element $z \in Z_!$. By 3.9.7, we have $(b, z)_{\rightarrow} \subseteq (b, y)_{\rightarrow}$.

Let $U \overset{\text{def}}{=} V \cup \prec_y$. With $L \overset{\text{def}}{=} M - \prec_y$, $C = (U, L)$ is a $\prec$-cut.

6.2.2 Sublemma We have

$$x\in\underline{B}\ ,\ \ u\in\underline{B}\ ,\ \ b\in\bar{B}\ ,\ \ z\in\underline{B}\ \text{ and }\ y\in\bar{B}\ .$$

Proof of 6.2.2 $x\in V$, hence $x\in U$. Assume $x\prec s$. Then $s\in M$, since $x\in E$. $x\prec s\prec y$ contradicts $x\rightarrow y$; $s\notin\prec_y$; $s\in L$. This shows $x\in\underline{B}$.

Next, we show that $u\in\underline{B}$.

Since $b\in E$, and $u\prec b$, we have $u\in V$, and $u\in U$. Assume $u\prec s$, to show that $s\in L$.

$s\underset{=}{\rightarrow}x$ is impossible since it would imply $s\underset{=}{\rightarrow}x\underset{=}{\rightarrow}u$, * to $u\prec s$. Likewise, $s\prec x$ is excluded: $u\prec s\prec x$, * to $x\underset{=}{\rightarrow}u$. Thus, either $x\prec s$ (case 1), or $x\rightarrow s$ (case 2). In case 1, $x\in B$ implies $s\in L$ as desired.

Assume case 2. Let's compare a and s. $s\rightarrow a$ is impossible: we would have $s\in[x,a)$, with $u\prec s$, contradicting $u\in\mu[x,a)_{\rightarrow}$. $a\underset{=}{\rightarrow}s$ would say $u\rightarrow a\underset{=}{\rightarrow}s$, * to $u\prec s$. $s\prec a$ would entail $u\prec s\prec a$, * to $u\rightarrow a$. We have proved $a\prec s$.

It follows that $s\in M$ (since $a\in E$). If $\neg(s\prec y)$, then $s\in M\text{-}\prec_y=L$; therefore, we may assume that $s\prec y$, with the intention to derive a contradiction. We do so by making a comparison of s and z.

Since $a\prec s\prec y$, we have $s\in Z$ (see 3.(13.1)). Since z is a $\rightarrow$-minimal element of Z (3.9.4), we conclude $\neg(s\rightarrow z)$. $s\neq z$ since $u\prec s$ and $u\rightarrow z$. ? $s\prec z$: $u\prec s\prec z$: * to $u\rightarrow z$. ? $z\prec s$: $z\prec s\prec y$: * to $z\prec!y$. ? $z\rightarrow s$: $u\rightarrow z\rightarrow s$: * to $u\prec s$. Contradiction!

This completes the proof that $u\in\underline{B}$.

b is in $\bar{E}$, and $b\notin\prec_y$. It follows that $b\in\bar{B}$.

Since $z\in\prec_y$, we have $z\in U$. If $z\prec s$, then $a\prec z\prec s$, thus $s\in L$; but $s\notin\prec_y$, since otherwise $z\prec s\prec y$, * to $z\prec!y$. We have proved that $s\in L$. This shows that $z\in\underline{B}$.

The definition of U as $V\cup\prec_y$, together with $y\in M$, directly shows that $y\in\bar{B}$. (**End of proof of 6.2.2**)

6.2.3 Sublemma We have

$$\delta_{\prec}(u,b)\leq\delta_{\prec}(x,b) \qquad (7)$$

Proof of 6.2.3 If $u\prec!b$, since $x\prec b$, (7) holds. Otherwise, there is u_1, $u\prec!u_1\prec b$, with $\delta_{\prec}(u,b)=\delta_{\prec}(u_1,b)+1$. We have $u_1\rightarrow a$, because

? $\quad u_1 \preceq a$: $\quad u \prec u_1 \preceq a$: * to $u \rightarrow a$;
? $\quad a \rightarrow u_1$: $\quad u \rightarrow a \rightarrow u_1$: * to $u \prec u_1$;
? $\quad a \prec u_1$: $\quad a \prec u_1 \prec b$: * to $a \prec ! b$.

We have $x \prec u_1$, because
? $\quad x \rightarrow u_1$: $\quad x \rightarrow u_1 \rightarrow a$ (!) & $u \prec u_1$: * to $u \in \mu[x, a)$;
? $\quad u_1 \preceq x$: $\quad u \prec u_1 \preceq x$: * to $x \underset{=}{\rightarrow} u$;
? $\quad u_1 \rightarrow x$: $\quad u_1 \rightarrow x \underset{=}{\rightarrow} u$: * to $u \prec u_1$.

$x \prec u_1 \prec b$ says that $\delta_{\prec}(x, b) \geq \delta_{\prec}(u_1, b)+1=\delta_{\prec}(u, b)$. (**End of proof of 6.2.3**).

Since $\delta_{\prec}(a, z) < \delta_{\prec}(a, y)$ and $\delta_{\prec}(u, b) \leq \delta_{\prec}(x, b)$ (6.2.3), the induction hypothesis can be applied to the pair (u, z) in place of the pair (x, y) . Accordingly, we have

$$S^u_z = \underline{S}^u_a \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot \bar{S}^b_z \ . \tag{8.1}$$

Applying decomposition to $C \restriction (x, \infty)_{\rightarrow}$ in two ways,

$$(S^x_\infty)^C = S^x_y \cdot \mathrm{d}y \cdot (S^y_\infty)^C = (S^x_u)^C \cdot \mathrm{c}u \cdot S^u_z \cdot \mathrm{c}z \cdot (S^z_\infty)^C \ . \tag{8.2}$$

Substituting (8.1) into (8.2):

$$S^x_y \cdot \mathrm{d}y \cdot (S^y_\infty)^C = (S^x_u)^C \cdot \mathrm{c}u \cdot \underline{S}^u_a \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot \bar{S}^b_z \cdot \mathrm{c}z \cdot (S^z_\infty)^C \ . \tag{8.3}$$

Looking at $C_{x,\, a}$ and noting that $C_{x,\, a} \restriction (x, u)_{\rightarrow} = C_{x,\, u}$, $C_{x,\, a} \restriction (u, a)_{\rightarrow} = C_{u,\, a}$:

$$\underline{S}^x_a = \underline{S}^x_u \cdot \mathrm{c}u \cdot \underline{S}^u_a \ .$$

$(x, u)_{\rightarrow} \subseteq (x, a)_{\rightarrow} \subseteq \prec_b$ by 3.9.1; hence, by $b \in E$, $(x, u)_{\rightarrow} \subseteq V \subseteq U$. Therefore, we have $C_{x,\, u} = C \restriction (x, u)_{\rightarrow}$, and so $\underline{S}^x_u = (S^x_u)^C$;

$$\underline{S}^x_a = (S^x_u)^C \cdot \mathrm{c}u \cdot \underline{S}^u_a \tag{8.4}$$

Look at $C \restriction (b, \infty)_{\rightarrow}$:

$$(S^b_\infty)^C = (S^b_y)^C \cdot \mathrm{d}y \cdot (S^y_\infty)^C = (S^b_z)^C \cdot \mathrm{c}z \cdot (S^z_\infty)^C \ .$$

$(b, y)_{\rightarrow} \subseteq \succ_a$ by 3.9.1; hence, by $a \in E$, $(b, y)_{\rightarrow} \subseteq M$, and then also $(b, y)_{\rightarrow} \subseteq L$.

Therefore, also $(b, z)_{\rightarrow} \subseteq L$ (see 3.9.7), and

$$(S^b_y)^C = \underline{S}^b_y \ , \quad (S^b_z)^C = \underline{S}^b_z \ ,$$

and

$$\underline{S}^b_y \cdot \mathrm{d}y \cdot (S^y_\infty)^C = \overline{S}^b_z \cdot \mathrm{c}z \cdot (S^z_\infty)^C \tag{8.5}$$

Comparing (8.3), (8.4), (8.5):

$$S^x_y \cdot \mathrm{d}y \cdot (S^y_\infty)^C = \underline{S}^x_a \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot \underline{S}^b_y \cdot \mathrm{d}y \cdot (S^y_\infty)^C \ .$$

Canceling:

$$S^x_y = \underline{S}^x_a \cdot \mathrm{c}a \cdot S^a_b \cdot \mathrm{d}b \cdot \underline{S}^b_y$$

as desired.

6.2 Lemma is proved.

§7 Completion of the proof of 5.1 Theorem

7.1 Lemma **a)** Suppose τ and θ are signed $(\prec\prec, \twoheadrightarrow)$-spans, $(\prec\prec, \twoheadrightarrow)$-defining the same cut of $(\prec\prec, \twoheadrightarrow)$. Then

$$\textbf{(i)} \qquad (s_{\infty}^{-\infty})^{\tau}=(s_{\infty}^{-\infty})^{\theta} \; ; \tag{1}$$

and more generally,

(ii) if $u\twoheadrightarrow v$, and both u and v are in both τ and θ (including the possibilities that $u=-\infty$ or $v=\infty$), we have $(s_{v}^{u})^{\tau}=(s_{v}^{u})^{\theta}$.

b) Any cut C of $(\prec, \to)$ is a cut for $(\prec\prec, \twoheadrightarrow)$ (obviously); the value $(s_{v}^{u})_{(\prec, \to)}^{C}$ in the old sense relative to $(\prec, \to)$ equals to the value $(s_{v}^{u})_{(\prec\prec, \twoheadrightarrow)}^{C}$ in the new sense relative to $(\prec\prec, \twoheadrightarrow)$.

Proof Let's note that, for (a), (i) is sufficient to prove. Namely, assuming (i) proved, for the general case of (ii), with data as given, let us define the signed $(\prec\prec, \twoheadrightarrow)$-span ζ by the equalities

$$\zeta\restriction(-\infty, u] \overset{\text{def}}{=} \tau\restriction(-\infty, u] \; , \quad \zeta\restriction[v, \infty) \overset{\text{def}}{=} \tau\restriction[v, \infty) \; , \quad \zeta\restriction(u, v) \overset{\text{def}}{=} \theta\restriction(u, v) \; .$$

Then, first of all, ζ is $(\prec\prec, \twoheadrightarrow)$-equivalent to θ as well as to τ , by 3.4 applied to the arrangement $(\prec\prec, \twoheadrightarrow)$. Apply the present (i) to τ and ζ . Then, appropriately using (4.0) in section 5, and by cancelling on both sides, we get (ii).

Any $(\prec\prec, \twoheadrightarrow)$-span consists of a sequence $\langle x_i\rangle_{i=0,\,...,\,n}$ such that

$$-\infty=x_0\twoheadrightarrow!x_1\twoheadrightarrow!x_2\twoheadrightarrow!\;...\twoheadrightarrow!x_{n-1}\twoheadrightarrow!x_n=\infty$$

Recall (3.5') that

$$x\twoheadrightarrow!y \iff (x\to!y \;\&\; \neg(x\overset{*}{\twoheadrightarrow}y)) \;\dot{\vee}\; (x=a \;\&\; y=b) \; ;$$

thus, in particular

$$x\twoheadrightarrow!y \implies x\to!y \;\dot{\vee}\; (x=a \;\&\; y=b) \; .$$

In other words, a $(\prec\prec, \twoheadrightarrow)$-span consists of "links" $x\twoheadrightarrow!y$ that are either "links" of the form $x\to!y$ (satisfying a further condiiton), or else equal to the one proper $\twoheadrightarrow!$-link $a\twoheadrightarrow!b$.

Assume the link $a\twoheadrightarrow!b$ does not appear in the signed $(\prec\prec, \twoheadrightarrow)$-span τ . Then, obviously, τ is a signed $(\prec, \to)$-span as well. As such, it determines *some* $(\prec, \to)$-cut;

call it $C=(U, L)$. In fact,

$$U = \{u : \exists x \in \tau . u \prec x\} \cup \underline{\tau} \;;$$

$$L = \{u : \exists x \in \tau . x \prec v\} \cup \bar{\tau} \;.$$

Let the $(\prec\prec, \twoheadrightarrow)$-cut $(\prec\prec, \twoheadrightarrow)$-determined by τ be $C^{*}=(V, M)$. We have

$$V = \{u : \exists x \in \tau . u \prec\prec x\} \cup \underline{\tau} \;;$$

$$M = \{u : \exists x \in \tau . x \prec\prec v\} \cup \bar{\tau} \;.$$

Since $\prec\prec \subset \prec$, we clearly have that $V \subseteq U$, $M \subseteq L$. But also: $\mathbf{N} = V \dot{\cup} M = U \dot{\cup} L$. It follows that $V=U$, $M=L$; that is, $C^{*}=C$. We conclude that τ determines the same cut, a cut of $(\prec, \to)$, in both arrangements: $(\prec, \to)$ and $(\prec\prec, \twoheadrightarrow)$.

Let us turn to the proof of (1).

Case 1. Both τ and θ are "old spans": the link $a \twoheadrightarrow ! b$ do not appear in them.

Since, by assumption, τ and θ $(\prec\prec, \twoheadrightarrow)$-determine the same $(\prec\prec, \twoheadrightarrow)$-cut, by what we just said, τ and θ $(\prec, \to)$-determine the same $(\prec, \to)$-cut; i.e., they are equivalent in the sense of the arrangement $(\prec, \to)$. Therefore, (1) holds by the properties of the given prescheme on $(\prec, \to)$.

Case 2. Both τ and θ contain the link $a \twoheadrightarrow ! b$.

We have $(S_{\infty}^{-\infty})^{\tau}=(S_{a}^{-\infty})^{\tau} \cdot \partial^{\tau} a \cdot S_{b}^{a} \cdot \partial^{\tau} b \cdot (S_{\infty}^{b})^{\tau}$ and $(S_{\infty}^{-\infty})^{\theta}=(S_{a}^{-\infty})^{\theta} \cdot \partial^{\theta} a \cdot S_{b}^{a} \cdot \partial^{\theta} b \cdot (S_{\infty}^{b})^{\theta}$. Since τ and θ define the same $(\prec\prec, \twoheadrightarrow)$-cut in $(-\infty, \infty)_{\twoheadrightarrow}$, they define the same $(\prec\prec, \twoheadrightarrow)$-cut in $(-\infty . a)_{\twoheadrightarrow}$ and $(b, \infty)_{\twoheadrightarrow}$. But in $(-\infty . a)_{\twoheadrightarrow}=(-\infty, a)_{\to}$ and $(b, \infty)_{\twoheadrightarrow}=(b, \infty)_{\to}$, the arrangements $(\prec\prec, \twoheadrightarrow)$ and $(\prec, \to)$ coincide; therefore, τ and θ define the same $(\prec, \to)$-cuts in $(-\infty . a)_{\to}$ and $(b, \infty)_{\to}$. By the properties of the given prescheme on $(\prec, \to)$ therefore $(S_{a}^{-\infty})^{\tau}=(S_{a}^{-\infty})^{\theta}$ and $(S_{\infty}^{b})^{\tau}=(S_{\infty}^{b})^{\theta}$. Since τ and θ are equivalent for the particular elements a and b , $\partial^{\tau} a=\partial^{\theta} a$ and $\partial^{\tau} b=\partial^{\theta} b$. It follows that $(S_{\infty}^{-\infty})^{\tau}=(S_{\infty}^{-\infty})^{\theta}$.

Case 3. Say, τ does, and θ does not, contain the link $a \twoheadrightarrow ! b$.

As before, now θ both $(\prec\prec, \twoheadrightarrow)$- and $(\prec, \to)$-determines the same $(\prec, \to)$-cut $C=(U, L)$; hence, in particular

$$(S_{\infty}^{-\infty})^{\theta} = (S_{\infty}^{-\infty})^{C}_{(\prec, \rightarrow)} = (S_{\infty}^{-\infty})^{C} \ ; \tag{2}$$

and also, C is $(\prec\!\prec, \rightarrow\!\!\!\rightarrow)$-determined by τ .

From now on, we may forget about the span θ . Instead, we have the $(\prec, \rightarrow)$-cut C , and the signed $(\prec\!\prec, \rightarrow\!\!\!\rightarrow)$-span τ , $(\prec\!\prec, \rightarrow\!\!\!\rightarrow)$-defining C .

What we want is

$$?: \qquad (S_{\infty}^{-\infty})^{\tau} = (S_{\infty}^{-\infty})^{C} \ . \tag{2.1}$$

where $(S_{\infty}^{-\infty})^{C}$ is meant, of course, in the sense of the given prescheme on $(\prec, \rightarrow)$.

Note that both a and b are on τ . We use the notation $\partial^{C}u=\mathrm{c}u$ if $u\in U$, $\partial^{C}u=\mathrm{d}u$ if $u\in L$.

Note that this will prove part b) of 7.1.

We make a series of preparatory remarks.

Let B be the $(\prec, \rightarrow)$-border of C , E the $(\prec\!\prec, \rightarrow\!\!\!\rightarrow)$-border of C . I claim that $B\subseteq E$. This becomes clear from the expresssions

$$u\in B \iff (u\in U \ \& \ \forall x(u\prec x \Rightarrow x\in L)) \vee (u\in L \ \& \ \forall x(x\prec u \Rightarrow x\in U)) \ ,$$
$$u\in E \iff (u\in U \ \& \ \forall x(u\prec\!\prec x \Rightarrow x\in L)) \vee (u\in L \ \& \ \forall x(x\prec\!\prec u \Rightarrow x\in U)) \ ,$$

and the fact that $\prec\!\prec \subset \prec$.

By assumption, a and b belong to E . Do they belong to B too? Not necessarily! However, we can say this much: if $a\in\bar{E}$, then $a\in\bar{B}$, and if $b\in\underline{E}$, then $b\in\underline{B}$. Namely, if

$v\prec a$, then $v\prec\!\prec a$; if also $a\in\bar{E}$, then $v\in L$; which shows the first assertion; the second is similar.

Taking contrapositives, we have

$$b\notin\underline{B} \Longrightarrow b\in\bar{E} \ ,$$

$$a\notin\bar{B} \Longrightarrow a\in\underline{E} \ .$$

Accordingly, we distinguish the exhaustive, but not necessarily mutually exclusive, subcases:

Case 3.1 $b\in B$.

Case 3.2 $a\in B$.

Case 3.3 $a\in \underline{E} - B$ and $b\in \bar{E} - B$.

Case 3.1: Assume $b\in B$.

Let us note first that, since $b\in B$ and $a\prec b$, it immediately follows that $a\in U$. On the other hand, it will remain throughout undecided whether b is in U or L . Also, we do not know if $a\in B$ (probably, $a\notin B$). On the positive side: we have that $a, b\in E$.

Let us consider the restriction of the arrangement $(\prec\!\prec, \twoheadrightarrow)$ to the interval

$$[a, \infty)_{\twoheadrightarrow} = [a, \infty)_{\rightarrow} \cup [b, \infty)_{\rightarrow} \quad .$$

We have, in general

$$x\prec\!\prec y \iff x\prec y \;\&\; \neg(x\overset{*}{\twoheadrightarrow}y) \;\&\; \neg(y\overset{*}{\twoheadrightarrow}x) \quad .$$

For $x, y\in [a, \infty)_{\twoheadrightarrow}$, also assuming $x\prec y$, we have

$$x\overset{*}{\twoheadrightarrow}y \iff x\underset{=}{\rightarrow}a \;\&\; b\underset{=}{\rightarrow}y \iff x=a \;\&\; b\underset{=}{\rightarrow}y \;,$$

and

$$y\overset{*}{\twoheadrightarrow}x \iff y\underset{=}{\rightarrow}a \;\&\; b\underset{=}{\rightarrow}x \;:\; \textit{false} \text{ by } x\prec y \;.$$

Thus, for $x, y\in [a, \infty)_{\twoheadrightarrow}$:

$$x\prec\!\prec y \iff x\prec y \;\&\; \neg(x=a \;\&\; b\underset{=}{\rightarrow}y) \;,$$

in particular,

$$x\prec y \;\&\; x\neq a \implies x\prec\!\prec y \quad .$$

As a consequence,

$$[b, \infty) = [b, \infty)_{\twoheadrightarrow} = [b, \infty)_{\rightarrow} \;,$$

and the arrangements $(\prec, \rightarrow)$ and $(\prec\!\prec, \twoheadrightarrow)$ coincide on the interval $[b, \infty)$. In particular, the notions of "cut", "border of a cut", "signed span defining a cut" for $(\prec, \rightarrow)$ and $(\prec\!\prec, \twoheadrightarrow)$, when restricted to $[b, \infty)$, mean the same things.

Similarly, we can see that the relation $x\overset{*}{\twoheadrightarrow}y$ is identically false when restricted to the interval

$$(-\infty, a] = (-\infty, a]_{\twoheadrightarrow} = (-\infty, a]_{\rightarrow} \;;$$

as a consequence, we have that the two arrangements $(\prec, \rightarrow)$ and $(\prec\!\prec, \twoheadrightarrow)$, when restricted to $(-\infty, a]$, are the same.

Next, I'll define a new $(\prec, \rightarrow)$-cut $\tilde{\hat{C}}=(\tilde{\hat{U}}, \tilde{\hat{L}})$. The motivation is to modify C minimally so that a becomes a member of the $(\prec, \rightarrow)$-border $\tilde{\hat{B}}$ of $\tilde{\hat{C}}$; note that a may very well not belong to B. The definition of $\tilde{\hat{C}}$ is as follows.

$$\tilde{\hat{U}} = U - \{u \in U : a \prec u\} \;;$$

i.e.,

$$u \in \tilde{\hat{U}} \iff u \in U \;\&\; \neg(a \prec u) \;.$$

$\tilde{\hat{U}}$ is up-closed: assume $x \prec v \in \tilde{\hat{U}}$. Then $v \in U$ and $x \in U$. If we had $a \prec x$, then $a \prec x \prec v$, and $a \prec v$, contradicting $v \in \tilde{\hat{U}}$; thus, $x \in U \,\&\, \neg(a \prec x)$, and $x \in \tilde{\hat{U}}$.

I claim that

$$u \in U \;\&\; a \prec u \implies b \underset{=}{\rightarrow} u \;. \tag{2.2}$$

Because: we have $\neg(a \overset{*}{\twoheadleftarrow\!\!\twoheadrightarrow} u)$, that is,

$$\neg(a \underset{=}{\rightarrow} a \;\&\; b \underset{=}{\rightarrow} u) \;\&\; \neg(u \underset{=}{\rightarrow} a \;\&\; b \underset{\substack{=\\ \uparrow\\ \perp}}{\rightarrow} a)$$

unless $b \underset{=}{\rightarrow} u$. Otherwise, $a \prec u$ implies $a \prec\prec u$, which, together with $u \in U$, contradicts $a \in E$.

Note that $b \underset{=}{\rightarrow} u$ is incompatible with both $u \longrightarrow b$ and $u \underset{=}{\rightarrow} a$. We can conclude that the $(\prec\prec, \twoheadrightarrow)$-cuts $\tilde{\hat{C}}$ and C coincide when restricted to the $(\prec\prec, \twoheadrightarrow)$-interval $(-\infty, b)_{\twoheadrightarrow} = (-\infty, a]_{\rightarrow} \cup (-\infty, b)_{\rightarrow}$. In particular, $\tilde{\hat{E}}$ and E coincide on the same interval. Since $a \in E$, we have $a \in \tilde{\hat{E}}$.

7.1.1 Sublemma

$$e \in (-\infty, b]_{\twoheadrightarrow} \implies (e \in E \iff e \in \tilde{\hat{E}} \iff e \in \tilde{\hat{B}})$$

Proof of 7.1.1 Let's take the point of view of *trying* to show that $\tilde{\hat{B}} = \tilde{\hat{E}}$ (?).

Of course, $\tilde{\hat{B}} \subseteq \tilde{\hat{E}}$ by the same general argument as before for C. Assume

$$e \in \underline{\tilde{\hat{E}}} \,,\;\; e \prec u \,,\;\; u \in \tilde{\hat{U}} \,, \tag{3.1}$$

or

$$e\in\overline{\tilde{E}}\ ,\ \ u\prec e\ ,\ \ u\in\tilde{L}\ . \qquad (3.2)$$

to get (hopefully) a contradiction. $e\prec\prec u$ would be a contradiction in case (3.1); and $u\prec\prec e$ would be a contradiction in case (3.2). So, we assume that $e \leftarrow\overset{*}{\longleftrightarrow}\rightarrow u$, i.e.,

$$e\xrightarrow[=]{}a\ \&\ b\xrightarrow[=]{}u \qquad (4.1)$$

or

$$u\xrightarrow[=]{}a\ \&\ b\xrightarrow[=]{}e \qquad (4.2)$$

Assume (3.1).

The second alternative (4.2) is an impossible picture:

$$u\xrightarrow[=]{}a\ \&\ b\xrightarrow[=]{}e\ \&\ e\prec u\ \&\ a\prec!b\ ;$$

as we see by making all possible comparisons of a and e :

$?:a\prec e\ :\ a\prec e\prec u\ :$ * to $u\xrightarrow[=]{}a$;
$?:e\prec a\ :\ e\prec a\prec b\ :$ * to $b\xrightarrow[=]{}e$;
$?:a\rightarrow e\ :\ u\xrightarrow[=]{}a\rightarrow e\ :$ * to $e\prec u$;
$?:e\xrightarrow[=]{}a\ :\ b\xrightarrow[=]{}e\xrightarrow[=]{}a\ :$ * to $a\prec b$.

Let's look at the first alternative (4.1).

In this case, we must have that $a\prec u$. (Namely, $?:u\xrightarrow[=]{}a$ gives $b\xrightarrow[=]{}u\xrightarrow[=]{}a$, * to $a\prec b$; $?:a\longrightarrow u$ gives $e\xrightarrow[=]{}a\longrightarrow u$, * to $e\prec u$; $?:u\prec a$ gives $u\prec a\prec b$, * to $b\xrightarrow[=]{}u$.)

However, with $a\prec u$, we cannot have $u\in\tilde{U}$ ($=U-\{u:a\prec u\}$) as we do. We are done in case (3.1).

Assume, second, (3.2).

Now, (4.1) is an impossible picture:

$$e\xrightarrow[=]{}a\ \&\ b\xrightarrow[=]{}u\ \&\ u\prec e\ \&\ a\prec!b\ ,$$

exactly as before. Thus, we have to deal with (4.2).

Alas, however, we cannot exclude this. We are left with the following conclusion, weaker than what we tried to show first:

$$e\in\tilde{E}-\tilde{B}\ \Longrightarrow\ b\xrightarrow[=]{}e\ .$$

In turn, this implies that

$$e\in(-\infty,b)_{\to\!\to} \implies (e\in E \iff e\in\tilde{\hat{E}} \iff e\in\tilde{\hat{B}})$$

(here, we write $A\iff B\iff C$ for $(A\iff B)\&(B\iff C)$. The first "$\iff$" on the right was noted before).

In fact, we can extend this to include the element b :

$$e\in(-\infty,b]_{\to\!\to} \implies (e\in E \iff e\in\tilde{\hat{E}} \iff e\in\tilde{\hat{B}}) \tag{5}$$

since $b\in E$, $b\in\tilde{\hat{E}}$ and $b\in B$; only $b\in\tilde{\hat{E}}$ requires checking.

We have $b\in\tilde{\hat{L}}=L\cup X$, $X=\{u\in U:a\prec u\}$; each $x\in X$ satisfies, in particular, that $b\underset{=}{\longrightarrow}x$ (see (2.2)); b itself belongs to X . Assume $v\prec b$, to conclude $v\in\tilde{\hat{U}}$. Since $b\in B$, $v\in U$. But $v\notin X$ since $v\prec b$ and $b\underset{=}{\longrightarrow}v$ are incompatible. Thus, $v\in U-X=\tilde{\hat{U}}$ as desired.

(End of proof 7.1.1)

Since $b\in B$ and $a\prec b$, we have $a\in U$, and also $a\in\tilde{\hat{U}}$.

By definition, we have that

$$(S_{\infty}^{-\infty})^{\tau} = (S_{a}^{-\infty})^{\tau}\cdot \mathrm{c}a\cdot S_{b}^{a}\cdot\partial^{C}b\cdot(S_{\infty}^{b})^{\tau} \;; \tag{7}$$

also, since $b\in B$,

$$(S_{\infty}^{-\infty})^{C} = (S_{b}^{-\infty})^{C}\cdot\partial^{C}b\cdot(S_{\infty}^{b})^{C} \;. \tag{8}$$

Since $b\in B$, $(S_{\infty}^{b})^{C}$ makes sense; and in fact, since $(\prec\!\prec,\to\!\to)$ and $(\prec,\to)$ coincide on $[b,\infty)_{\to}=[b,\infty)_{\to\!\to}$, we have

$$(S_{\infty}^{b})^{\tau} = (S_{\infty}^{b})^{C} \;. \tag{9}$$

On the other hand, $a\in\tilde{\hat{B}}$ and $b\in\tilde{\hat{B}}$ by (5). Therefore, by Lemma 6.1, we have

$$(S_{b}^{-\infty})^{\tilde{\hat{C}}} = (S_{a}^{-\infty})^{\tilde{\hat{C}}}\cdot \mathrm{c}a\cdot S_{b}^{a} \;. \tag{10}$$

The crucial facts are the following two:

$$(s_a^{-\infty})^{\tau} = (s_a^{-\infty})^{\tilde{\tilde{C}}} \qquad ?(11)$$

$$(s_b^{-\infty})^{C} = (s_b^{-\infty})^{\tilde{\tilde{C}}} \qquad ?(12)$$

(notice that $(s_a^{-\infty})^C$ does not make sense since, very likely, $a\notin B$).

For (11): By assumption, $\tau\restriction(-\infty, a]$ $(\prec\!\prec, \to\!\!\to)$-defines the cut $C\restriction(-\infty, a]$. But $C\restriction(-\infty, a]$ is the same as $\tilde{\tilde{C}}\restriction(-\infty, a]$. Therefore, $\tau\restriction(-\infty, a]$ $(\prec\!\prec, \to\!\!\to)$-defines the cut $\tilde{\tilde{C}}\restriction(-\infty, a]$. However, the $(\prec, \to)$-border $\tilde{\tilde{B}}\cap(-\infty, a]$ and the $(\prec\!\prec, \to\!\!\to)$-border $\tilde{\tilde{E}}\cap(-\infty, a]$ of $\tilde{\tilde{C}}\restriction(-\infty, a]$ coincide (see (5)). τ is a span for $\tilde{\tilde{E}}\cap(-\infty, a]$; thus, τ is a span for $\tilde{\tilde{C}}\restriction(-\infty, a]$. The assertion follows.

For (12), one notes that C and $\tilde{\tilde{C}}$ coincide on the interval $(-\infty, b)_{\to}$.

We have enough:

$$\begin{aligned}
(s_\infty^{-\infty})^{\tau} &\overset{(7)}{=} (s_a^{-\infty})^{\tau}\cdot ca\cdot s_b^a\cdot \partial^C b\cdot (s_\infty^b)^{\tau} \\
&\overset{(9),(11)}{=} (s_a^{-\infty})^{\tilde{\tilde{C}}}\cdot ca\cdot s_b^a\cdot \partial^C b\cdot (s_\infty^b)^{C} \\
&\overset{(10)}{=} (s_b^{-\infty})^{\tilde{\tilde{C}}}\cdot \partial^C b\cdot (s_\infty^b)^{C} \\
&\overset{(12)}{=} (s_b^{-\infty})^{C}\cdot \partial^C b\cdot (s_\infty^b)^{C} \\
&\overset{(8)}{=} (s_\infty^{-\infty})^{C} \quad .
\end{aligned}$$

This completes the proof in Case 3.1.

Case 3.2 is "dual" to Case 3.1.

Case 3.3: Assume $a\in \underline{E} - B$ and $b\in \bar{E} - B$.

Now $a\in U$ since $a\in\underline{E}$, and $b\in L$ since $b\in\overline{E}$.

For any pair (x, y) for which $x \underset{=}{\to} y$, $\delta_{\to}(x, y)$ denotes the maximal n for which there is a $\to$-chain $x=x_0\to x_1\to ...\to x_n=y$. $\delta_{\to}(x, y)=0$ iff $x=y$, $\delta_{\to}(x, y)=1$ iff $x\to !y$.

Since $a\in U\text{-}B$, there is $y\in\underline{B}$ such that $a\prec y$. Since $a\in\underline{E}$, we must have $\neg(a\prec\prec y)$; $a\overset{*}{\twoheadrightarrow}y \vee y\overset{*}{\twoheadrightarrow}a$; $(a\xrightarrow[=]{}a\&b\xrightarrow[=]{}y)\vee(y\xrightarrow[=]{}a\&b\xrightarrow[=]{}a)$; $b\xrightarrow[=]{}y$; therefore, $b\to y$ since $b\in L$, $y\neq b$.

Similarly, let x be such that $x\prec b$, $x\in\bar{B}$ and $x\to a$.

Note that since $x\in L$ and $y\in U$, we cannot have $x\underline{\prec}y$. But $y\prec x$ is excluded since it would give $y\prec x\prec b$, contradicting $b\to y$; $y\to x$ is out since $y\to x\to a$ contradicts $a\prec y$. We conclude that $x\to y$.

We have shown that there are pairs (x, y) with the properties

$$x\in B,\ y\in B,\ x\to a\prec y,\ x\prec b\to y \text{ and } x\to y. \tag{13}$$

Call such pairs *appropriate*. We minimize first the distance $\delta_{\to}(b, y)$, then the distance $\delta_{\to}(x, a)$, for an appropriate pair (x, y); let the appropriate pair (x, y) be so minimally chosen.

7.1.2 Sublemma $x\to!y$.

Proof of 7.1.2 Assume otherwise. We can take a span (maximal $\to$-chain) entirely within B (see section 3); therefore, there must be $z\in B$ such that $x\to z\to y$.

Note that each of $a\xrightarrow[=]{}z$ and $z\prec a$ is excluded: $a\xrightarrow[=]{}z$ gives $a\xrightarrow[=]{}z\to y$, contradicting $a\prec y$; $z\prec a$ gives $z\prec a\prec y$, contradicting $z\to y$. Hence, either $z\to a$ or $a\prec z$.

Similarly, either $z\prec b$, or $b\to z$.

Assume $z\prec b$. We must have $z\to a$, since $a\prec z$ gives $a\prec z\prec b$, contradicting $a\prec!b$. But now we have

$$z\in B,\ y\in B,\ z\to a\prec y,\ z\prec b\to y \text{ and } z\to y;$$

that is, (z, y) is an appropriate pair; however, since $x\to z\to a$, $\delta_{\to}(z, a)<\delta_{\to}(x, a)$; thus, we have gotten into contradiction with the minimal choice of $\delta_{\to}(x, a)$ (given that $\delta_{\to}(b, y)$ have remained the same).

Finally, assume $b\to z$. Now, we must have $a\prec z$, since $z\to a$ gives $b\to z\to a$, contradicting $a\prec b$. In this case,

$$x\in B,\ z\in B,\ x\to a\prec z,\ x\prec b\to z \text{ and } x\to z,$$

and a contradiction is reached, since $\delta_{\to}(b, z)<\delta_{\to}(b, y)$, to the minimal choice of $\delta_{\to}(b, y)$.

(**End of proof of 7.1.2**).

We have (13) , and $x \to !y$; in particular, $x \to a$ and $b \to y$. This is the situation that Lemma 6.2 and 3.9.1 apply to. We obtain

$$s_y^x = \underline{s}_a^x \cdot \mathrm{c}a \cdot s_b^a \cdot \mathrm{d}b \cdot \overline{s}_y^b \ . \tag{14}$$

(Here, s_y^x is a basic 1-pd of the assumed prescheme on $(\prec, \to)$, on the basis of the fact $x \to !y$.)

Let's look at (13) again. Since $a \prec y \in B$, we have $a \in U$; since $x \prec b$, $x \in B$, we have $b \in L$. More generally,

Let us repeat what 3.9.1 says:

$$u \in [x, a]_{\to} \ \& \ v \in [b, y]_{\to} \ \& \ (u, v) \neq (x, y) \implies u \prec v \ . \tag{15}$$

$$[x, y]_{\to\to} = [x, a]_{\to} \ \dot{\cup} \ [b, y]_{\to} \ . \tag{16}$$

We want to show (2.1). Note that $x, y \in B$; thus,

$$(s_\infty^{-\infty})^C = (s_x^{-\infty})^C \cdot \partial^C x \cdot s_y^x \cdot \partial^C y \cdot (s_\infty^y)^C \ ;$$

thus, by (14),

$$(s_\infty^{-\infty})^C = (s_x^{-\infty})^C \cdot \partial^C x \cdot \underline{s}_a^x \cdot \mathrm{c}a \cdot s_b^a \cdot \mathrm{d}b \cdot \overline{s}_y^b \cdot \partial^C y \cdot (s_\infty^y)^C \ .$$

On the other hand, the case assumption for Case 3.3 says, in particular, that a is signed $\underline{a}$, b is signed $\overline{b}$, for τ ; thus

$$(s_\infty^{-\infty})^\tau = (s_a^{-\infty})^\tau \cdot \mathrm{c}a \cdot s_b^a \cdot \mathrm{d}b \cdot (s_\infty^b)^\tau$$

Therefore, for (2.1) , all I need is

$$(s_a^{-\infty})^\tau \stackrel{?}{=} (s_x^{-\infty})^C \cdot \partial^C x \cdot \underline{s}_a^x \tag{17.1}$$

$$(s_\infty^b)^\tau \stackrel{?}{=} \overline{s}_y^b \cdot \partial^C y \cdot (s_\infty^y)^C \tag{17.2}$$

We show the first of these equalities; the other one is symmetric.

Consider the interval $I=(-\infty, a)_{\rightarrow}$ of $(\mathbf{N}, \prec, \rightarrow)$. Define the cut $\hat{C}=(\hat{U}, \hat{L})$ in $(I, \prec\restriction I, \rightarrow\restriction I)$ by $\hat{U}=U\cap I$, $\hat{L}=L\cap I$; $\hat{C}=C\restriction I$.

First note that a is on the border E, in the sense of $(\mathbf{N}, \prec\!\prec, \twoheadrightarrow)$, of the cut C. Therefore, $E\cap I$ is the border for $\hat{C}=C\restriction I$ in $(I, \prec\!\prec\restriction I, \twoheadrightarrow\restriction I)$ (3.2 applied to $(\mathbf{N},\prec\!\prec,\twoheadrightarrow)$). However, $\prec\!\prec\restriction I=\prec\restriction I$, $\twoheadrightarrow\restriction I=\rightarrow\restriction I$. Therefore,

(*) $E\cap I$ is the border of $\hat{C}$ in $(I, \prec\restriction I, \rightarrow\restriction I)$.

Since $x\in B$, we have $x\in E$, and $x\in E\cap I$. Therefore, (*) lets us write

$$(S_a^{-\infty})^{\hat{C}}=(S_x^{-\infty})^{\hat{C}}\cdot\partial^{\hat{C}}x\cdot(S_a^x)^{\hat{C}},$$

by section 5, clause (iii) for the given prescheme . But $x\in B$, thus $(S_x^{-\infty})^{C}$ makes sense, and equals $(S_x^{-\infty})^{\hat{C}}$.

What is $(S_a^x)^{\hat{C}}$?

By (15), every element of $J=[x, a)_{\rightarrow}$ is $\prec y$; since $y\in B$, with B the $(\prec, \rightarrow)$-border of C, it follows that $J\subseteq U$, and thus $J\subseteq\hat{U}$. Since $J=[x, a)_{\twoheadrightarrow}$, and a is on the $(\prec\!\prec, \twoheadrightarrow)$-border E of C, by 3.2, applied to $(\prec\!\prec, \twoheadrightarrow)$, the $(\prec\!\prec, \twoheadrightarrow)$-border of $\hat{C}\restriction J$ is $E\cap J$, and since $(\prec\!\prec, \twoheadrightarrow)=(\prec, \rightarrow)$ on J,

(**) the $(\prec, \rightarrow)$-border of $\hat{C}\restriction J$ is $E\cap J$.

But $\hat{C}\restriction J=(J, \emptyset)$. Therefore, $E\cap J=\mu_{\prec}[x, a)_{\rightarrow}$. It follows by (**) that $(S_a^x)^{\hat{C}}=\underline{S}_a^x$.

We conclude that

$$(S_a^{-\infty})^{\hat{C}} = (S_x^{-\infty})^{C}\cdot\partial^{C}x\cdot\underline{S}_a^x \quad . \tag{18}$$

In $(\mathbf{N}, \prec\!\prec, \twoheadrightarrow)$, C is a cut, and τ is a signed span for it; also, a is on the border, in the sense of $(\mathbf{N}, \prec\!\prec, \twoheadrightarrow)$, of the cut C. Therefore, $\xi \overset{\text{def}}{=} \tau\cap I$ is a signed span for the cut $\hat{C}=C\restriction I$ in $(I, \prec\!\prec\restriction I, \twoheadrightarrow\restriction I)$ by 3.3, applied to $(\mathbf{N}, \prec\!\prec, \twoheadrightarrow)$. However, $\prec\!\prec\restriction I=\prec\restriction I$, $\twoheadrightarrow\restriction I=\rightarrow\restriction I$. Therefore,

(***) $\xi=\tau\cap I$ is a signed span for $\hat{C}$ in the sense of $(I, \prec\restriction I, \rightarrow\restriction I)$.

It follows by section 5, cluase (iii) for the given prescheme that

$$(s_a^{-\infty})^{\tau} \overset{\text{DEF}}{=} (s_a^{-\infty})^{\xi} = (s_a^{-\infty})^{\hat{c}} . \tag{19}$$

(18) and (19) entail (17.1).

§8 Proof of Theorem 1.1 and Corollary 1.2

Since we have to deal with "supp" and "Supp" often, I will abbreviate the former by s, the latter by S.

Let $\underline{a}=(\boldsymbol{X}, a)$ be a 1-Pd. I will write $\mathrm{s}^{\circ}(a)$ for the "inner support" of a that ignores the end-point zero-cells. More precisely, we can write a as the composite of 1-cells in a diagram

$$X_1 \xrightarrow{r_1} X_2 \xrightarrow{r_2} X_3 \xrightarrow{r_3} \cdots \xrightarrow{r_{m-1}} X_m \ . \qquad (1)$$

Here, all displayed items are uniquely determined from a itself; in case $m=1$, the minimal value, a is $\mathrm{id}_{X_1}: X_1 \to X_1$. Note that a itself is *not* a formal entity like a molecule, or a diagram, it is an actual cell in an actual ω-category.

We make the definition

$$\mathrm{s}^{\circ}(a) \overset{\mathrm{DEF}}{=} \{X_i : 1 \le i \le m\} \cup \{r_i : 1 \le i \le m-1\} \ .$$

$\mathrm{s}_1^{\circ}(a)$ (the subscript 1 indicates we take 1-indets only) is empty if and *only if* a is an identity.

It may be tempting to say that $\mathrm{s}^{\circ}(a) \underset{?}{\overset{\mathrm{DEF}}{=}} \mathrm{s}(a) - (\mathrm{s}(\mathrm{d}a) \cup \mathrm{s}(\mathrm{c}a))$, except that it would be incorrect. However, we do have $\mathrm{s}^{\circ}(a) \cup \mathrm{s}(\mathrm{d}a) \cup \mathrm{s}(\mathrm{c}a)) = \mathrm{s}(a)$.

The 1-Pd a displayed in (1) is separated (as defined in section 1) iff the zero-cells $X_1, \ldots, X_m$ are (pairwise) distinct; note that as a consequence, the one-cells $r_1, \ldots, r_{m-1}$ are (pairwise) distinct as well.

I will say that a as in (1) is *semi-separated* if $X_i \ne X_j$ whenever $i \ne j$ and $\{i, j\} \ne \{1, m\}$; in other words, we allow the end-points X_1, X_m to coincide. If (1) is semi-separated, the r_j still will necessarily be all different.

We have discussed what we mean by the fact that 1-pd's are uniquely typed. To repeat, in down-to-earth terms, this means that for any a as in (1), there is a diagram, unique up to unique isomorphism, of the form

$$\hat{X}_1 \xrightarrow{\hat{r}_1} \hat{X}_2 \xrightarrow{\hat{r}_2} \hat{X}_3 \xrightarrow{\hat{r}_3} \cdots \xrightarrow{\hat{r}_{m-1}} \hat{X}_m \ ; \qquad (\hat{1})$$

of 0-cells $\hat{x}_i$ and 1-indets $\hat{r}_j$, where the zero-cells $\hat{x}_1, \ldots, \hat{x}_m$ are pairwise distinct, and, as a consequence, the one-cells $\hat{r}_1, \ldots, \hat{r}_{m-1}$ are pairwise distinct. For the composite $(\hat{1})$, $\hat{a}$, we have a unique morphism $\underline{\hat{a}} \to \underline{a}$ of Pd's. (This fact looks so trivial that is hard to see why one would even mention it ... .)

A *molecule* (or *Molecule*) here always means a 2-m(M)olecule, and *atom* a 2-atom.

Let *either*

$$\Phi = (\varphi_1, \ldots \varphi_N) = (\varphi_1[u_1], \ldots \varphi_N[u_N]) \tag{1.1}$$

be a Molecule of positive length $N \geq 1$, *or*

$$\Phi = (\mathrm{d}\Phi) \ , \tag{1.2}$$

one of length $N=0$.

We want to make a simple observation. Recall from §1 : $\mathrm{s}(\Phi) \stackrel{\text{def}}{=} \bigcup_{i=1}^{N} \mathrm{s}(\varphi_i)$ when $n \geq 1$, and $\mathrm{s}(\Phi) \stackrel{\text{def}}{=} \mathrm{s}(\mathrm{d}\Phi)$ when $N=0$. The observation is this: writing subscript ≤ 1 in $s_{\leq 1}$ in the sense of restriction to ≤ 1-indets only, we have

$$\mathrm{s}_{\leq 1}(\Phi) = \mathrm{s}(\mathrm{d}\Phi) \cup \bigcup_{i=1}^{N} \mathrm{s}^{\circ}(\mathrm{c}u_i) \quad . \tag{2}$$

(when $N=0$, we just have $\mathrm{s}_{\leq 1}(\Phi) = \mathrm{s}(\mathrm{d}\Phi)$).

Of course, it is essential for this that Φ is, by definition of "molecule", *composable*. In the "composition" of the atoms in Φ, going from left to right (actually, from up to down), we obtain new 0-and 1-indets only because the next 2-indet u_i introduces new *inner* 0-indets and (all) 1-indets within its codomain, $\mathrm{c}u_i$. The formal proof of (2) is a straight-forward induction on the length N.

It is clear that (2) says something directly about the 2-pd Γ defined by Φ. We have, for any 2-Pd $\underline{\Gamma} = (\boldsymbol{X}, \Gamma)$, $\mathbf{N} = |\boldsymbol{X}|_2$, that

$$|\boldsymbol{X}|_{\leq 1} = \mathrm{s}(\mathrm{d}\Gamma) \cup \bigcup_{u \in \mathbf{N}} \mathrm{s}^{\circ}(\mathrm{c}u) \ . \tag{2*}$$

From now on, we assume that molecules and atoms are *anchored*; our subsequent definitions would be *incorrect* (because of not serving the purpose at hand) otherwise.

We say that the (anchored) molecule Φ in (1.1) or (1.2) is *separated* if

1) Φ is 2-separated (the u_i are pairwise distinct),

2) the union in (2) is disjoint:

$$\mathrm{s}_{\le 1}(\Phi) = \mathrm{s}(\mathrm{d}\Phi) \,\dot\cup\, \dot{\bigcup_{i=0}^{N}} \mathrm{s}^{\circ}(\mathrm{c}u_i) \tag{3}$$

and

3) $\mathrm{d}\Phi$ is separated, and each $\mathrm{c}u_i$ ($i=0, \ldots, N$) is semi-separated.

(In particular, a length-0 molecule Φ, given by a 1-pd $\mathrm{d}\Phi$, is *separated* if the 1-pd $\mathrm{d}\Phi$ is separated).

We cannot fail to see that this directly gives a definition of "separatedness" for a 2-Pd. However, we already have a notion called "separated" for Pd's (see section 2). Although the one being suggested now turns out to be equivalent to the original one (in the case of an anchored 2-Pd), we distinguish the new notion, at least temporarily, by calling it *-separated. Thus, an anchored 2-Pd $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$, $\mathbf{N}=|\boldsymbol{X}|_2$, is **-separated* if

1*) $\underline{\Gamma}$ is top-separated,

2*) the union in (2*) is disjoint:

$$|\boldsymbol{X}|_{\le 1} = \mathrm{s}(\mathrm{d}\Gamma) \,\dot\cup\, \dot{\bigcup_{u\in\mathbf{N}}} \mathrm{s}^{\circ}(\mathrm{c}u) \ . \tag{3*}$$

and

3*) $\mathrm{d}\Gamma$ is separated, and each $\mathrm{c}u$ ($u\in\mathbf{N}$) is semi-separated.

Alternatively, the definition for a molecule can be put in a recursive form.

A length-0 molecule $\Phi=(\mathrm{d}\Phi)$ is separated if the 1-pd $d\Phi$ is separated. A length-N, $N\ge 1$, molecule (1.1) is separated if

(a) the molecule $\Phi{\restriction}(N-1) \overset{\mathrm{def}}{=} (\varphi_1, \ldots\varphi_{N-1})$ is separated,

(b) $\mathrm{s}^{\circ}(\mathrm{c}(u_N))$ is disjoint from $\mathrm{s}(\Phi{\restriction}(N-1))$,

(c) $c(u_N)$ is semi-separated,

and, finally,

(d) u_N is distinct from each of $u_1, \ldots u_{N-1}$.

An (anchored) atom is *separated* if it is separated as a 1-term molecule.

As an example, let X be a 0-indet, $f{:}X\to X$ a 1-indet, and $u{:}\mathrm{id}_X\to f$ a 2-indet. Then

u is an anchored, separated 2-indet, the 1-pd cu is semi-separated, but not separated.

Extending the example, let, in addition, $g:X\to X$ and $v:f\to g$ be additional indets. Then the molecule (u, v) is separated, but the atom v is not separated.

The following proposition is easy to prove; still, we give the details, since we are interested in what we are up against in a future generalization to higher dimensions.

8.1 Proposition **(i)** For every anchored 2-Molecule $\underline{\Phi}$, there is a map $\underline{\hat{\Phi}}\to\Phi$ from some separated Molecule $\underline{\hat{\Phi}}$.

(ii) Every separated anchored 2-Molecule is projective.

(For "projective Molecule", see §2.)

Proof (i) Since for length 0 the assertion is obvious, we may assume that the length of $\underline{\Phi}$ is positive. Assume Φ given as in (1.1). To define the separated $\hat{\Phi}=(\hat{\varphi}_1[\hat{u}_1], \ldots, \hat{\varphi}_N[\hat{u}_N])$ and the map $\hat{\Phi}\to\Phi$, for $i=1, \ldots, N$, by recursion we construct the atom $\hat{\varphi}_i[\hat{u}_i]$, together with a map

$$\hat{\Phi}_i = (\hat{\varphi}_1[\hat{u}_1], \ldots, \hat{\varphi}_i[\hat{u}_i]) \xrightarrow{f_i} \Phi_i \overset{\text{def}}{=} (\varphi_1[u_1], \ldots, \varphi_i[u_i]) \ .$$

We start by letting $f_0:\hat{D}\to d\Phi=d\varphi_1$ be a map from a separated 1-Pd $\hat{D}$ to $d\Phi$. By convention, $\hat{\Phi}_0 \overset{\text{def}}{=} (\hat{D})$, $\Phi_0 \overset{\text{def}}{=} (d\Phi)$.

Suppose $i\geq1$, and $f_{i-1}:\underline{\hat{\Phi}}_{i-1}\to\underline{\Phi}_{i-1}$ have been defined.

Drop the subscript $i-1$: write $\hat{\varphi}[\hat{u}]$ for $\hat{\varphi}_{i-1}[\hat{u}_{i-1}]$, $\varphi[u]$ for $\varphi_{i-1}[u_{i-1}]$, $f:\underline{\hat{\Theta}}\to\underline{\Theta}$ for $f_{i-1}:\underline{\hat{\Phi}}_{i-1}\to\underline{\Phi}_{i-1}$. Write $\psi[v]$ for $\varphi_i[u_i]$.

The map $f:|\underline{\hat{\Theta}}|\to|\underline{\Theta}|$ of underlying computads restricts to a map $\dot{f}:\underline{c\hat{\varphi}}\longrightarrow\underline{c\varphi}$. But $c\varphi=d\psi$, although ψ is not in $|\underline{\Theta}|$. With $\psi=b\cdot v\cdot e$, and, as a consequence, $c\varphi=d\psi=b\cdot dv\cdot e$, in $|\underline{\hat{\Theta}}|$ we can write $c\hat{\varphi}=\hat{b}\cdot\hat{\gamma}\cdot\hat{e}$ uniquely such that $\dot{f}(\hat{b})=b$, $\dot{f}(\gamma)=dv$, $\dot{f}(\hat{e})=e$. In particular, we have the zero-cells $\hat{X}=c\hat{b}=d\hat{\gamma}$, $\hat{Y}=d\hat{e}=c\hat{\gamma}$ in $|\underline{\hat{\Theta}}|$.

In a computad $\boldsymbol{X}$ extending $|\underline{\hat{\Theta}}|$, we construct a 1-pd $\hat{C}$ such that $\hat{C}$ has the same length as $\mathrm{c}v$, $\mathrm{d}\hat{C}=\hat{X}$, $\mathrm{c}\hat{C}=\hat{Y}$, $\hat{C}$ is semi-separated, and $\mathrm{s}^{\circ}(\hat{C})$ is disjoint from $s(|\underline{\hat{\Theta}}|)$. This is done by successively adjoining to $|\underline{\hat{\Theta}}|$ new 0-cells and 1-cells forming $\mathrm{s}^{\circ}(\hat{C})$; the result is a computad $\boldsymbol{X}$ for which $|\boldsymbol{X}|=||\underline{\hat{\Theta}}||\dot{\cup}\mathrm{s}^{\circ}(\hat{C})$. Note that the only obstacle to this would be if $\mathrm{c}v$ were of zero length and, at the same time, $\hat{X}\neq\hat{Y}$; however, since $\mathrm{c}v$ is not of zero-length (Φ is anchored), this does not occur.

By the universal property of adjunction of indeterminates, we have the unique map $g:\boldsymbol{X}\longrightarrow|\underline{\Phi}_i|$ extending $f:|\underline{\hat{\Theta}}|\to|\underline{\Theta}|$ such that $g(\hat{C})=\mathrm{c}v$.

In $\boldsymbol{X}$, we have that the 1-pd's $\hat{\gamma}$ and $\hat{C}$ are parallel; hence, to $\boldsymbol{X}$, we can adjoin the new 2-indet $\hat{v}$ with $\mathrm{d}\hat{v}=\hat{\gamma}$ and $\mathrm{c}\hat{v}=\hat{C}$; we obtain the computad $\boldsymbol{X}[\hat{v}]$. In $\boldsymbol{X}[\hat{v}]$, we have the atom $\hat{\varphi}_i \overset{\mathrm{def}}{=} \hat{b}\cdot\hat{v}\cdot\hat{e}$, and the molecule $\hat{\Phi}_i \overset{\mathrm{def}}{=} \hat{\Theta}^\wedge\hat{\varphi}_i=(\hat{\varphi}_1,\,...,\,\hat{\varphi}_{i-1},\,\hat{\varphi}_i)$. Clearly, $\boldsymbol{X}[\hat{v}]=|\underline{\hat{\Phi}}_i|$.

Since $g(\hat{\gamma})=f(\hat{\gamma})=\mathrm{d}v$ and $g(\hat{C})=\mathrm{c}v$, we can extend g to $h:\boldsymbol{X}[\hat{v}]\longrightarrow|\underline{\Phi}_i|$ such that $h(\hat{v})=v$. The construction ensures that $h(\hat{\varphi}_i)=\varphi_i$, and thus $h(\hat{\Phi}_i)=\Phi_i$; we have our desired map $h:\underline{\hat{\Phi}}_i\longrightarrow\underline{\Phi}_i$ of Pd's.

The construction ensures that $\underline{\Phi}_i$ is an (anchored) separated molecule, according to clauses (a) to (d) above.

This completes the inductive proof of 8.1(i).

(ii) By induction on the length N of the separated Molecule $\underline{\hat{\Phi}}$.

Let
$$\Phi=(\varphi_1,\,...\varphi_N)=(\varphi_1[u_1],\,...\varphi_N[u_N]),$$
$$\hat{\Phi}=(\hat{\varphi}_1,\,...\hat{\varphi}_N)=(\hat{\varphi}_1[\hat{u}_1],\,...\varphi_N[\hat{u}_N]),$$
$$\tilde{\Phi}=(\tilde{\varphi}_1,\,...\tilde{\varphi}_N)=(\tilde{\varphi}_1[\tilde{u}_1],\,...\tilde{\varphi}_N[\tilde{u}_N])$$
$$\underline{\Phi}=(\boldsymbol{X},\Phi),\ \ \underline{\hat{\Phi}}=(\hat{\boldsymbol{X}},\hat{\Phi}),\ \ \underline{\tilde{\Phi}}=(\tilde{\boldsymbol{X}},\tilde{\Phi})\ .$$

Suppose $\underline{\hat{\Phi}}$ is separated, $f:\underline{\hat{\Phi}}\to\underline{\Phi}$ and $g:\underline{\tilde{\Phi}}\to\underline{\Phi}$, to show the existence of $h:\underline{\hat{\Phi}}\to\underline{\tilde{\Phi}}$ such that $g\circ h\overset{?}{=}f$.

If $N=0$, the assertion follows from unique typing for 1-pd's.

Assume $N\geq 1$. Denote the operation $(-)\restriction(N-1)$ by putting a bar on top. We have that $\underline{\bar{\hat{\Phi}}}$ is separated, we have $\bar{f}:\underline{\bar{\hat{\Phi}}}\to\underline{\bar{\Phi}}$ and $\bar{g}:\underline{\bar{\tilde{\Psi}}}\to\underline{\bar{\Phi}}$. By the induction hypothesis, we have $k:\underline{\bar{\hat{\Phi}}}\to\underline{\bar{\tilde{\Phi}}}$ such that $\bar{g}\circ k=\bar{f}$.

Drop the subscript N ; let $\varphi=\varphi_N$, $u=u_N$, $\hat{\varphi}=\hat{\varphi}_N$, etc.

Let the 1-pd's $\hat{b}, \hat{e}$ be defined by $\hat{\varphi}=\hat{b}\cdot\hat{u}\cdot\hat{e}$; b, e by $\varphi=b\cdot u\cdot e$; $\tilde{b}, \tilde{e}$ by $\tilde{\varphi}=\tilde{b}\cdot\tilde{u}\cdot\tilde{e}$.

Claim (e) We have that $\hat{b}, \mathrm{d}\hat{u}, \hat{e}$ belong to $\|\bar{\hat{X}}\|$, $b, \mathrm{d}u, e$ to $\|\bar{X}\|$, and $\tilde{b}, \mathrm{d}\tilde{u}, \tilde{e}$ to $\|\bar{\tilde{X}}\|$; moreover,

(f) $k(\hat{b})=\tilde{b}$, $k(\mathrm{d}\hat{u})=\mathrm{d}\tilde{u}$, $k(\hat{e})=\tilde{e}$.

(In the list

$f(\hat{b})=b$, $k(\hat{b})=\tilde{b}$, $g(\tilde{b})=b$,
$f(\mathrm{d}\hat{u})=\mathrm{d}u$, $k(\mathrm{d}\hat{u})=\mathrm{d}\tilde{u}$, $g(\mathrm{d}\tilde{u})=\mathrm{d}u$,
$f(\hat{e})=e$, $k(\hat{e})=\tilde{e}$, $g(\tilde{e})=e$,

the equalities involving f and g hold true, since f and g are defined on the levels of $\hat{\boldsymbol{X}}$, $\tilde{\boldsymbol{X}}$, not just their restrictions; only the ones involving k are still to be shown.)

For the proof, note first that

$$\mathrm{c}(\hat{\varphi}_{N-1}) = \mathrm{d}(\hat{\varphi}) = \hat{b}\cdot\mathrm{d}\hat{u}\cdot\hat{e} ,$$
$$\mathrm{c}(\tilde{\varphi}_{N-1}) = \mathrm{d}(\tilde{\varphi}) = \tilde{b}\cdot\mathrm{d}\tilde{u}\cdot\tilde{e} ,$$
$$\mathrm{c}(\varphi_{N-1}) = \mathrm{d}(\varphi) = b\cdot\mathrm{d}u\cdot e ,$$

and the items subscripted with $N-1$ are in the corresponding restrictions $\bar{\hat{\boldsymbol{X}}}$, $\bar{\tilde{\boldsymbol{X}}}$, $\bar{\boldsymbol{X}}$; this shows (e). Since k maps $\mathrm{c}(\hat{\varphi}_{N-1})$ to $\mathrm{c}(\tilde{\varphi}_{N-1})$, we have

$$k(\hat{b})\cdot k(\mathrm{d}\hat{u})\cdot k(\hat{e}) = \tilde{b}\cdot\mathrm{d}\tilde{u}\cdot\tilde{e} . \tag{4}$$

On the other hand, $k(\hat{b})$ and $\tilde{b}$ are both mapped to b , the first by f , the second by g ; and similarly for the other pairs of factors in the last equality. Now, consider the following obvious fact for 1-pd's, that, alas, is less obvious in higher dimensions: with $r, s, \dot{r}, \dot{s}$

1-pd's, suppose $r\cdot s=\dot{r}\cdot\dot{s}$ in a computad $\boldsymbol{Z}$, $g:\boldsymbol{Z}\to\boldsymbol{W}$ is a computad map, and $g(r)=g(\dot{r})$, $g(s)=g(\dot{s})$; then we must have $r=\dot{r}$, $s=\dot{s}$; in fact, one of the equalities, say $g(r)=g(\dot{r})$, is enough; one notes that r and $\dot{r}$ must have the same length, and an initial segment of a 1-pd is determined by its length.

The fact just mentioned implies, by the presence of the map $g:|\tilde{\Phi}|\to|\Phi|$, that, in (4), the factors on the two sides have to be pairwise equal. This proves part (f) of the claim.

We need to extend k to a map $h:\hat{\boldsymbol{X}}\to\tilde{\boldsymbol{X}}$ such that $g\circ h=f$. We need to define the effect of h on the set $\mathrm{s}^{\circ}(\mathrm{c}\hat{u})$, a set disjoint from $\mathrm{s}(\mathrm{dom}(h))$.

Look at the zero-cells $\hat{X}=\mathrm{dd}(\hat{u})=\mathrm{c}(\hat{b})$, $\hat{Y}=\mathrm{cc}(\hat{u})=\mathrm{d}(\hat{e})$ in $\bar{\hat{\boldsymbol{X}}}$, $\tilde{X}=\mathrm{dd}(\tilde{u})=\mathrm{c}(\tilde{b})$, $\tilde{Y}=\mathrm{cc}(\tilde{u})=\mathrm{d}(\tilde{e})$ in $\bar{\tilde{\boldsymbol{X}}}$, $X=\mathrm{dd}(u)=\mathrm{c}(b)$, $Y=\mathrm{cc}(u)=\mathrm{d}(e)$ in $\bar{\boldsymbol{X}}$. Since $k(\hat{b})=\tilde{b}$, $k(\hat{e})=\tilde{e}$ by the Claim, we have $k(\hat{X})=\tilde{X}$, $k(\hat{Y})=\tilde{Y}$. Write $\mathrm{c}\hat{u}$ in the form $(\hat{1})$ above; we have $\hat{X}_1=\hat{X}$, $\hat{X}_m=\hat{Y}$; the other items in $(\hat{1})$ are the elements of the set $\mathrm{s}^{\circ}(\mathrm{c}\hat{u})$. In $\tilde{\boldsymbol{X}}$, $\mathrm{c}\tilde{u}$ has the same form, with $\tilde{\ }'$s rather than $\hat{\ }'$s, with the same length m, and $\tilde{X}_1=\tilde{X}$, $\tilde{X}_m=\tilde{Y}$.

There is a unique extension $\hat{k}$ of k to the computad $\hat{\boldsymbol{X}}[\mathrm{s}^{\circ}(\mathrm{c}\hat{u})]$,

$$\hat{k}:\hat{\boldsymbol{X}}[\mathrm{s}^{\circ}(\mathrm{c}\hat{u})]\longrightarrow\tilde{\boldsymbol{X}}$$

such that $\hat{k}(\mathrm{c}\hat{u})=\mathrm{c}\tilde{u}$: for $i=2,\ldots,m-1$, we define $\hat{k}(\hat{X}_i)=\tilde{X}_i$; we then have $\hat{k}(\hat{X}_i)=\tilde{X}_i$ for $i=1,2,\ldots,m-1,m$; after which, for $j=1,\ldots,m-1$, we define $\hat{k}(\hat{f}_i)=\tilde{f}_i$.

Finally, we adjoin the new 2-indet $\hat{u}$, and define $h:\hat{\boldsymbol{X}}=\hat{\boldsymbol{X}}[\mathrm{s}^{\circ}(\mathrm{c}\hat{u})][\hat{u}]\longrightarrow\tilde{\boldsymbol{X}}$ extending $\hat{k}$, by setting $h(\hat{u})=\tilde{u}$; this is legitimate since $\hat{k}(\mathrm{d}\hat{u})=k(\mathrm{d}\hat{u})=\mathrm{d}\tilde{u}$, and $\hat{k}(\mathrm{c}\hat{u})=\mathrm{c}\tilde{u}$ by construction. Clearly, we have made the extension h so that $g\circ h=f$.

The fact that h is a map of Molecules, $h:\underline{\hat{\Phi}}\to\underline{\tilde{\Phi}}$, follows from $h(\hat{\varphi})=\tilde{\varphi}$, which is a consequence of the Claim.

This completes the proof of 8.1(ii).

We have proved 2.8 Elementary Lemma; and now we know the results of section 2 up to and including 2.11.

Before turning to the more mundane task of proving assertions made in section 1, we make some general remarks.

The notion of projective Pd is very general: a Pd $\underline{\Gamma}$, of an arbitrary dimension, is *projective* if whenever $\underline{\Lambda}\xleftarrow{f}\underline{\Gamma}\xrightarrow{g}\underline{\Xi}$, we have at least one $\underline{\Gamma}\xrightarrow{h}\underline{\Xi}$ such that $g\circ h=f$. Note that if $\underline{\Gamma}$ is of dimension n, then, for testing projectivity of $\underline{\Gamma}$, it suffices to look only at $\underline{\Lambda}$, $\underline{\Xi}$ also of dimension exactly n; and if $\underline{\Gamma}$ is an Indet (Γ itself is an indet), then $\underline{\Lambda}$, $\underline{\Xi}$ can also be restricted to be Indets of the same dimension as $\underline{\Gamma}$. When one restricts Pd's further, such as "anchored", or belonging to a "standard class" (see section 1), the definition further relativizes to the class.

In section 1, we mentioned *separated* Pd's, and *computopes*; the two are closely related. We note here that *every projective Pd is separated*: let $\underline{\Gamma}$ be projective; by the Corollary to Theorem[M](i) in section 1, there is a separated $\hat{\underline{\Gamma}}$ with a morphism $f:\hat{\underline{\Gamma}}\to\underline{\Gamma}$; by projectivity of $\underline{\Gamma}$, there is $g:\underline{\Gamma}\to\hat{\underline{\Gamma}}$; but by definition of "separated", g must be an isomorphism; thus, $\underline{\Gamma}$, being isomorphic to a separated Pd, is itself separated.

I do not know whether the converse is true.

Moreover, if $\underline{\Gamma}$ has a "projective cover" $\hat{\underline{\Gamma}}\to\underline{\Gamma}$, the condition 2) in section 1, "uniqueness of the type" in $\underline{\Gamma}$ being uniquely typed, is satisfied, since for any separated $\underline{\Lambda}$, if there is $\underline{\Lambda}\to\underline{\Gamma}$, Λ must be isomorphic to $\underline{\Gamma}$.

Let us now restrict ourselves to the anchored 2-dimensional case.

We have seen, in the proof of 2.9, that if $\underline{\Phi}$ is a projective (anchored 2-)Molecule, then the 2-Pd $[\![\underline{\Phi}]\!]$ defined by $\underline{\Phi}$ is projective as well. The converse is also true: if $[\![\underline{\Phi}]\!]$ is projective, then $\underline{\Phi}$ is too: by 8.1(i), there is a projective cover $f:\hat{\underline{\Phi}}\longrightarrow\underline{\Phi}$, inducing a map $f: [\![\hat{\underline{\Phi}}]\!]\longrightarrow[\![\underline{\Phi}]\!]$; $[\![\underline{\Phi}]\!]$ being projective, it is separated; hence, f is an isomorphism; $f:\hat{\underline{\Phi}}\longrightarrow\underline{\Phi}$ being an isomorphism, and $\hat{\underline{\Phi}}$ projective, $\underline{\Phi}$ is projective.

We now see that the notions "projective" and "separated" for anchored 2-Molecules coincide; and the three notions "separated", "projective", and "*-separated" for anchored 2-Pd's coincide.

The following proposition and its proof could have been included already in section 1. It is a general categorical argument based on the universal property defining the notion of computad. It infers the existence of enough projectives among $(n+1)$-Indets, on the basis of the same assumption for n-Pd's (the latter is a special case of the former), plus strong properties of $(n-1)$-Pd's. Since the hypotheses are verified for $n=2$, we will obtain that there are enough projectives among 2-anchored 3-Indets, which, in turn, enables us to infer 1.1 and 1.2 from 2.10.

We fix a standard class of computads (see section 1). All computads, Indets and Pd's are to be taken from the fixed class (which will remain nameless).

8.2 Proposition (We restrict ourselves to a fixed, although arbitrary, standard class of computads.) Let n be a fixed integer at least 2. We make the following assumptions:

(a) All $(n\text{-}1)$-Pd's are uniquely typed.
(b) There are enough projective n-Pd's: for any n-Pd $\underline{\Gamma}$, there is a projective n-Pd $\hat{\underline{\Gamma}}$ together with a map $\hat{\underline{\Gamma}} \to \underline{\Gamma}$.

Then:

There are enough projective $(n+1)$-Indet's: for any $(n+1)$-Indet $\underline{U}$, there are a projective $(n+1)$-Indet $\hat{\underline{U}}$ and a map $\hat{\underline{U}} \to \underline{U}$.

Proof For this proof, I'll simplify the notation somewhat. I will drop the underlining from Pd's; thus, e.g., given $\underline{U}=(\boldsymbol{X}, U)$, U may ambiguously mean either $\underline{U}=(\boldsymbol{X}, U)$, or U as an element of $\boldsymbol{X}$. On the other hand, I'll write $|U|$ for $\boldsymbol{X}$, the *underlying computad* of U.

For any Pd Ξ, we have $\mathrm{d}\Xi$, both as a pd and as a Pd. If we return to our pedantic ways, we have $\underline{\Xi}=(\boldsymbol{Z}, \Xi)$ ($\boldsymbol{Z}=|\underline{\Xi}|$), and $d\Xi$ a pd in $\boldsymbol{Z}$; we put

$$\mathrm{d}\underline{\Xi} \overset{\mathrm{def}}{=} (\mathrm{Supp}_{\boldsymbol{Z}}(\mathrm{d}\Xi), \mathrm{d}\Xi) ;$$

similarly for $\mathrm{c}\underline{\Xi}$. We again drop underlining, and just write $d\Xi$ in both senses involved. Note that we have the inclusion map $|\mathrm{d}\Xi| \rightarrowtail |\Xi|$; but writing $\mathrm{d}\Xi \rightarrowtail \Xi$ is (too) incorrect; $f:\Gamma\to\Lambda$ also implies that $f(\Gamma)=\Lambda$ (!).

Let U be an $(n+1)$-Indet; $\mathrm{d}U$ and $\mathrm{c}U$ are n-Pd's. By assumption (b), there are projective n-Pd's Γ and Λ, with maps $\Gamma \xrightarrow{f} \mathrm{d}U$, $\Lambda \xrightarrow{g} \mathrm{c}U$. We have the inclusions

$$|\mathrm{d}\Gamma| \overset{\alpha}{\rightarrowtail} |\Gamma| , \quad |\mathrm{c}\Gamma| \overset{\beta}{\rightarrowtail} |\Gamma| , \quad |\mathrm{d}\Lambda| \overset{\gamma}{\rightarrowtail} |\Lambda| , \quad |\mathrm{c}\Lambda| \overset{\delta}{\rightarrowtail} |\Lambda| ,$$

$$|\mathrm{d}U| \overset{\varepsilon}{\rightarrowtail} |U| , \qquad |\mathrm{c}U| \overset{\eta}{\rightarrowtail} |U| .$$

The composites $\varepsilon\circ f\circ\alpha: |\mathrm{d}\Gamma| \longrightarrow |U|$, $\varepsilon\circ f\circ\gamma: |\mathrm{d}\Lambda| \longrightarrow |U|$ give rise to (are) the respective maps $\varepsilon f\alpha: \mathrm{d}\Gamma \longrightarrow \mathrm{dd}U$, $\varepsilon f\gamma: \mathrm{d}\Lambda \longrightarrow \mathrm{dc}U$ of n-Pd's. We have $\mathrm{dd}U=\mathrm{dc}U$. Since $(n\text{-}1)$-Pd's are uniquely typed (assumption (a)), with D the unique type of $\mathrm{dd}U=\mathrm{dc}U$, D must also be the type of $\mathrm{d}\Gamma$ and $\mathrm{d}\Lambda$; we conclude that there is a separated $(n\text{-}1)$-pd D with maps $D \xrightarrow{\varphi} \mathrm{d}\Gamma$, $D \xrightarrow{\psi} \mathrm{d}\Lambda$ such that $\varepsilon\circ f\circ\alpha\circ\varphi=\eta\circ g\circ\gamma\circ\psi$. Similarly, we have a separated $(n\text{-}1)$-pd C with maps $C \xrightarrow{\rho} \mathrm{c}\Gamma$, $C \xrightarrow{\sigma} \mathrm{c}\Lambda$ such that $f\circ\beta\circ\rho=g\circ\delta\circ\sigma$.

We define the computad $\mathbf{X}$ by the following colimit diagram:

$$
\begin{tikzcd}
& {|D|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
{|\mathrm{d}\Gamma|} \arrow[d, "\alpha"'] & \circ & {|\mathrm{d}\Lambda|} \arrow[d, "\gamma"] \\
{|\Gamma|} \arrow[r, "\xi"] & \mathbf{X} & {|\Lambda|} \arrow[l, "\zeta"'] \\
{|\mathrm{c}\Gamma|} \arrow[u, "\beta"] & \circ & {|\mathrm{c}\Lambda|} \arrow[u, "\delta"'] \\
& {|C|} \arrow[ul, "\rho"] \arrow[ur, "\sigma"'] &
\end{tikzcd}
$$

We have taken the colimit of the diagram which is the one above without the items $\mathbf{X}$, ξ and ζ; $\mathbf{X}$ is the colimit object, ξ and ζ are colimit coprojections (and so are the composite arrows from the objects other than $|\Gamma|$, $|\Lambda|$).

In the computad $\mathbf{X}$, we have the n-pd's $\hat{\Gamma} \overset{\mathrm{def}}{=} \xi(\Gamma)$, $\hat{\Lambda} \overset{\mathrm{def}}{=} \zeta(\Lambda)$. Let $\hat{D}=(\xi\circ\alpha\circ\varphi)(D)=(\zeta\circ\gamma\circ\psi)(D)$, an $(n-1)$-pd in $\mathbf{X}$. Since $\mathrm{d}\Gamma=(\alpha\circ\varphi)(D)$, $\hat{D}=\xi(\mathrm{d}\Gamma)=\mathrm{d}(\xi\Gamma)=\mathrm{d}(\hat{\Gamma})$. Similarly, $\hat{D}=\mathrm{d}(\hat{\Lambda})$. For $\hat{C}=(\xi\circ\beta\circ\rho)(C)=(\zeta\circ\delta\circ\sigma)(C)$, we similarly have $\hat{C}=\mathrm{c}(\hat{\Gamma})=\mathrm{c}(\hat{\Lambda})$. We conclude that the n-pd's $\hat{\Gamma}$, $\hat{\Lambda}$ are parallel. Therefore, we may adjoin a $(n+1)$-indet $\hat{U}$ to $\mathbf{X}$ with the specification $\mathrm{d}(\hat{U})=\hat{\Gamma}$, $\mathrm{c}(\hat{U})=\hat{\Lambda}$, and form $\mathbf{X}[\hat{U}]$. I claim that $U=(\mathbf{X}[\hat{U}],\hat{U})$ is the desired item.

The commutativity of the following diagram:

$$
\begin{tikzcd}
& & {|D|} \arrow[dll, "\varphi"'] \arrow[drr, "\psi"] & & \\
{|\mathrm{d}\Gamma|} \arrow[d, "\alpha"'] & & \circ & & {|\mathrm{d}\Lambda|} \arrow[d, "\gamma"] \\
{|\Gamma|} \arrow[r, "f"] & {|\mathrm{d}U|} \arrow[r, "\varepsilon"] & {|U|} & {|\mathrm{c}U|} \arrow[l, "\eta"'] & {|\Lambda|} \arrow[l, "g"'] \\
{|\mathrm{c}\Gamma|} \arrow[u, "\beta"] & & \circ & & {|\mathrm{c}\Lambda|} \arrow[u, "\delta"'] \\
& & {|C|} \arrow[ull, "\rho"] \arrow[urr, "\sigma"'] & &
\end{tikzcd}
$$

and the colimit property of the previous one shows that there is a unique arrow $h:\mathbf{X}\to|U|$ such that the following commutes:

$$\begin{tikzcd}
 & & \boldsymbol{X} \arrow[d, "h"] & & \\
|\Gamma| \arrow[urr, "\xi"] \arrow[r, "f"] & |\mathrm{d}U| \arrow[r, "\varepsilon"] \arrow[u, phantom, "\circ"] & |U| & |\mathrm{c}U| \arrow[l, "\eta"'] \arrow[u, phantom, "\circ"] & |\Lambda| \arrow[l, "g"'] \arrow[ull, "\zeta"']
\end{tikzcd} .$$

From these two commutativites we infer that $h(\hat{\Gamma})=\mathrm{d}U$, $h(\hat{\Lambda})=\mathrm{c}U$. Therefore, it is legitimate to require of a map $k:\boldsymbol{X}[\hat{U}]\longrightarrow|U|$ that it extend h and maps $\hat{U}$ to U; we do so! We have obtained the map $k:\hat{U}\to U$ of $(n+1)$-Indets.

To verify that $\hat{U}$ is projective, assume we have $\hat{U}\xrightarrow{\ell}V\xleftarrow{m}W$, to show that there is $\hat{U}\underset{?}{\xrightarrow{n}}W$ such that $m\circ n=\ell$.

We have the induced maps

$$\Gamma\xrightarrow{\ell\xi}\mathrm{d}V,\ \mathrm{d}W\xrightarrow{\hat{m}}\mathrm{d}V,\ \Lambda\xrightarrow{\ell\zeta}\mathrm{c}V,\ \mathrm{c}W\xrightarrow{\tilde{m}}\mathrm{c}V.$$

Γ and Λ are projective: there are $\Gamma\xrightarrow{p}dW$, $\Lambda\xrightarrow{q}cW$ such that $\hat{m}\circ p=\ell\xi$, $\tilde{m}\circ q=\ell\zeta$. Let's write $\theta:|\mathrm{d}W|\to|W|$, $\tau:|\mathrm{c}W|\to|W|$ for the inclusions.

The following diagram commutes:

$$\begin{tikzcd}
 & & |D| \arrow[dll, "\varphi"'] \arrow[drr, "\psi"] & & \\
|\mathrm{d}\Gamma| \arrow[d, "\alpha"'] & & & & |\mathrm{d}\Lambda| \arrow[d, "\gamma"] \\
|\Gamma| \arrow[r, "p"] & |\mathrm{d}W| \arrow[r, "\theta"] & |W| \arrow[u, phantom, "\circ"] \arrow[d, phantom, "\circ"] & |\mathrm{c}W| \arrow[l, "\tau"'] & |\Lambda| \arrow[l, "q"'] \\
|\mathrm{c}\Gamma| \arrow[u, "\beta"] & & & & |\mathrm{c}\Lambda| \arrow[u, "\delta"'] \\
 & & |C| \arrow[ull, "\rho"] \arrow[urr, "\sigma"'] & &
\end{tikzcd}$$

The reason is that, for instance, the two upper composites, $c_1=\theta\circ p\circ\alpha\circ\varphi$ and $c_2=\tau\circ q\circ\gamma\circ\psi$, when applied to the element D in $|D|$, give the same value, namely $\tilde{D}\overset{\mathrm{def}}{=}\mathrm{dd}W=\mathrm{dc}W$; we have two parallel maps $D\underset{c_2}{\overset{c_1}{\rightrightarrows}}\tilde{D}$ of $(n-1)$-Pd's; they must be equal: $c_1=c_2$ (follows from assumption (a)).

We again apply the colimit definition of $\boldsymbol{X}$: we obtain $r:\boldsymbol{X}\longrightarrow|W|$ such that the following commutes:

$$\begin{tikzcd}
 & & \boldsymbol{X} \arrow[d, "r"] & & \\
|\Gamma| \arrow[urr, "\xi"] \arrow[r, "p"] & |\mathrm{d}W| \arrow[r, "\theta"] \arrow[ur, phantom, "\circ"] & |W| & |\mathrm{c}W| \arrow[l, "\tau"'] \arrow[ul, phantom, "\circ"] & |\Lambda| \arrow[ull, "\zeta"'] \arrow[l, "q"']
\end{tikzcd} \quad .$$

It follows that $r(\hat{\Gamma})=\mathrm{d}W$, $r(\hat{\Lambda})=\mathrm{d}W$; we have $n:|\hat{U}|=\boldsymbol{X}[\hat{U}]\longrightarrow|W|$ extending r and mapping $\hat{U}$ to W. We have $n:\hat{U}\to W$.

When the maps $m\circ n$ and ℓ (whose equality we want) are restricted to $\boldsymbol{X}$, they are equal, by the uniqueness part of the colimit definition of $\boldsymbol{X}$, and $\hat{m}\circ p=\ell\xi$, $\tilde{m}\circ q=\ell\zeta$. As a result, $m\circ n=\ell$ holds since the one remaining indet $\hat{U}$ in $|U|$ is mapped by both to W.

The proof is complete.

I repeat that for the standard class of 2-anchored computads, and for $n=2$, we have (a) trivially, and (b) by 2.9. Therefore, the conclusion holds for 2-anchored 3-Indets.

2.6 (that we already know) immediately implies

8.3 Proposition Parallel maps $\underline{V}\underset{g}{\overset{f}{\rightrightarrows}}\underline{U}$ of 2-anchored 3-Indet's must coincide: $f=g$.

Proof The reason is that we have the induced maps $\mathrm{d}\underline{V}\underset{g\restriction \mathrm{S}(\mathrm{d}V)}{\overset{f\restriction \mathrm{S}(\mathrm{d}V)}{\rightrightarrows}}\mathrm{d}\underline{V}$, $\mathrm{c}\underline{V}\underset{g\restriction \mathrm{S}(\mathrm{c}V)}{\overset{f\restriction \mathrm{S}(\mathrm{c}V)}{\rightrightarrows}}\mathrm{c}\underline{V}$ of anchored 2-Pds, which, by 2.6, must pairwise coincide; $f=g$ follows since the only item beyond $\mathrm{s}(\mathrm{d}V)\cup\mathrm{s}(\mathrm{c}V)$ on which they act is the indet V itself, which they both map to U.

We have now proved 1.1 Theorem and 1.2 Corollary. Indeed, consider conditions 1) and 2) (in section 1) defining "uniquely typed" for 2-anchored 3-Pd's: 1) is a special case of 8.3; and 2) holds by 8.2 and the remarks after 8.1.

§9 Higher-dimensional pasting preschemes

Substitution and up-substitution

We need to recall some things essentially dealt with in sections 8 and 9 in [M].

Suppose $\boldsymbol{X}$ is a computad of dimension n, Γ is an n-pd in $\boldsymbol{X}$, and $\bar{u}$ is an n-indet in $\boldsymbol{X}$. We say that $\bar{u}$ *occurs in* Γ *exactly once if*, for some, or equivalently, for all, n-molecule $\Phi=(\varphi_1[v_1],\ \ldots,\ \varphi_M[v_M])$ representing Γ, there is exactly one $j\in\{1,\ \ldots,\ M\}$ such that $v_j=\bar{u}$. (By section 2, it is indeed true that if the condition holds for Φ, it holds for any Ψ such that $[\![\Phi]\!]=[\![\Psi]\!]$.) .

Assume that, indeed, $\bar{u}$ occurs in Γ exactly once. Now, let u be a (new) indeterminate of dimension $n{+}1$ attached to $\boldsymbol{X}$ by $\mathrm{d}u\|\mathrm{c}u$, both n-pd's $\mathrm{d}u$, $\mathrm{c}u$ in $\boldsymbol{X}$, such that, in addition, $\mathrm{dd}u=\mathrm{d}\bar{u}$ and $\mathrm{cc}u=\mathrm{c}\bar{u}$. We are going to define the result of substituting u for $\bar{u}$ in Γ, despite the fact that u is of one-higher dimension than $\bar{u}$. I will denote the result of this substitution by $\Gamma[u/\bar{u}]$, or sometimes, sloppily, $\Gamma[u]$, when $\bar{u}$ in $\Gamma=\Gamma[\bar{u}]$ is "understood". $\Gamma[u]$ is going to be an $(n{+}1)$-cell in $\boldsymbol{X}[u]$. Moreover, we will have that

$$\mathrm{d}(\Gamma[u])=\Gamma[\mathrm{d}u] \qquad \text{and} \qquad \mathrm{c}(\Gamma[u])=\Gamma[\mathrm{c}u]\ ; \tag{1}$$

where $\Gamma[\mathrm{d}u]$, $\Gamma[\mathrm{c}u]$ are ordinary substitutions, defined by the universal property of $\boldsymbol{X}=\boldsymbol{Y}[\bar{u}]$, via the mappings

$$\boldsymbol{Y}[\bar{u}] \overset{f}{\underset{g}{\rightrightarrows}} \boldsymbol{X}$$

which are the identity on $\boldsymbol{Y}$, f maps $\bar{u}$ to $\mathrm{d}u$, g to $\mathrm{c}u$; these are legitimate by $\mathrm{d}\bar{u}=\mathrm{dd}u$, $\mathrm{c}\bar{u}=\mathrm{cd}u$ ($=\mathrm{cc}u$), etc.; $\Gamma[\mathrm{d}u]\overset{\mathrm{def}}{=}f(\Gamma)$, $\Gamma[\mathrm{c}u]\overset{\mathrm{def}}{=}g(\Gamma)$.

The definition of $\Gamma[u]$ seems fairly obvious, at the first sight at least. Take any $\Phi=(\varphi_1[v_1],\ \ldots,\ \varphi_M[v_M])$ representing Γ; let j be the one subscript for which $v_j=\bar{u}$; write $\varphi=\varphi_j$; write $\Lambda=\varphi_1[v_1]\cdot\ldots\cdot\varphi_{j-1}[v_{j-1}]$, $\Xi=\varphi_{j+1}[v_{j+1}]\cdot\ldots\cdot\varphi_M[v_M]$; thus $\Gamma=\Gamma[\bar{u}]=\Lambda\cdot\varphi[\bar{u}]\cdot\Xi$. Recall that, as an atom, $\varphi[\bar{u}]$ can be written in the form

$$\varphi[\bar{u}]=b_{n-1}\cdot(b_{n-2}\cdot(\ldots b_1\cdot\bar{u}\cdot e_1\ldots)\cdot e_{n-2})\cdot e_{n-1} \tag{2.1}$$

for some b_i, e_i such that $\dim(b_i)=\dim(e_i)=i$. Therefore, we can define

$$\varphi[u] \overset{\text{DEF}}{\underset{?}{=}} b_{n-1}\cdot(b_{n-2}\cdot(\ldots b_1\cdot u\cdot e_1\ldots)\cdot e_{n-2})\cdot e_{n-1} \tag{2.2}$$

and

$$\Gamma[u] \overset{\text{DEF}}{\underset{?}{=}} \Lambda\cdot\varphi[u]\cdot\Xi \quad . \tag{3}$$

Although this calculation will be important, as a definition it has the fault that it is not clear that the value $\Gamma[u]$ is independent from the choice of the molecule representing $\Gamma[u]$, and worse, from the choice of the presentation of the atom $\varphi[\bar{u}]$ which we used.

However, in section 8 of [M], there is a construction of the so-called *collapse* of the (n+1)-dimensional computad $\boldsymbol{X}[u]$ to an n-dimensional one, $\overline{\boldsymbol{X}[u]}$. The set of n-indets of the latter is $\boldsymbol{X}_n \dot\cup \{u\}$; also, $\boldsymbol{X}$ is a subcomputad of $\overline{\boldsymbol{X}[u]}$. Moreover, as an n-cell in $\overline{\boldsymbol{X}[u]}$, u is parallel to $\bar{u}$ (since $\mathrm{d}_{\overline{\boldsymbol{X}[u]}}(u)=\mathrm{dd}u=\mathrm{d}\bar{u}$, and similarly for c). Thus, we can map $\boldsymbol{X}=\boldsymbol{Y}[\bar{u}]$ to $\overline{\boldsymbol{X}[u]}$ by mapping $\boldsymbol{Y}$ identically, and mapping $\bar{u}$ to u; write $f:\boldsymbol{X}\longrightarrow\overline{\boldsymbol{X}[u]}$ for this map, $\varphi[u]$ is defined to be $f(\varphi)$, and $\Gamma[u]$ is defined to be $f(\Gamma)$. Once $\Gamma[u]$ is thus well-defined, we see that, with the data for the molecule Φ and for the atom $\varphi[\bar{u}]$ in (2.1) chosen in any way, (2.2) and (3) are true, simply because f is a map of ω-categories.

Moreover, the equalities (1), with d and c understood now in the sense of $\boldsymbol{X}[u]$ of course, are true too: clearly, since $\dim(\Lambda)=\dim(\Xi)\le n$, $\mathrm{d}(\Lambda\cdot\varphi[u]\cdot\Xi) = \Lambda\cdot\mathrm{d}(\varphi[u])\cdot\Xi$; and by (2.2), $\mathrm{d}(\varphi[u]) = \varphi[\mathrm{d}u]$, etc.

Expected facts such as $(b\cdot\Gamma)[u/\bar{u}] = b\cdot(\Gamma[u/\bar{u}])$ ($b\in\boldsymbol{X}_n$, $\Gamma\in\boldsymbol{X}_{n+1}$, u and $\bar{u}$ as before) are immediate from (2.2) and (3).

There are certain obvious commutativity and associativity rules concerning substitution, including up-substitution; they follow directly from the universal-property-induced definition of substitution; we tend to use them without comment.

Pasting preschemes

Let $\boldsymbol{X}$ be a computad, $\mathbf{N}$ a finite set of (n+1)-indets in $\boldsymbol{X}$. As in section 9 of [M], we associate with the elements of $\mathbf{N}$ appropriate *new* n-indets in a bijective manner: we have a map $(u\in\mathbf{N})\longmapsto\bar{u}$ such that $\bar{u}$ is an n-indet, $\bar{u}\notin|\boldsymbol{X}|$, and $\mathrm{d}\bar{u}=\mathrm{dd}u$, $\mathrm{c}\bar{u}=\mathrm{cc}u$. We write $\bar{\mathbf{N}}$ for the set $\{\bar{u}:u\in\mathbf{N}\}$; $\bar{\mathbf{N}}$ has the obvious attachment to $\boldsymbol{X}$; we can consider the computad $\boldsymbol{X}[\bar{\mathbf{N}}]$.

The elements of $\bar{\mathbf{N}}$ are called *niches*. They are to be distinguished from the other

n-indeterminates in $\boldsymbol{X}$. The set of niches in a pd Γ is written $\mathbf{n}(\Gamma)$.

For a subset S of $\mathbf{N}$, $\bar{S}$ (of course) denotes $\{\bar{u}: u \in S\}$.

Suppose that we have an irreflexive partial order $\prec$ on $\mathbf{N}$.

Let S be an (unsigned) span of $(\mathbf{N}, \prec)$, that is, a maximal $\prec$-antichain.

Let Γ be an n-(dimensional) pd in $\boldsymbol{X}[\bar{\mathbf{N}}]$. We call Γ an S-*frame* if $\mathbf{n}(\Gamma) = \bar{S}$ and every $\bar{u} \in \bar{S}$ occurs in Γ exactly once.

Let Γ be an S-frame. Consider any signing of S: a partition $S = \underline{S} \dot{\cup} \bar{S}$; let's write $\tilde{S}$ for the signed span so obtained. For $u \in \mathbf{N}$, let's write $\partial^{\tilde{S}} u = \mathrm{d}u$ if $u \in \underline{S}$, and $\partial^{\tilde{S}} u = \mathrm{c}u$ if $u \in \bar{S}$.

Let R be a subset of S; of course, R is necessarily a $\prec$-antichain. We are going to define an n-pd, denoted by $\Gamma\langle\tilde{S}\rangle[\bar{R}]$, in the computad $\boldsymbol{X}[\bar{\mathbf{N}}]$; it will be obtained by "partially filling" the frame Γ. The notation indicates that we will have $\mathbf{n}(\Gamma\langle\tilde{S}\rangle[\bar{R}]) = \bar{R}$.

Define $\Gamma\langle\tilde{S}\rangle[\bar{R}]$ by:

$$\Gamma\langle\tilde{S}\rangle[\bar{R}] \overset{\mathrm{DEF}}{=} \Gamma[\partial^{\tilde{S}} u/\bar{u}]_{u \in S-R}$$

(repeated, or simultaneous, substitution; it is legitimate since $\partial u \| \bar{u}$).

We have

$$Q \subseteq R \implies \Gamma\langle\tilde{S}\rangle[\bar{Q}] = \Gamma\langle\tilde{S}\rangle[\bar{R}][\partial^{\tilde{S}} u/\bar{u}]_{u \in R-Q} \tag{4}$$

When $R = \emptyset$, we omit it from the notation: $\Gamma\langle\tilde{S}\rangle[\emptyset] = \Gamma\langle\tilde{S}\rangle = \Gamma[\partial^{\tilde{S}} u/\bar{u}]_{u \in S}$; $\Gamma\langle\tilde{S}\rangle$ is an n-pd. The n-pds of the form $\Gamma\langle\tilde{S}\rangle$ are called the n-*cuts*, or the *hyperplanes*, of the PPS $\vec{\Gamma}$.

We are ready to define the notion of an $(n+1)$-*dimensional pasting prescheme*, $(n+1)$-PPS. It is an object of the form $\Theta = (\boldsymbol{Y}, \mathbf{N}, \prec, \Theta\langle S\rangle)_{S\ \prec\text{-span}}$; it is given by

a n-dimensional computad $\boldsymbol{Y}$;
a finite set $\mathbf{N}$ of $(n+1)$-indeterminates attached to $\boldsymbol{Y}$; we write $\boldsymbol{X} = \boldsymbol{Y}[\mathbf{N}]$;
an irreflexive partial order $\prec$ on the set $\mathbf{N}$;

and

for each span (maximal $\prec$-antichain) S in $\mathbf{N}$, an S-frame $\Theta\langle S\rangle$, an n-pd in the computad $\mathbf{Y}[\bar{\mathbf{N}}]$;

such that the following condition, the "*matching equality*", is satisfied:

$$C[\tilde{S}] = C[\tilde{T}] \implies \Theta\langle\tilde{S}\rangle[\overline{S\cap T}]=\Theta\langle\tilde{T}\rangle[\overline{S\cap T}] \ . \tag{5}$$

(we have abbreviated $\Theta\langle S\rangle\langle\tilde{S}\rangle$ as $\Theta\langle\tilde{S}\rangle$; and similarly for T).

As a consequence, we will have

$$R\subseteq S \ \& \ R\subseteq T \ \& \ C[\tilde{S}] = C[\tilde{T}] \implies \Theta\langle\tilde{S}\rangle[\bar{R}]=\Theta\langle\tilde{T}\rangle[\bar{R}] \ ; \tag{5'}$$

this is because of (4): note that if $C[\tilde{S}] = C[\tilde{T}]$, then for any $u\in S\cap T$, $\partial^{\tilde{S}}u=\partial^{\tilde{T}}u$.

Let's note that the case of (5') when R is a maximal antichain is vacuously true: the antecedent of (5') implies that $R=T$ and $\tilde{S}=\tilde{T}$.

Using the matching equalities, we can define, for a cut C of $(\mathbf{N},\prec)$ and R an $\prec$-antichain in the boundary $B[C]$ of C , the quantity $\Theta\langle C\rangle[\bar{R}]$ by

$$\Theta\langle C\rangle[\bar{R}] \overset{\mathrm{DEF}}{=} \Theta\langle\tilde{S}\rangle[\bar{R}]$$

for some/any signed span $\tilde{S}$ such that $C=C[\tilde{S}]$ and R a subset of S , the underlying span of $\tilde{S}$.

We write $\Theta\langle C\rangle$ for $\Theta\langle C\rangle[\emptyset]$. $\Theta\langle C\rangle = \Theta\langle\tilde{S}\rangle$ for any signed span $\tilde{S}$ defining C .

[Before proceeding, let us (again) adopt the following notational conventions. C , $\tilde{C}$, D denote cuts in $(\mathbf{N},\prec)$; $C=(U,L)$, $\tilde{C}=(\tilde{U},\tilde{L})$, $D=(V,M)$; $B.\ \tilde{B}$, E are the borders of C , $\tilde{C}$ and D , respectively.]

There is a further equality implied by the logic of the situation.

Suppose R is a $\prec$-antichain, C , D cuts. Let us say that $C\equiv D\ (\mathrm{mod}\ R)$ if R is a subset of the boundaries of both cuts, and the cuts coincide outside R : $R\subseteq B$, $R\subseteq E$ and $U-R=V-R$ (and, equivalently, $L-R=M-R$). Note the special case when R is maximal: in this case, if both $R\subseteq B$ and $R\subseteq E$, we always have $C\equiv D\ (\mathrm{mod}\ R)$.

We claim that

9.1 Lemma $C\equiv D\ (\mathrm{mod}\ R)$ implies $\Theta\langle C\rangle[\bar{R}] = \Theta\langle D\rangle[\bar{R}]$.

Proof To see this, first note

9.1.1 Sublemma $C\equiv D\ (\mathrm{mod}\ R)$ implies that, for any x such that $R\cup\{x\}$ is (still) a $\prec$-antichain, $x\in B$ iff $x\in E$.

As we know

$$x\in B \iff \forall y\in\mathbf{N}.[(y\prec x \Rightarrow y\in U)\ \&\ (y\succ x \Rightarrow y\in L)]\ , \tag{5.1}$$
$$x\in E \iff \forall y\in\mathbf{N}.[(y\prec x \Rightarrow y\in V)\ \&\ (y\succ x \Rightarrow y\in M)]\ . \tag{5.2}$$

Assume $C\equiv D\ (\mathrm{mod}\ R)$, $R\cup\{x\}$ is a $\prec$-antichain and $x\in B$, to show $x\in E$. If $x\in R$, the assertion is true since $R\subseteq E$. Assume $x\in\mathbf{N}-R$. The RHS of (5.1) holds. Then the RHS of (5.2) holds, since if either $y\prec x$ or $y\succ x$, we must have $y\notin R$ ($R\cup\{x\}$ is a $\prec$-antichain), thus, $y\in U$ iff $y\in V$, and $y\in L$ iff $y\in M$. (**End of proof of 9.1.1**)

Assume R is an antichain and $C\equiv D\ (\mathrm{mod}\ R)$. Let S be any maximal $\prec$-antichain such that $R\subseteq S\subseteq B$. By 9.1.1, $R\subseteq S\subseteq E$. Let $\tilde{\tilde{S}}$ and $\tilde{\tilde{T}}$ be the signed spans with underlying span S such that $\tilde{\tilde{S}}$ defines C, $\tilde{\tilde{T}}$ defines D: to each element of S give the sign according to where it lies with respect to the cuts. Since $C\equiv D\ (\mathrm{mod}\ R)$, the signing of each $u\in S-R$ is the same in $\tilde{\tilde{S}}$ as in $\tilde{\tilde{T}}$, and thus $\partial^{\tilde{\tilde{S}}}u=\partial^{\tilde{\tilde{T}}}u$ for $u\in S-R$. Looking at the formulas for $\Theta\langle S\rangle\langle\tilde{\tilde{S}}\rangle[\bar{R}]$ and $\Theta\langle S\rangle\langle\tilde{\tilde{T}}\rangle[\bar{R}]$, we now see that these two quantities are equal. This means precisely that $\Theta\langle C\rangle[\bar{R}]=\Theta\langle D\rangle[\bar{R}]$.

The following is an elementary fact, properly belonging to section 3.

9.2 Lemma Let $\prec$ be an irreflexive partial order on the set $\mathbf{N}$. Let S be a maximal $\prec$-antichain, and let $\tilde{\tilde{S}}_1$, $\tilde{\tilde{S}}_2$ be signed spans, both with underlying span S, given by the partitions $S=\underline{S}_1\dot{\cup}\bar{S}_1=\underline{S}_2\dot{\cup}\bar{S}_2$, and let $C_i=(U_i, L_i)$ be the cut determined by $\tilde{\tilde{S}}_i$. Then $C_1\equiv C_2\ (\mathrm{mod}\ S)$.

Proof Immediate from the equivalences

$$u\in U_i \iff u\in\underline{S}_i\ \dot{\vee}\ \exists s\in S.u\prec s\ ,$$
$$u\in L_i \iff u\in\bar{S}_i\ \dot{\vee}\ \exists s\in S.u\succ s\ .$$

Note that an $(n+1)$-PPS is entirely given within the context of the n-computad $\boldsymbol{Y}$. Although there are references in it to the n-indets $\bar{u}$, which are not in $\boldsymbol{Y}$, they are purely formal, and derived from the $(n-1)$-dimensional information inherent in the $(n+1)$-indets $u\in\mathbf{N}$. In the definition of the $\Theta\langle\tilde{\tilde{S}}\rangle$, we use the n-dimensional information given in the $u\in\mathbf{N}$; that information is part of $\boldsymbol{Y}$.

On the other hand, an $(n+1)$-PPS's can be *pasted* to obtain a well-defined $(n+1)$-pd in the computad $\boldsymbol{Y}[\mathbf{N}]$, the *pasting* of the PPS. This, of course, is the justification of the expression "pasting scheme". We will arrive at the pasting of the pasting prescheme in due course.

Examples and constructions of PPS's

It is easy -- in fact, "too easy"; this is the reason for the "pre" in "prescheme"; see also below -- to give examples for PPS's.

[1] Molecules are PPS's

The simplest ones are the top-separated molecules. In fact, in the next few paragraphs, we realize that an $(n+1)$-PPS whose underlying "backbone" order $\prec$ is a total order is the same thing as a top-separated $(n+1)$-molecule.

Let's start with a total order $\prec$ on the set $\mathbf{N}$ of N distinct $(n+1)$-indets, with an enumeration $\mathbf{N}=\{u_1, ..., u_N\}$ chosen so that $u_i \prec u_j \Longleftrightarrow i<j$. We have the following:

There are $N+1$ $\prec$-cuts $C_k=(U_k, L_k)$, $U_k=\{u_i : 1\le i\le k\}$, $L_k=\{u_i : k<i\le N\}$, $k=0, ..., N$, and N spans $S_k=\{u_k\}$, $k=1, ..., N$;

each span S_k has two signed versions S^ℓ_k and S^{u}_k : $\underline{S}^\ell_k=\emptyset$, $\bar{S}^\ell_k=\{u_k\}$, $\underline{S}^{\mathrm{u}}_k=\{u_k\}$, $\bar{S}^{\mathrm{u}}_k=\emptyset$;

S^ℓ_k defines the cut C_{k-1} , S^{u}_k the cut C_k ; the pairs of signed spans defining the same cut are $(S^{\mathrm{u}}_k, S^\ell_{k+1})$, corresponding to the cuts C_k , $k=1, ..., N-1$ (all cuts except the top and the bottom).

Therefore, for a PPS Θ based on $(\mathbf{N}, \prec)$, the data will consist of frames $\Theta\langle S_k\rangle$ $(k=0, ..., N)$. The matching equalities for R a non-empty, necessarily singleton, $\prec$-antichain are vacuous (since they are maximal antichains); the remaining matching equalities are for $R=\emptyset$, and they are

$$\Theta\langle S^{\mathrm{u}}_k\rangle = \Theta\langle S^\ell_{k+1}\rangle \qquad\qquad (k=1, ..., N-1)$$

We have $C_{k-1}\equiv C_k\,(\mathrm{mod}\{u_k\})$; and these are the only non-equality instances of the relation $C\equiv D\,(\mathrm{mod}\, R\,)$.

What we have is precisely an $(n+1)$-molecule

$$\Phi = (\varphi_1[u_1], ..., \varphi_N[u_N])$$

where

$$\varphi_k[u_k] = \Theta\langle S_k\rangle[u_k/\bar{u}_k] = \Theta\langle S_k\rangle[u_k] = \Theta\langle C_{k-1}\rangle[u_k] = \Theta\langle C_k\rangle[u_k] .$$

In terms of the atoms φ_k, the frames are

$$\Theta\langle S_k\rangle = \varphi_k[\bar{u}_k] \ .$$

We have

$$\Theta\langle S_k^{\ell}\rangle \ (= \Theta\langle S_k\rangle\langle S_k^{\ell}\rangle\ ) = \mathrm{d}(\varphi_k[u_k]) = \varphi_k[\mathrm{d}u_k]\ ;$$
$$\Theta\langle S_k^{\mathrm{u}}\rangle \ (= \Theta\langle S_k\rangle\langle S_k^{\mathrm{u}}\rangle\ ) = \mathrm{c}(\varphi_k[u_k]) = \varphi_k[\mathrm{c}u_k]\ ;$$

and the matching equalities become

$$(\mathrm{c}\varphi_k =)\ \varphi_k[\mathrm{c}u_k] = \varphi_{k+1}[\mathrm{d}u_{k+1}]\ (= \mathrm{d}\varphi_{k+1}\ )\ ,$$

which are the defining property of a molecule.

[2] Planar PPS's are PPS's

The planar pasting preschemes of sections 4 and 5 also give rise to pasting preschemes, in this case 2-dimensional ones.

Let $(\mathbf{N}, \prec, \rightarrow, \mathbf{M}, \mathbf{P}, \vec{S})$ be a planar pasting prescheme (see section 4). Let S be any maximal $\prec$-antichain in $\mathbf{N}$. S is a maximal $\rightarrow$-chain, pictured as

$$-\infty = u_0 \rightarrow !\, u_1 \rightarrow !\, u_2 \rightarrow !\ ...\ \rightarrow !\, u_{m-1} \rightarrow !\, u_m = \infty\ ;$$

(S is the set $\{u_1, ..., u_{m-1}\}$; u_0 and u_m are mere symbols.)

For each $k=0, ..., m-1$, for $x=u_k$, $y=u_{k+1}$, and the empty interval $(x, y)_{\rightarrow}$, we have the 1-pd $S_y^x \overset{\mathrm{DEF}}{=} (S_y^x)^C$ given in the planar pasting scheme, with the unique cut C in $(x, y)_{\rightarrow}$ (thanks to the empty set!). For our new notation for PPS's, we define the S-frame $\Theta\langle S\rangle$ by

$$\Theta\langle S\rangle \overset{\mathrm{DEF}}{=} S_{u_1}^{u_0}\cdot\bar{u}_1\cdot S_{u_2}^{u_1}\cdot\bar{u}_2\cdot\ ...\ \cdot\bar{u}_{m-2}\cdot S_{u_{m-1}}^{u_{m-2}}\cdot\bar{u}_{m-1}\cdot S_{u_m}^{u_{m-1}}\ . \tag{6}$$

Note that under this definition, with any signed version $\xi=\tilde{S}$ of S, we obtain

$$\Theta\langle\xi\rangle = S_{u_1}^{u_0}\cdot\partial^{\xi}u_1\cdot S_{u_2}^{u_1}\cdot\partial^{\xi}u_2\cdot\ ...\ \cdot\partial^{\xi}u_{m-2}\cdot S_{u_{m-1}}^{u_{m-2}}\cdot\partial^{\xi}u_{m-1}\cdot S_{u_m}^{u_{m-1}}\ ,$$

a quantity that was denoted by $(S_{\infty}^{-\infty})^{\xi}$ in section 5 (after **(ii)**).

Moreover, for any u and v such that $u \to v$, for

$$u \to !v_1 \to !v_2 \to ! \ldots \to !v_\ell \to !v$$

a span in the interval $(u, v)_\to$ and ξ a signed span with underlying span $\{v_1, \ldots v_\ell\}$, the quantity

$$\Theta\langle\xi\rangle(u, v) \overset{\text{DEF}}{=} s^u_{v_1} \cdot \partial^\xi v_1 \cdot s^{v_1}_{v_2} \cdot \partial^\xi v_2 \cdot \ldots \cdot \partial^\xi v_{\ell-2} \cdot s^{v_{\ell-2}}_{v_{\ell-1}} \cdot \partial^\xi v_{\ell-1} \cdot s^{v_{\ell-1}}_v$$

equals $(s^u_v)^\xi$. Thus, if ξ and ζ define the same cut in $(u, v)_\to$, then, by the definition of "planar pasting prescheme", we have $\Theta\langle\xi\rangle(u, v) = \Theta\langle\zeta\rangle(u, v)$.

Now, let $R \subseteq S$, $R \subseteq T$, ξ, ζ signed spans with respective underlying spans S, T, and assume that $\xi \sim \zeta$ (ξ and ζ define the same cut). Let $\{w_1 \to w_2 \to \ldots \to w_m\}$ enumerate the set $S - R$. We have, directly by the definitions, that

$$\Theta\langle\xi\rangle[\bar{R}] = \Theta\langle\xi\rangle(-\infty, w_1) \cdot \partial^\xi w_1 \cdot \Theta\langle\xi\rangle(w_1, w_2) \cdot \ldots \cdot \partial^\xi w_m \cdot \Theta\langle\xi\rangle(w_m, \infty),$$
$$\Theta\langle\zeta\rangle[\bar{R}] = \Theta\langle\zeta\rangle(-\infty, w_1) \cdot \partial^\zeta w_1 \cdot \Theta\langle\zeta\rangle(w_1, w_2) \cdot \ldots \cdot \partial^\zeta w_m \cdot \Theta\langle\zeta\rangle(w_m, \infty);$$

hence, by the above, $\Theta\langle\xi\rangle[\bar{R}] = \Theta\langle\zeta\rangle[\bar{R}]$ (note that $\xi \restriction (w_j, w_{j+1}) \sim \zeta \restriction (w_j, w_{j+1})$ and $\partial^\xi w_j = \partial^\zeta w_j$). This shows that the matching equalities hold, and we indeed have a 2-PPS.

[3] Restriction by extending the backbone order

We construct a new PPS out of a given one by *restriction*.

Let $\prec, \lll$ be two irreflexive partial orders on the same finite set $\mathbf{N}$, $\lll$ extending $\prec$; $\prec \subseteq \lll$. Obviously, any antichain for $\lll$ is an antichain for $\prec$; any cut for $\lll$ is a cut for $\prec$. However, a maximal $\lll$-antichain is not necessarily a maximal $\prec$-antichain.

Let Θ denote an $(n+1)$-PPS; we use the notation developed above. Let $\lll$ be any (irreflexive) partial order on the set $\mathbf{N}$ *extending* $\prec$: $\prec \subseteq \lll$. Then one can *restrict* Θ to a new $(n+1)$-PPS denoted

$$\Theta\restriction\lll = (\mathbf{Y}, \mathbf{N}, \lll, (\Theta\restriction\lll)\langle R\rangle)_{R\ \lll\text{-span}};$$

the frames $(\Theta\restriction\lll)\langle R\rangle$ are defined as follows.

Let R be a maximal $\lll$-antichain, and choose an arbitrary signing $\tilde{R}$ of R, given by a partition $R = \underline{R} \dot\cup \bar{R}$. $\tilde{R}$ $\lll$-determines a $\lll$-cut C. But then, C is also a $\prec$-cut. Moreover, $B^{\lll}[C] \subseteq B^{\prec}[C]$; this is clear from (5.1), (5.20 , applied to both $\prec$ and $\lll$. Therefore, R is an $\prec$-antichain such that $R\subseteq B^{\prec}[C]$.

Since C is a $\prec$-cut, R is an $\prec$-antichain contained in $B^{\prec}[C]$, $\Theta\langle C\rangle[\bar{R}]$ is well-defined.

Moreover, if $\tilde{\hat{R}}_1$ and $\tilde{\hat{R}}_2$ are both signed $\ll$-spans with underlying $\ll$-span R , and C_i is the $\ll$-cut $\ll$-determined by $\tilde{\hat{R}}_i$, then $C_1\equiv C_2\ (\mathrm{mod}\ R)$ with reference to $(\mathbf{N},\ll)$: see 9.2, applied to $(\mathbf{N},\ll)$; since R is an $\prec$-antichain such that $R\subseteq B^{\prec}[C]$, we have that $C_1\equiv C_2\ (\mathrm{mod}\ R)$ holds with reference to $(\mathbf{N},\prec)$; by 9.1, $\Theta\langle C_1\rangle[\bar{R}]=\Theta\langle C_2\rangle[\bar{R}]$.

We conclude that, for any $\ll$-span R , the definition

$$(\Theta\restriction\ll)\langle R\rangle \overset{\mathrm{DEF}}{=} \Theta\langle C\rangle[\bar{R}]\ \ (C=C^{\ll}[\tilde{\hat{R}}]\ ,\ \tilde{\hat{R}}\ \text{signed}\ \ll\text{-span based on}\ R\ ) \qquad (7)$$

is unambiguous.

To verify the matching equalities, let us calculate the quantity $(\Theta\restriction\ll)\langle\tilde{\hat{R}}\rangle[\bar{Q}]$ provided $Q\subseteq R$, $\tilde{\hat{R}}$ signed span based on the $\ll$-span R . Let C be the $\ll$-cut $\ll$-defined by $\tilde{\hat{R}}$.

As we noted above, $B^{\ll}[C]\subseteq B^{\prec}[C]$: the $\ll$-border of C is contained in the $\prec$-border. Since $R\subseteq B^{\ll}[C]\subseteq B^{\prec}[C]$, we can take the maximal $\prec$-antichain S in $B^{\prec}[C]$ such that $R\subseteq S$. Let $\tilde{\hat{S}}$ be the signed version of S corresponding to C ; clearly, $\tilde{\hat{S}}$ extends $\tilde{\hat{R}}$. We have

$$\begin{aligned}(\Theta\restriction\ll)\langle\tilde{\hat{R}}\rangle[\bar{Q}] &= (\Theta\restriction\ll)\langle\tilde{\hat{R}}\rangle[\partial^{\tilde{\hat{R}}}u/\bar{u}]_{u\in R-Q} = \Theta\langle\tilde{\hat{S}}\rangle[\partial^{\tilde{\hat{S}}}v/\bar{v}]_{u\in S-R}[\partial^{\tilde{\hat{R}}}u/\bar{u}]_{u\in R-Q} = \\ &\overset{!}{=} \Theta\langle\tilde{\hat{S}}\rangle[\partial^{\tilde{\hat{S}}}v/\bar{v}]_{u\in S-Q} = \Theta\langle\tilde{\hat{S}}\rangle[\bar{Q}] = \Theta\langle C\rangle[\bar{Q}]\ .\end{aligned}$$

The $\ll$-version of (5'),

$$Q\subseteq R_1\ \&\ Q\subseteq R_2\ \&\ C^{\ll}[\tilde{\hat{R}}_1]=C^{\ll}[\tilde{\hat{R}}_2]=C \implies (\Theta\restriction\ll)\langle\tilde{\hat{R}}_1\rangle[\bar{R}]=(\Theta\restriction\ll)\langle\tilde{\hat{R}}_2\rangle[\bar{Q}]$$

is true, since both quantities in the last equality are equal to $\Theta\langle C\rangle[\bar{Q}]$.

The pasting of a pasting prescheme

Start with an $(n+1)$-PPS $\Theta=(\boldsymbol{Y},\mathbf{N},\prec,\Theta\langle S\rangle_{S\ \prec\text{-span}})$. Recall the operation $(\Theta,\ll)\mapsto\Theta\restriction\ll$ of restriction. Recall that we concluded that if $\ll$ is a total order extending $\prec$, then $\Theta\restriction\ll$ is, essentially, an $(n+1)$-molecule; let us write $\Theta\restriction\ll$ too for the $(n+1)$-molecule which is "essentially" the restriction $\Theta\restriction\ll$.

9.3 Proposition Let Θ be any $(n+1)$-PPS .

(i) Θ defines a unique $(n+1)$-pd $[\![\Theta]\!]$, the *pasting* of the pasting scheme Θ , by the formula $[\![\Theta]\!]=[\![\Theta\restriction\ll]\!]$ for some/any total order $\ll$ extending $\prec$.

(ii) We have $\mathrm{d}[\![\Theta]\!]=\Theta\langle C_{\mathrm{d}}\rangle=\Theta\langle C_{\mathrm{d}}\rangle[\emptyset]$, $\mathrm{c}[\![\Theta]\!]=\Gamma\langle C_{\mathrm{c}}\rangle$, where C_{d} , C_{c} are the cuts $C_{\mathrm{d}}=(\emptyset,\mathbf{N})$, $C_{\mathrm{c}}=(\mathbf{N},\emptyset)$ (these are equalities of n-pd's).

(iii) All frames $\Theta\langle S\rangle$ are parallel to each other: $\mathrm{d}\Gamma\langle S\rangle=\mathrm{dd}[\![\Theta]\!]$, $\mathrm{c}\Gamma\langle S\rangle=\mathrm{cc}[\![\Theta]\!]$ (the frames are n-pd's; we have equalities of $(n-1)$-pd's here).

(iv) For any partial order $\prec\!\prec$ extending $\prec$, the restriction $\Theta\restriction\prec\!\prec$ has the same value as Θ : $[\![\Theta]\!]=[\![\Theta\restriction\prec\!\prec]\!]$.

Proof We have to show that

$$[\![\Theta\restriction\ll]\!]=[\![\Theta\restriction\ll']\!] \tag{8?}$$

for any two total orders $\ll$, $\ll'$ extending $\prec$.

Let us say (again?) that $\ll$ *is switched to* $\ll'$ *at* (u,v) , in notation $\mathcal{S}_{u,v}(\ll,\ll')$, if $u\ll!v$ and

$$\ll'=\ll-\{(u,v)\}\,\dot\cup\,\{(v,u)\}\,. \tag{9}$$

Write $\mathcal{S}(\ll,\ll')$ for $\exists u,v.\mathcal{S}_{u,v}(\ll,\ll')$.

Note that if $\ll$ is a total order on $\mathbf{N}$ and $u\ll!v$, then (9) defines another total order on $\mathbf{N}$: there is unique $\ll'$ to which $\ll$ is switched at (u,v) . If $\mathcal{S}_{u,v}(\ll,\ll')$ then $\mathcal{S}_{v,u}(\ll',\ll)$. If $\mathcal{S}_{u,v}(\ll,\ll')$, $\ll$ is compatible with the partial order $\prec$, and u,v are incomparable in $\prec$, then $\ll'$ is also compatible with $\prec$. It is an elementary fact that for any two total orders $\ll$, $\ll'$ on $\mathbf{N}$, both compatible with $\prec$, there is a finite sequence

$$\ll\,=\,\ll_1\,,\ \ll_2\,,\ \ldots\,,\ \ll_m\,=\,\ll'$$

of total orders on $\mathbf{N}$, all compatible with $\prec$, such that $\mathcal{S}(\ll_k,\ll_{k+1})$ $(k=1,\ldots,m-1)$.

(Proof: define the distance of $\ll$ and $\ll'$, $\delta(\ll,\ll')$, as the number of pairs (u,v) such that $(u,v)\in\ll$ and $(v,u)\in\ll'$. Note that if $\mathcal{S}(\ll,\ll')$, then $\delta(\ll,\ll')=1$. Moreover, if $\delta(\ll,\ll')>0$, $u\ll!v$, $(v,u)\in\ll'$, and $\mathcal{S}_{u,v}(\ll,\ll'')$, then $\delta(\ll'',\ll')=\delta(\ll,\ll')-1$. Assume $\ll$, $\ll'$ are compatible with $\prec$, and $\delta(\ll,\ll')>0$, that is, $\ll\neq\ll'$. There must be (u,v) such that $u\ll!v$, $(v,u)\in\ll'$; otherwise, the list $u_1\ll!u_2\ll!\ldots\ll!u_N$ enumerating all of $\mathbf{N}$ in the order $\ll$ is also a correct order for $\ll'$, which then must be the total enumeration of $\mathbf{N}$ in the order $\ll'$, and $\ll'=\ll$. Take such (u,v) . Since both $\ll$ and $\ll'$ are compatible with $\prec$, we must have that $\neg(u\prec v)$ and $\neg(v\prec u)$; thus, for the $\ll'$ for which $\mathcal{S}_u(\ll,\ll'')$, $\ll''$ is also compatible with $\prec$. $\ll''$ is $\ll_2$ in the the sequence above; clearly, in fact, $m=\delta(\ll,\ll')+1$.)

Therefore, it is enough to show (8) when $\mathcal{S}_{u,\,v}(\ll,\ll')$; we assume this now. We have that u and v are $\prec$-incomparable.

Let $u_1,\ ...,\ u_N$ be the enumeration of $\mathbf{N}$ such that $u_j\ll u_k \Longleftrightarrow j<k$. Let i be such that $u=u_i$; then $v=u_{i+1}$.

We now put together what we have in [3] on a restriction $\Theta\restriction\ll$, and what we know from [1] about a PPS such as $\Theta\restriction\ll$ whose backbone-order is total.

The molecule $\Phi=\Theta\restriction\ll$ is of the form $\Phi=(\varphi_1[u_1],\ ...,\ \varphi_N[u_N])$, with the already fixed indexing u_i , $i=1,\ ...,\ N$. With any $w\in\mathbf{N}$, we use the notation $\varphi_w=\varphi_w[w] \stackrel{\text{DEF}}{=} \varphi_j[u_j]$ for the specific j such that $w=u_j$.

The molecule $\Phi'=\Theta\restriction\ll' = (\varphi'_1[u'_1],\ ...,\ \varphi'_N[u'_N])$ has $u'_j=u_j$ except for $j=i$ and $j=i+1$; $u'_i=u_{i+1}=v$, $u'_{i+1}=u_i=u$. $\varphi'_w=\varphi'_w[w] \stackrel{\text{DEF}}{=} \varphi'_j[u'_j]$ for j such that $u'_j=w$.

We have

$$\varphi_w[w] = \Theta\langle\ll^{\underline{w}}\rangle[w] = \Theta\langle\ll^{\overline{w}}\rangle[w] \quad ; \tag{10.1}$$

(we have dropped $\{\ ,\ \}$ in $\Theta\langle\ll^{\underline{w}}\rangle[\{w\}]$); here, the $\prec$-cut $\ll^{\underline{w}}=(U,\,L)$ is the one for which $z\in U \Longleftrightarrow z\underline{\ll}w$, $z\in L \Longleftrightarrow z\gg w$; and $\ll^{\overline{w}}=(V,\,M)$ has $z\in V \Longleftrightarrow z\ll w$, $z\in M \Longleftrightarrow z\underline{\gg}w$. Similarly,

$$\varphi'_w[w] = \Theta\langle\ll'^{\underline{w}}\rangle[w] = \Theta\langle\ll'^{\overline{w}}\rangle[w] \tag{10.2}$$

with the appropriate meaning for the cuts $\ll'^{\underline{w}}$, $\ll'^{\overline{w}}$.

Clearly, $\ll^{\underline{w}} = \ll'^{\underline{w}}$ and $\ll^{\overline{w}} = \ll'^{\overline{w}}$ unless $w=u$ or $w=v$. Therefore,

$$\varphi'_w[w]=\varphi_w[w] \text{ unless } w=u \text{ or } w=v\ . \tag{11}$$

Let us denote $\ll^{\underline{u}} = \ll^{\overline{v}}$ by $C=(U,\,L)$, $\ll'^{\underline{v}} = \ll'^{\overline{u}}$ by $D=(V,\,M)$. They are both $\prec$-cuts. To repeat, we have

$$U = \{x:x\underline{\ll}u\} = \{x:x\ll v\}\ ,$$
$$V = \{x:x\underline{\ll}'v\} = \{x:x\ll' u\}\ ,$$

$$L = \{y : y \underline{\gg} v\} = \{y : y \gg u\} ,$$
$$M = \{y : y \underline{\gg}' v\} = \{y : y \gg' u\} \; .$$

C and D are obtained from each other by "switching" u and v :

$$V = U - \{u\} \cup \{v\} , \; M = L - \{v\} \cup \{u\} ,$$
$$U = V - \{v\} \cup \{u\} , \; L = M - \{u\} \cup \{v\} \; .$$

u and v are on the border of both C and D .

By (10.1) and (10.2),

$$\varphi_u[u] = \Theta\langle C\rangle[u] \; , \qquad \varphi_v[v] = \Theta\langle C\rangle[u] \; ,$$

$$\varphi'_v[v] = \Theta\langle D\rangle[v] \; , \qquad \varphi'_u[u] = \Theta\langle D\rangle[u] \; .$$

Let S be a span (maximal $\prec$-antichain in $\mathbf{N}$) such that S is a subset of the border of C , and S contains both u and v (see section 3). Let $\tilde{S}$ be the signed span with underlying span S that defines C : $\underline{S}=S\cap U$, $\bar{S}=S\cap L$. Of course, $u\in\underline{S}$, $v\in\bar{S}$. We have (see section 3)

$$x\in U \iff x\in\underline{S} \mathbin{\dot\vee} \exists s\in S.\ x\prec s$$

$$x\in L \iff x\in\bar{S} \mathbin{\dot\vee} \exists s\in S.\ x\succ s \; .$$

Let us "switch" $\tilde{S}$ to $\tilde{S}'$ by putting $\overline{S'} = \bar{S}-\{u\}\cup\{v\}$, $\underline{S'} = \underline{S}-\{v\}\cup\{u\}$; $\tilde{S}'$ is a signed span with the same underlying span S ; $\tilde{S}$ and $\tilde{S}'$ are identical except on u and v , on which they are of opposite signs.

When we take the definition of the cut $C'=(U', L')$ defined by S' , we see that we get $C'=D$:

$$x\in V \iff x\in\underline{S}' \mathbin{\dot\vee} \exists s\in S.\ x\prec s$$

$$y\in M \iff y\in\bar{S}' \mathbin{\dot\vee} \exists s\in S.\ y\succ s \; .$$

The definitions tell us that

$$\varphi_u[u] = \Theta\langle S\rangle[\partial^{\tilde{S}}w/\bar{w}]_{w\in S-\{u\}} \; , \qquad \varphi_v[v] = \Theta\langle S\rangle[\partial^{\tilde{S}}w/\bar{w}]_{w\in S-\{v\}} \; , \tag{12.1}$$

$$\varphi'_v[v] = \Theta\langle S\rangle[\partial^{\tilde{S}'}w/\bar{w}]_{w\in S-\{v\}} \; , \qquad \varphi'_u[u] = \Theta\langle S\rangle[\partial^{\tilde{S}'}w/\bar{w}]_{w\in S-\{u\}} \; . \tag{12.2}$$

Consider the frame $\Theta\langle S\rangle$. We have an n-molecule $\Psi=(\psi_1[x_1], \ldots, \psi_M[x_M])$ such that $\Theta\langle S\rangle = [\![\Psi]\!]$; for every $s\in S$, $\bar{s}$ occurs as x_p for exactly one p ; and $\{x_1, \ldots x_M\}\cap\bar{\mathbf{N}}=\bar{S}$. In particular, we have unique q and r in $\{1, \ldots, M\}$ such that $x_q=\bar{u}$, $x_r=\bar{v}$.

We have two cases to distinguish: $q<r$ or $r<q$; the two, however, are mirror images of each other; thus we may assume $q<r$.

We have the n-atoms

$$\rho[\bar{u}] = \psi_q[x_q] \ ,$$
$$\sigma[\bar{v}] = \psi_r[x_r]$$

and the n-pd's

$$\Lambda_1 = \psi_1[x_1]\cdot \ldots \cdot\psi_{q-1}[x_{q-1}] \ ,$$
$$\Lambda_2 = \psi_{q+1}[w_{q+1}]\cdot \ldots \cdot\psi_{r-1}[w_{r-1}] \ ,$$
$$\Lambda_3 = \psi_{r+1}[w_{r+1}]\cdot \ldots \cdot\psi_M[w_M] \ ;$$

of course, any of the latter may be equal to an identity n-cell. Thus,

$$\Theta\langle S\rangle = \Lambda_1\cdot\rho[\bar{u}]\cdot\Lambda_2\cdot\sigma[\bar{v}]\cdot\Lambda_3 \ .$$

$\bar{u}$ and $\bar{v}$ do not occur in the n-pd's Λ_1 , Λ_2 and Λ_3 . Let, for $i=1, 2, 3$,

$$\hat{\Lambda}_i = \Lambda_i[\partial^{\tilde{\hat{S}}}w/\bar{w}]_{w\in S-\{u, v\}} = \Lambda_i[\partial^{\tilde{\hat{S}}'}w/\bar{w}]_{w\in S-\{u, v\}} \ .$$

Then, by (12.1) and (12.2), since $\partial^{\tilde{\hat{S}}}u=\mathrm{c}u$, $\partial^{\tilde{\hat{S}}}v=\mathrm{d}v$, $\partial^{\tilde{\hat{S}}'}u=\mathrm{d}u$, $\partial^{\tilde{\hat{S}}'}v=\mathrm{c}v$, we have

$$\varphi_u[u] = \hat{\Lambda}_1\cdot\rho[u]\cdot\hat{\Lambda}_2\cdot\sigma[\mathrm{d}v]\cdot\hat{\Lambda}_3 \ , \qquad \varphi_v[v] = \hat{\Lambda}_1\cdot\rho[\mathrm{c}u]\cdot\hat{\Lambda}_2\cdot\sigma[v]\cdot\hat{\Lambda}_3 \ ,$$

$$\varphi'_v[v] = \hat{\Lambda}_1\cdot\rho[\mathrm{d}u]\cdot\hat{\Lambda}_2\cdot\sigma[v]\cdot\hat{\Lambda}_3 \ , \qquad \varphi'_u[u] = \hat{\Lambda}_1\cdot\rho[u]\cdot\hat{\Lambda}_2\cdot\sigma[\mathrm{c}v]\cdot\hat{\Lambda}_3 \ .$$

Let $\alpha \overset{\mathrm{DEF}}{=} \hat{\Lambda}_1\cdot\rho[u]\cdot\hat{\Lambda}_2$, and $\beta \overset{\mathrm{DEF}}{=} \sigma[v]\cdot\hat{\Lambda}_3$. Then $\mathrm{d}\alpha = \hat{\Lambda}_1\cdot\rho[\mathrm{d}u]\cdot\hat{\Lambda}_2$, and similar equalities hold for $\mathrm{c}\alpha$, $\mathrm{d}\beta$, $\mathrm{c}\beta$. We obtain that

$$\varphi_u[u] = \alpha\cdot\mathrm{d}\beta\ , \ \ \varphi_v[v] = \mathrm{c}\alpha\cdot\beta\ , \ \ \varphi'_v[v] = \mathrm{d}\alpha\cdot\beta\ , \ \ \varphi'_u[u] = \alpha\cdot\mathrm{c}\beta\ ,$$

and the equality

$$\varphi_U[u] \cdot \varphi_V[v] = \varphi'_V[v] \cdot \varphi'_U[u] \tag{13}$$

becomes an instance of the commutative law

$$(\alpha \cdot \mathrm{d}\beta) \cdot (\mathrm{c}\alpha \cdot \beta) = (\mathrm{d}\alpha \cdot \beta) \cdot (\alpha \cdot \mathrm{c}\beta) \tag{14}$$

(whose precondition, $cc\alpha = dd\beta$, holds as a consequence of the fact that the two sides of the equality (14) are known to be well-defined).

(11) and (13) tell us that $[\![\Phi]\!] = [\![\Phi']\!]$, and therefore that (8?) is indeed true.

This completes the proof of 9.3(i).

9.3(ii) and (iii) are consequences of (i). 9.3(iv) is obvious.

(**Constructions of PPS's, continued**)

[4] Slices of a PPS

There is another way of restricting any given $(n+1)$-PPS $\Theta = (\boldsymbol{Y}, \mathbf{N}, \prec, \Theta\langle S\rangle)_S$, this time to the slice of a slicing (C_1, C_2) of $(\mathbf{N}, \prec)$ (see "Convex sets" in section 3). The result depends not only on the slice, however, but also on the slicing itself. The result is called the (C_1, C_2)-*slice* of Θ, and it is denoted $\Theta\restriction(C_1, C_2)$. The set of $(n+1)$-indets of $\Theta\restriction(C_1, C_2)$ will, as expected, be the slice $P(C_1, C_2)$ of $(\mathbf{N}, \prec)$.

As particular cases of the construction, we will be able to restrict Θ to any upward closed set U, and to any downward closed set L; letting $C=(U, L)$, $C_{\mathrm{min}}=(\emptyset, \mathbf{N})$, $C_{\mathrm{max}}=(\mathbf{N}, \emptyset)$, we will have $\Theta\restriction U = \Theta\restriction(C_{\mathrm{min}}, C)$, $\Theta\restriction L = \Theta\restriction(C, C_{\mathrm{max}})$.

Let (C_1, C_2) be a slicing in $(\mathbf{N}, \prec)$, $P = P(C_1, C_2) = L_1 \cap U_2$ the corresponding slice.

Let R be a maximal antichain in $(P, \prec\restriction P)$. Let $\tilde{D} = (\tilde{V}, \tilde{M})$ be any cut in $(P, \prec\restriction P)$ such that $R \subseteq \tilde{E} = B[\tilde{D}]$; any two such $\tilde{D}$ are $\equiv (\mathrm{mod}\ R)$ in $(P, \prec\restriction P)$. Define $V = U_1 \dot{\cup} \tilde{V}$, $M = \tilde{M} \dot{\cup} L_2$. Then V is up-closed, M is down-closed in $\mathbf{N}$, and $V \dot{\cup} M = \mathbf{N}$; $D = (V, M)$ is a cut in $(\mathbf{N}, \prec)$. Any two such D obtained from R are $\equiv (\mathrm{mod}\ R)$ in $(\mathbf{N}, \prec)$.

R is a subset of $E = B[D]$; indeed, if $u \prec r \in R$, then we must have $u \in V$, and if $v \succ r \in R$, then $v \in M$. Therefore, the expression $\Theta\langle D\rangle[R]$ is well-defined.

Abbreviating $\Psi = \Theta\restriction(C_1, C_2)$, we define

$$\Psi\langle R\rangle \overset{\mathrm{DEF}}{=} \Theta\langle D\rangle[R] .$$

In this formula, D is determined up to $\equiv (\mathrm{mod}\ R)$; therefore, $\Psi\langle R\rangle$ is unambiguously defined.

As to the matching equality, let $\tilde{\hat{R}}_1$ and $\tilde{\hat{R}}_2$ be two signed $\prec$-spans in $(P, \prec\restriction P)$ with respective underlying spans R_1 and R_2, $\tilde{\hat{R}}_1$ and $\tilde{\hat{R}}_2$ defining the same cut $\tilde{\hat{D}}$ in $(P, \prec\restriction P)$, and let $Q=R_1\cap R_2$. Let $\tilde{\hat{S}}_1$ and $\tilde{\hat{S}}_2$ be signed spans containing $\tilde{\hat{R}}_1$ and $\tilde{\hat{R}}_2$, respectively, defining D, where D is derived from $\tilde{\hat{D}}$ as above. Then

$$\Psi\langle\tilde{\hat{R}}_i\rangle[Q] = \Psi\langle R_i\rangle[\partial^{\tilde{\hat{R}}_i}u/\bar{u}]_{u\in R_i\text{-}Q} = \Theta\langle\tilde{\hat{S}}_i\rangle[R][\partial^{\tilde{\hat{R}}_i}u/\bar{u}]_{u\in R_i\text{-}Q} =$$

$$= \Theta\langle\tilde{\hat{S}}_i\rangle[\partial^{\tilde{\hat{S}}_i}v/\bar{v}]_{v\in S_i\text{-}R}[\partial^{\tilde{\hat{R}}_i}u/\bar{u}]_{u\in R_i\text{-}Q} \underset{\substack{\uparrow \\ \partial^{\tilde{\hat{R}}_i}u=\partial^{\tilde{\hat{S}}_i}u}}{=} \Theta\langle\tilde{\hat{S}}_i\rangle[\partial^{\tilde{\hat{S}}_i}v/\bar{v}]_{v\in S_i\text{-}Q}$$

$$= \Theta\langle\tilde{\hat{S}}_i\rangle[Q] = \Theta\langle D\rangle[Q] \quad ,$$

independently from $i=1, 2$. This shows what we want.

Furthermore, we have the following expected equalities:

$$(\Theta\restriction U_2)\restriction U_1 = \Theta\restriction U_1 \ ,$$
$$(\Theta\restriction L_1)\restriction L_2 = \Theta\restriction L_2 \ ,$$
$$\Theta\restriction(C_1, C_2) = (\Theta\restriction U_2)\restriction(L_1\cap U_2) = (\Theta\restriction L_1)\restriction(L_1\cap U_2)$$

($L_1\cap U_2$ is up-closed in U_2, down-closed in L_1. However, as we said, it is not possible to write $\Theta\restriction(L_1\cap U_2)$, since for a general convex set P, $\Theta\restriction P$ is not defined unambiguously);

$$(\Theta\restriction\lll)\restriction(C_1, C_2) = (\Theta\restriction(C_1, C_2))\restriction(\lll\restriction(L_1\cap U_2))$$

where $\lll\supseteq\prec$, and $C_1\leq C_2$ are cuts for $(\mathbf{N}, \lll)$ (and, hence, for $(\mathbf{N}, \prec)$ as well).

Moreover,

$$\mathrm{d}[\![\Theta\restriction(C_1, C_2)]\!] = \Theta\langle C_1\rangle \ ,$$
$$\mathrm{c}[\![\Theta\restriction(C_1, C_2)]\!] = \Theta\langle C_2\rangle \ ,$$
$$[\![\Theta]\!] = [\![\Theta\restriction U]\!]\cdot[\![\Theta\restriction L]\!] \qquad (15.1)$$
$$\text{(} (U, L) \text{ is a cut in } (\mathbf{N}, \prec) \text{)}$$
$$[\![\Theta]\!] = [\![\Theta\restriction U_1]\!]\cdot[\![\Theta\restriction(C_1, C_2)]\!]\cdot[\![\Theta\restriction L_2]\!]$$
$$[\![\Theta]\!] = \Theta\langle\emptyset\rangle \qquad \text{when } \mathbf{N}=\emptyset \ ; \qquad (15.2)$$
$$[\![\Theta]\!] = \Theta\langle\{u\}\rangle[u/\bar{u}] \qquad \text{when } \mathbf{N}=\{u\} \ . \qquad (15.3)$$

9.4 Proposition Under the indicated definitions of $\Theta{\restriction}U$, $\Theta{\restriction}L$ (U upclosed, L downclosed), but with the definition of $\llbracket\text{-}\rrbracket$ for PPS's removed, the equalities (15.1), (15.2), (15.3) define a unique evaluation operation $\Theta\mapsto\llbracket\Theta\rrbracket$ from $(n+1)$-PPS's to $(n+1)$-pds.

[5] Substitution of PPS's

The most interesting construction of a new PPS is by *substituting* one into another.

Let $\Theta=(\boldsymbol{Y}, \mathbf{N}, \prec, \Theta\langle S\rangle)_{S\ \prec\text{-span}}$ be an $(n+1)$-PPS; $\Psi=(\boldsymbol{Y}, \mathbf{P}, \lll, \Psi\langle R\rangle)_{R\ \lll\text{-span}}$ another one. Let $u\in\mathbf{N}$ be a particular indet; we are going to substitute Ψ for u in Θ. We assume the *framing condition*: $\mathrm{d}\llbracket\Psi\rrbracket=\mathrm{d}u$, $\mathrm{c}\llbracket\Psi\rrbracket=\mathrm{c}u$ (for $\mathrm{d}\llbracket\Psi\rrbracket$ and $\mathrm{c}\llbracket\Psi\rrbracket$, see 9.3(ii) and 9.3(iii)).

We will define the PPS $\Xi=\Theta[\Psi/u]=(\boldsymbol{Y}, \mathbf{Q}, \prec\!\prec, \Xi\langle Q\rangle)_{Q\ \prec\prec\text{-span}}$.

$$\mathbf{Q} \overset{\text{DEF}}{=} \mathbf{N}-\{u\}\ \dot\cup\ \mathbf{P}$$; we are assuming that $\mathbf{N}$ and $\mathbf{P}$ are disjoint.

For $v\in\mathbf{Q}$, $x\in\mathbf{Q}$:

$$\begin{array}{rl}(v,x)\in\prec\!\prec\ \Longleftrightarrow & v,x\in\mathbf{N}-\{u\}\ \&\ v\prec x\\ & \vee\ v,x\in\mathbf{P}\ \&\ v\prec\!\prec x\\ & \vee\ v\in\mathbf{N}-\{u\}\ \&\ x\in\mathbf{P}\ \&\ v\prec u\ (!)\\ & \ \ v\in\mathbf{P}\ \&\ x\in\mathbf{N}-\{u\}\ \&\ u\prec x\ .\end{array}$$

In other words, two elements of $\mathbf{N}-\{u\}$ are related in $\prec\!\prec$ as they are in $\prec$; two elements of $\mathbf{P}$ as they are in $\lll$; and an element of $\mathbf{N}-\{u\}$ and another one of $\mathbf{P}$ are related as the first is related to the fixed "slot" u in $\prec$. It is clear that $\prec\!\prec$ is a partial order on $\mathbf{Q}$.

As for cuts, we have a similar "substitutional" situation. A cut $C=(U, L)$ of $(\mathbf{Q}, \prec\!\prec)$ may be of two kinds. A cut $C=(U, L)$ of $(\mathbf{Q}, \prec\!\prec)$, is given as follows:

either there is a cut $D=(V, M)$ of $(\mathbf{N}, \prec)$ and we have $u\in V$ & $U=(V-\{u\})\dot\cup\mathbf{P}$ & $L=M$ *or* $u\in M$ & $U=V$ & $L=(M-\{u\})\dot\cup\mathbf{P}$ [" C does not cut through $\mathbf{P}$ "]
or there are: a cut $D=(V, M)$ of $(\mathbf{N}, \prec)$ and a cut $E=(W, N)$ of $(\mathbf{P}, \lll)$ such that u belongs to the border B of D, and we have $U=(V-\{u\})\dot\cup W$, $L=(M-\{u\})\dot\cup N$. (The two cases are not exclusive of each other: the one as in the first case is obtained in the second case only if $u\in B$; in that case, we take $W=\emptyset$ when $u\in M$, and $N=\emptyset$ when $u\in V$.)

Similarly, a span S of $(\mathbf{Q}, \prec\!\prec)$ is either a span in $(\mathbf{N}, \prec)$ such that $u\notin S$ (**case 1**) or $S=Q\dot\cup R$, where $Q\dot\cup\{u\}$ is a span in $(\mathbf{N}, \prec)$, and R is a span in $(\mathbf{P}, \lll)$ (**case 2**).

In **case 1**, $\Xi\langle S\rangle \overset{\text{DEF}}{=} \Theta\langle S\rangle$; in **case 2** , $\Xi\langle S\rangle \overset{\text{DEF}}{=} \Theta\langle Q\dot\cup\{u\}\rangle[\Lambda\langle R\rangle/\bar{u}]$.

9.5 Proposition The substitution operation is well-defined. Moreover, $\llbracket\Theta[\Psi/u]\rrbracket = \llbracket\Theta\rrbracket[\llbracket\Psi\rrbracket/u]$.

The **proof** is left to the reader.

The name "pasting scheme" should be given to a pasting prescheme Θ only when Θ is "complete" in some sense. Let me say that a pasting prescheme Θ is *complete*, and that is a *pasting scheme*, if every molecule Φ which represents the value of Θ, $[\![\Phi]\!]=[\![\Theta]\!]$, appears as one of the restrictions $\Theta\restriction\lll$, for a total order $\lll$ extending the backbone order $\prec$ of Θ. (Note that since $\lll$ is the order of indeterminates in $\Theta\restriction\lll$, for two different $\lll$ and $\lll'$, $\Theta\restriction\lll$ and $\Theta\restriction\lll'$ are different; there can be at most one $\lll$ such that $\Theta\restriction\lll$ is a given Φ.)

A pd is *displayable* if there is a pasting scheme, a *display* of the pd, defining it; I don't know if the display is necessarily unique if it exists.

Note that, in particular, a top-separated pd Θ can be displayable only if Γ has unique factorization (see section 2).

Let us call the PPS Ψ an *expansion* of the PPS Θ if Θ is a restriction of Ψ : $\Theta=\Psi\restriction\prec$, where $\prec$ is the backbone order of Θ; Ψ is a *proper* expansion of Θ is $\Psi\neq\Theta$, that is $\prec\prec\subsetneqq\prec$, where $\prec\prec$ is the backbone order of Ψ.

Let us write $\Theta\blacktriangleleft\Psi$ to indicate that Ψ is a proper expansion of Θ.

Let's call a PPS Θ *maximal* if it has no proper expansion.

Note that "complete" implies "maximal", or, what is the same, "non-maximal" implies "non-complete": if $\prec\prec\subsetneqq\prec$, then there is at least one total extension $\lll$ of $\prec\prec$, which is not total extension of $\prec$, giving rise to a molecule representing $[\![\Psi]\!]$ which is not among the restrictions of Θ.

It is a triviality that every pd has at least one maximal PPS representing it: a chain of PPS's $\Theta_1\blacktriangleleft\Theta_2\blacktriangleleft\ldots\blacktriangleleft\Theta_k\blacktriangleleft\ldots$ induces a strictly decreasing sequence of partial orders $\prec_1\supset\prec_2\supset\ldots\supset\prec_k\supset\ldots$ on the fixed finite set $\mathbf{N}$.

The top-separated 2-pd $\rho\cdot\sigma$ of the Example after item (vii) in section 4 cannot be displayable, since it is not uniquely factorable. In particular, "maximal" for PPS's does not in general imply "complete".

9.6 Proposition **(i)** For Θ a complete PPS, every slice $\Theta\restriction(C_1,C_2)$ is complete.

(ii) A displayable pd has strong unique factorization.

Proof (i): Let $[\![\Phi]\!]=[\![\Theta\restriction(C_1,C_2)]\!]$, and let Φ_1, Φ_2 be molecules such that $[\![\Phi_1]\!]=[\![\Theta\restriction U_1]\!]$, $[\![\Phi_2]\!]=[\![\Theta\restriction U_2]\!]$. Then for the molecule $\Psi=\Phi_1\hat{}\Phi\hat{}\Phi_2$, $[\![\Phi]\!]=[\![\Theta]\!]$. By the completeness of Θ, there is $\lll$ such that $\Theta\restriction\lll=\Phi$. Then

$$\Phi=\Psi\restriction(C_1,C_2)=(\Theta\restriction\lll)\restriction(C_1,C_2)=(\Theta\restriction(C_1,C_2))\restriction(\lll\restriction P)\ ,$$

where $P=L_1 \cap U_2$. This completes the proof.

(ii): Suppose $\Gamma = [\![\Phi]\!] \cdot [\![\Psi]\!]$, $U \overset{\text{DEF}}{=} [\Phi]$, $L \overset{\text{DEF}}{=} [\Psi]$. Let $\ll = <_{\Phi\wedge\Psi}$. Since Θ is complete, as the backbone order of the molecule $\Phi\wedge\Psi$ defining Γ , $\ll$ is compatible with the backbone order $(\mathbf{N}, \prec)$ of Θ , in particular, $C=(U, L)$ is a cut for $(\mathbf{N}, \ll)$ and for $(\mathbf{N}, \prec)$; and $\Theta \restriction \ll = \Phi\wedge\Psi$. It follows that

$$\Phi = (\Theta \restriction \ll) \restriction U = (\Theta \restriction U) \restriction (\ll \restriction U) ,$$

$$[\![\Phi]\!] = [\![(\Theta \restriction U) \restriction (\ll \restriction U)]\!] = [\![(\Theta \restriction U)]\!] ,$$

the last equality being the definition of $[\![(\Theta \restriction U)]\!]$, the main point being that $[\![(\Theta \restriction U) \restriction (\ll \restriction U)]\!]$ does not depend on $\ll \restriction U$ (9.3).

We have shown that in a factorization $\Gamma = [\![\Phi]\!] \cdot [\![\Psi]\!]$, $[\![\Phi]\!]$ depends only on the set $[\Phi]$. Similar statement holds for Ψ . This completes the proof.

§10. Final arguments

We first discuss the possibilities of a converse of the construction [2] in section 9: getting a planar prescheme out of a 2-PPS.

A 2-dimensional pasting prescheme does not necessarily arise from a planar one, since not every partial order $\prec$ of $\mathbf{N}$ can be made into a planar arrangement $(\mathbf{N},\prec,\rightarrow)$, and, on the other hand, by restriction, any $\prec$ will appear as the backbone order of a 2-PPS.

Given 2-PPS $\Theta=(\boldsymbol{Y}, \mathbf{N}, \prec, \Theta\langle S\rangle)_{S\ \prec\text{-span}}$. Let S be any span; then $\Theta\langle S\rangle$ is of the form

$$\Theta\langle S\rangle \ :=:\ s_{u_1}^{-\infty}\cdot\bar{u}_1\cdot s_{u_2}^{u_1}\cdot\bar{u}_2\cdot\ldots\cdot s_{u_m}^{u_{m-1}}\cdot\bar{u}_k\cdot s_{\infty}^{u_m} \tag{1}$$

for some, distinct, elements u_i of $\mathbf{N}$, and 1-pd's s_w^v in the computad $\boldsymbol{Y}$; every item in (1) is uniquely determined from $\Theta\langle S\rangle$ itself; we have that $\langle u_i : 1\le i\le m\rangle$ is a resulting repetition-free enumeration of S.

We define the relation $\rightarrow_S$ on $\mathbf{N}$ by

$$u\rightarrow_S v \iff \exists i, j.\ i<j\ \&\ u_i=u\ \&\ u_j=v$$

with reference to (1) (of course, $\rightarrow_S\subseteq S\times S$).

It is clear that $\rightarrow_S$ is an irreflexive relation.

We show that the definition "does not depend on S":

$$u, v\in S_1\cap S_2 \implies (u\rightarrow_{S_1} v \iff u\rightarrow_{S_2} v). \tag{2}$$

The proof is an argument similar to the one used for 6.1 in section 6.

To do the proof, we consider the expressions $\Theta\langle C\rangle[\bar{u}, \bar{v}]$, that is, $\Theta\langle C\rangle[\bar{R}]$ with $R=\{u, v\}$; $\Theta\langle C\rangle[\bar{u}, \bar{v}]$ is defined iff $u, v\in B$ (=boundary of the cut C).

Let us fix $u\neq v$, both in $\mathbf{N}$. Consider the set $\mathcal{C}_{u,v}$ of cuts C such that $u, v\in B(=B[C])$.

For $C\in\mathcal{C}_{u,v}$, $\Theta\{C\} \overset{\text{DEF}}{=} \Theta\langle C\rangle[\bar{u}, \bar{v}]$ will have one of the following two *types*, type-1 or type-2:

$$\Theta\{C\}= \qquad a_1^C\cdot\bar{u}\cdot a_2^C\cdot\bar{v}\cdot a_3^C \tag{3.1}$$
$$\Theta\{C\}= \qquad a_1^C\cdot\bar{v}\cdot a_2^C\cdot\bar{u}\cdot a_3^C\ ; \tag{3.2}$$

of course, only one of the two forms can be present for any one $\Theta\{C\}$; and the ingredients a_i^C are uniquely determined. Note that (2) will follow if we can show that the type is constant throughout $\mathcal{C}_{u,\,v}$; this is because, while C runs through $\mathcal{C}_{u,\,v}$, the $\Theta\{C\}$ run through appropriate substitution instances of $\Theta\langle S\rangle$ for *all* S such that $u,\,v\in S$.

We make two observations.

One is that if $C,\,\tilde{C}\in\mathcal{C}_{u,\,v}$, C and $\tilde{C}$ are shifts of one another, then they are of the same type: $\Theta\{\tilde{C}\}$ is obtained from $\Theta\{C\}$ by replacing a consecutive part of one of the a_i^C by another 1-pd.

Two is that is that if $C,\,D\in\mathcal{C}_{u,\,v}$, and $\rho(C,\,D)>0$ (for $\rho(C,\,D)$, see section 3), then there is $\tilde{C}\in\mathcal{C}_{u,\,v}$ such that $\rho(C,\,\tilde{C})=1$ ($\tilde{C}$ is a shift of C), and $\rho(\tilde{C},\,D)=\rho(C,\,D)-1$. (*Two* is shown by taking $w\in\mu U\cap M$ or $w\in\nu L\cap V$, and letting $\tilde{C}$ be the w-shift of C ; w is $\prec$-incomparable to u and v , thus by 9.1.1 for $R=\{w\}$, $\tilde{C}\in\mathcal{C}_{u,\,v}$).

The desired assertion follows by induction.

We can thus define, for $u,\,v\in\mathbf{N}$,

$$u\to v \iff \exists S.\,u\to_S v \iff \forall S(u,\,v\in S\Rightarrow u\to_S v)\ . \tag{4}$$

10.1 The 2-dimensional PPS Θ arises from a planar pasting prescheme if and only the relation $\to$ defined in (4) is transitive.

Sketch of proof Assume that $\to$ is transitive.

It is clear that for any $u\neq v$ in $\mathbf{N}$, exactly one of the relations $u\prec v$, $v\prec u$, $u\to v$, $v\to u$ holds. $(\mathbf{N},\,\prec,\,\to)$ is a planar arrangement.

For any span S , in the expression (1) for $\Theta\langle S\rangle$, we have that $i<j$ implies $u_i\to u_j$. Since S is a maximal $\prec$-antichain, we also have $u_i\to !u_{i+1}$ and $-\infty\to !u_1$, $u_m\to !\infty$ in the obvious senses. Thus, (1) is the same as (6) in section 9, except that we should see that, in (1), the expressions S_w^v for $v\to !w$, now allowing also $v=-\infty$, $w=\infty$, are *independent* of S .

More generally (but actually, equivalently), we need the following (we assume that $v\neq-\infty$, $w\neq\infty$ in the formulation given next; but we need suitable versions with $v=-\infty$ and/or $w=\infty$, which we leave to the reader to formulate) :

given an closed interval $[v,\,w]_\to$ (see section 3: we now do have a planar arrangement, thus we can use what we know about such) , a maximal span

$R=\{v=u_1 \to ! u_2 \to ! \ldots \to ! u_k = w\}$ in $[v, w]_{\to}$, there is a 1-pd $[S_w^v]^R$ of the form

$$[S_w^v]^R = \bar{u}_1 \cdot S_{u_2}^{u_1} \cdot \bar{u}_2 \cdot \ldots \cdot \bar{u}_{k-1} \cdot S_{u_k}^{u_{k-1}} \cdot \bar{u}_k \tag{5}$$

with each S_y^x a 1-pd in $\boldsymbol{Y}$, such that for every $C \in \mathcal{C}_R = \{C : R \subseteq B[C]\}$,

$$\Theta\langle C\rangle[\bar{R}] = b_1^C \cdot [S_w^v]^R \cdot b_2^C \tag{6}$$

for suitable b_1^C, $b_2^C \in \boldsymbol{Y}$. This is proved by a similar "continuity" argument as was done above, as follows.

Certainly, we do have, for any $\Theta\langle C\rangle[\bar{R}]$, $C \in \mathcal{C}_R$, a uniquely determined expression of the form (6) , with (5) for the middle factor, if we allow the middle factor $[S_w^v]^R$ and its ingredients in (6) to vary with $C \in \mathcal{C}_R$: having $[S_w^v]^{R,\, C}$ instead of $[S_w^v]^R$. What we need is $[S_w^v]^{R,\, C}$ is constant; or what is the same, $[S_w^v]^{R,\, C} = [S_w^v]^{R,\, D}$ for any $C, D \in \mathcal{C}_R$.

We recall 9.1 , 3.4, 3.4'. By 3.4', any C is determined by its restrictions to $(-\infty, v)_{\to}$, $[v, w]_{\to}$, $(w, \infty)_{\to}$; and these restrictions can be arbitrarily and independently prescribed; for brevity, call these restrictions C_1, C_2 and C_3, in the given order. Note that $C \equiv D \pmod{R}$ iff $C_1 = D_1$ and $C_3 = D_3$. Given any $C, D \in \mathcal{C}_R$, we can find a connecting sequence $C = C_1, C_2, \ldots, C_\ell = D$ consisting of adjacent pairs C_i, C_{i+1} of cuts that are shifts of each other by an element either in $(-\infty, v)_{\to}$, or in $(w, \infty)_{\to}$. The desired conclusion is now fairly clear.

Consolidating a practice found in previous parts of this paper, we reserve the notation $A[\bar{S}]$ for a n-pd A such that every $\bar{u} \in \bar{S}$ occurs in A exactly once. We use $\bar{S}$, as before, as a set of distinct and "new" n-indets $\bar{v}$, one for each $v \in S$, S a set of $(n+1)$-indets, such that $\mathrm{d}\bar{v} = \mathrm{dd}v$, $\mathrm{c}\bar{v} = \mathrm{cc}v$.

We will prove two lemmas, 10.2 and 10.3, for the purposes of 10.4 Theorem, which is intended as the main theorem of the paper.

10.2 Suppose S is a finite set of anchored 2-indets, $A = A[\bar{S}]$ and $B = B[\bar{S}]$ are 1-pd's such that for every $u \in S$, $A[\mathrm{c}v/\bar{v}]_{v\in S-\{u\}} = B[\mathrm{c}v/\bar{v}]_{v\in S-\{u\}}$. (7)

Then $A=B$.

Remark This is false without the condition "anchored"; let u and v be two distinct 2-indets such that $\mathrm{c}u=\mathrm{d}u=\mathrm{c}v=\mathrm{d}v=1_X$, and let $A=\bar{u}\cdot\bar{v}$, $B=\bar{v}\cdot\bar{u}$.

Proof of 10.2 By induction on $\#S$. For $\#S=0$, the assertion is obvious.

Suppose $\#S\geq 1$, and

$$A = A_0\cdot\bar{u}_1\cdot A_1\cdot\bar{u}_2\cdot\ldots\cdot\bar{u}_\ell\cdot A_{\ell+1} \ , \tag{8.1}$$

$$B = B_0\cdot\bar{v}_1\cdot B_1\cdot\bar{v}_2\cdot\ldots\cdot\bar{v}_\ell\cdot B_{\ell+1} \ . \tag{8.2}$$

Suppose $u_1\neq v_1$, to reach a contradiction. We have $\ell\geq i>1$ such that $u_i=v_1$ and $\ell\geq j>1$ such that $u_1=v_j$.

Let us write $u=u_1$ and $v=v_1$.

Let us make the two substitutions $[cx/\bar{x}]_{x\neq u}$ and $[cx/\bar{x}]_{x\neq v}$; we obtain from (7) :

$$A_0\cdot\bar{u}\cdot(A_1\cdot\mathrm{c}u_2\cdot\ldots\cdot\mathrm{c}u_\ell\cdot A_{\ell+1}) = (B_0\cdot\mathrm{c}v\cdot B_1\cdot\ldots\cdot B_{j-1})\cdot\bar{u}\cdot\ldots\cdot B_{\ell+1}$$

and

$$(A_0\cdot\mathrm{c}u\cdot A_1\cdot\ldots\cdot A_{i-1})\cdot\bar{v}\cdot\ldots\cdot A_{\ell+1}) = B_0\cdot\bar{v}\cdot(B_1\cdot\mathrm{c}v_2\cdot\ldots\mathrm{c}v_\ell\cdot B_{\ell+1}) \ .$$

and thus, since $\bar{u}$ and $\bar{v}$ occur only at the places indicated,

$$A_0 = B_0\cdot\mathrm{c}v\cdot B_1\cdot\ldots\cdot B_{j-1}, \qquad B_0 = A_0\cdot\mathrm{c}u\cdot A_1\cdot\ldots\cdot A_{i-1} \ .$$

Since $\mathrm{c}u$, $\mathrm{c}v$ are non-identity 1-pd's, we got that A_0 and B_0 are proper initial segments of each other; contradiction.

Returning to (8.1) and (8.2), we now have that $u_1=v_1=u$. Making the substitution $[cx/\bar{x}]_{x\neq u}$, from (7) we obtain that

$$A_0=B_0 \tag{9}$$

and $\{u_2, \ldots, u_\ell\} = \{v_2, \ldots, v_\ell\} = S' = S-\{u\}$.

Let

$$A' = A_1 \cdot \bar{u}_2 \cdot \ldots \cdot \bar{u}_\ell \cdot A_{\ell+1} ,$$
$$B' = B_1 \cdot \bar{v}_2 \cdot \ldots \cdot \bar{v}_\ell \cdot B_{\ell+1} .$$

We see that the condition (7) for S', A' and B' in place of S, A and B, is a consequence (7) for S, A, B, by using (9) and cancellation. The induction hypothesis tells us that $A'=B'$; $A=B$ follows by (9). (**End of proof of 10.2**)

10.3 Suppose that Θ and Θ^* are 2-PPS's both based on the "backbone" $(\mathbf{N}, \prec)$ such that $\Xi \overset{\text{DEF}}{=} [\![\Theta]\!] = [\![\Theta^*]\!]$ is an anchored 2-pd. Then $\Theta \approx \Theta^*$, that is, $\Theta\langle S\rangle = \Theta^*\langle S\rangle$ for all $\prec$-spans S.

Proof of 10.3 Let $\Theta = (\mathbf{N}, \prec, \Theta\langle S\rangle)_S$ be any PPS. Let $u \in \mathbf{N}$, and $\ll$ a total order of $\mathbf{N}$ extending $\prec$. The $(\ll, u)$-*atom* of Θ, denoted $\varphi^\Theta(\ll, u)$, is defined to be $\Theta\langle C\rangle[\{u\}]$, where $C=(U, L)$ is the $\prec$-cut "modulo $\{u\}$" for which $v \in U - \{u\} \Longleftrightarrow v \ll u$, $v \in L - \{u\} \Longleftrightarrow v \gg u$. If $\Theta\restriction\ll$ is the molecule Φ, then $\varphi^\Theta(\ll, u) = \varphi^\Phi[u]$, in our earlier notation for atoms in molecules.

Let's make the assumptions of the lemma.

Since Ξ has unique factorization (2.1), it follows that $\varphi^\Theta(\ll, u) = \varphi^{\Theta^*}(\ll, u)$ for all $u \in \mathbf{N}$ and total extensions $\ll$ of $\prec$.

Write $A = \Theta\langle S\rangle$, $B = \Theta^*\langle S\rangle$. Let $u \in S$. Since S is an $\prec$-antichain, there is a total order $\ll$ of $\mathbf{N}$ such that $v \ll u$ for all $v \in S - \{u\}$ (take the ordered sum of the following four total orderings: a total order of the set $\{x \in \mathbf{N} : \exists v \in S . x \prec v\}$, then one of the set $S - \{u\}$, then the one of the singleton $\{u\}$, and finally a total ordering of the set $\{x \in \mathbf{N} : \exists v \in S . x \succ v\}$).

For this pair $(\ll, u)$, $\varphi^\Theta(\ll, u) = A[cv/\bar{v}]_{v \in S - \{u\}}$, and $\varphi^{\Theta^*}(\ll, u) = B[cv/\bar{v}]_{v \in S - \{u\}}$. By the above, the hypothesis of 10.2 is satisfied, and we have our desired conclusion. (**End of proof of 10.3**)

10.4 Theorem **(i)** Every PPS defining a top-separated anchored 2-Pd can be extended to a planar pasting prescheme.

(ii) Every PPS defining a top-separated anchored 2-Pd can be extended to a complete PPS.

(iii) Any top-separated anchored 2-Pd has a unique maximal PPS defining it, and this PPS is both complete and planar.

Proof This could, probably, be done directly by modifying the proof of 4.2 Theorem. But, fortunately, there is a shortcut to the result that *uses* 4.2.

Let $\underline{\Gamma}$ be a top-separated anchored 2-Pd.

4.2 constructs a planar pasting prescheme $\mathrm{p}\Theta$ associated with $\underline{\Gamma}$. $\mathrm{p}\Theta$ gives rise to a PPS Θ by the construction [2] in section 9. By definition, the backbone order $\prec$ of $\mathrm{p}\Theta$ and of Θ is the intersection of the total orders $<_\Phi$ associated with all molecules $\Phi\in\mathbf{G}^\Gamma$ defining Γ; and 4.2(ii) says that every $\Phi\in\mathbf{G}^\Gamma$ equals the restriction $\Theta\restriction<_\Phi$. In other words, Θ is a complete PPS defining Γ.

Given any PPS Θ^* defining Γ, for its backbone order $\prec^*$, we have $\prec\subseteq\prec^*$, since every total extension $\lll$ of $\prec^*$ is $<_\Phi$ for some $\Phi\in\mathbf{G}^\Gamma$, namely $\Phi=\Theta^*\restriction\lll$, and therefore every total extension of $\prec^*$ is a total extension of $\prec$. Consider the restriction $\Theta\restriction\prec^*$. By 10.3, we must have $\Theta\restriction\prec^*=\vec{\Gamma}$. We conclude that Θ extends Θ^*. This proves both (i) and (ii).

If Θ^* is a maximal PPS defining Γ, we still have that the given Θ extends it; but then, we must have $\Theta^*=\Theta$. This proves (iii).

To deal with the last assertions of section 2, let us start with a complete planar pasting prescheme, $(\mathbf{N},\prec,\rightarrow,\mathrm{dd}\Gamma,\mathrm{cc}\Gamma,\vec{S})$ in the notation of 4.2 in section 4, defining the anchored 2-Pd $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$. We use the notation and terminology of section 8, as well as the notational conventions put down at the start of section 6: $C=(U,L)$, $D=(V,M)$, etc.

All indets and pd's are in the computad $\boldsymbol{X}$.

Let C be any cut. I claim that

10.5

$$\mathrm{s}(\llbracket C\rrbracket)\subseteq\mathrm{s}(\mathrm{d}\Gamma)\cup\bigcup_{v\in U}\mathrm{s}^\circ(\mathrm{c}v)\ . \qquad (10)$$

This is seen by induction on the cardinality $\#U$ of U as follows.

When $\#U=0$, $(S_\infty^{-\infty})^C=\mathrm{d}\Gamma$, and the assertion holds.

Let $\#U>0$. There is $\tilde{C}$ such that $\tilde{C}<!C$, $\tilde{C}$ is the u-shift of C (see section 3), $u\in\underline{B}$, $u\in\overline{\tilde{B}}$, $\#V=\#U-1$.

Since C and $\tilde{C}$ agree on the open intervals $(-\infty,u)_\rightarrow$, $(u,\infty)_\rightarrow$, we have

$$S_1\overset{\mathrm{DEF}}{=}(S_u^{-\infty})^C=(S_u^{-\infty})^{\tilde{C}},\ S_2\overset{\mathrm{DEF}}{=}(S_\infty^u)^C=(S_\infty^u)^{\tilde{C}}\ .$$

Also,

$$(S_{\infty}^{-\infty})^{\tilde{C}} = S_1 \cdot \mathrm{d}u \cdot S_2 \tag{11.1}$$

$$(S_{\infty}^{-\infty})^{C} = S_1 \cdot \mathrm{c}u \cdot S_2 \ \ , \ . \tag{11.2}$$

By the induction hypothesis, (10) holds for $\tilde{C}$ in place of C; thus, by (11.1), $\mathrm{s}(S_1)$, $\mathrm{s}(S_2)$ are subsets of $\mathrm{s}(\mathrm{d}\Gamma) \cup \bigcup_{v\in\tilde{U}=U-\{u\}} \mathrm{s}^{\circ}(\mathrm{c}v)$. By (11.2),

$$\mathrm{s}((S_{\infty}^{-\infty})^{C}) = \mathrm{s}(S_1)\cdot\mathrm{s}^{\circ}(\mathrm{c}u)\cdot\mathrm{s}(S_2)$$

since the the left and right end-0-cells of $\mathrm{c}u$ are included in S_1, resp. S_2.

(10) for C follows.

The inequality, going in the opposite direction,

$$\bigcup_{u\in\underline{B}} \mathrm{s}(\mathrm{c}u) \ \subseteq \ \mathrm{s}((S_{\infty}^{-\infty})^{C}) \ \ , \tag{12}$$

follows (11.2) being true, with suitable 1-pd's S_1 and S_2, for any $u\in\underline{B}$.

More generally, with the same proof, we have

10.5' If $D<C$ then

$$[\![D]\!] \subseteq [\![C]\!] \cup \bigcup_{w\in U-V} \mathrm{s}^{\circ}(\mathrm{c}w) \ \ . \tag{12'}$$

Now, *assume that* Γ *is separated*.

10.5" For any cut $C=(U,L)$, $v\in L$, we have $\mathrm{s}^{\circ}(\mathrm{c}v)\cap[\![C]\!]=\emptyset$.

This is immediate from 10.5, and the definition of "separated".

Take two different cuts C and D. There is v such that either $v\in U\cap M$ or $v\in V\cap L$. Assume, for instance, the first alternative. Take $u\in\underline{B}=\mu U$ such that $v\prec u$. Then, of course, $u\in\underline{B}\cap M$.

Remember (item (3) in section 8) that the sets $\mathrm{s}(\mathrm{d}\Gamma)$, $\mathrm{s}^{\circ}(\mathrm{c}x)$ $(x\in\mathbf{N})$ are pairwise disjoint.

Since Γ is anchored, the set $\mathrm{s}^{\circ}(\mathrm{c}u)$ is non-empty.

By (12), $\mathrm{s}^{\circ}(\mathrm{c}u)$ is a subset of $\mathrm{s}((S_{\infty}^{-\infty})^{C})$; and the same set is disjoint from $\mathrm{s}((S_{\infty}^{-\infty})^{D})$ by (10) applied to D in place of C . We conclude

10.6 (Γ is anchored and separated) The mapping $C\mapsto(S_{\infty}^{-\infty})^{C}$, from cuts to 1-pd's is one-to-one. In fact, the mapping $C\mapsto\mathrm{s}((S_{\infty}^{-\infty})^{C})$ from cuts to sets of 0- and 1-cells is one-to-one.

The 1-pd's in $\boldsymbol{X}$ of the form $(S_{\infty}^{-\infty})^{C}$, $C\in\mathcal{C}$, are called the 1-*cuts* of $\underline{\Gamma}$.

A 1-pd a being 1-separated has been defined; it is the same as being top-separated. It means that, a being the composite of the 1-pd's r_i's as in

$$X_1\xrightarrow{r_1}X_2\xrightarrow{r_2}X_3\xrightarrow{r_3}\dots\xrightarrow{r_{m-1}}X_m\ , \tag{13}$$

that the r_i are pairwise distinct, without saying anything about the zero-cells.

$\mathrm{s}_1(a)$ denotes the set $\mathrm{supp}(a)\cap|\boldsymbol{X}|_1$.

10.7 **(i)** In a separated anchored 2-Pd Γ , all 1-cuts are 1-separated.

(ii) In a separated positive 2-Pd Γ , all 1-cuts are separated.

Proof (i) Let $C=(U,L)\in\mathcal{C}$, $a=(S_{\infty}^{-\infty})^{C}$. If $U=\emptyset$, the assertion is true by the definition of Γ being separated. We prove the assertion by induction on $\#U$, the base case $\#U=0$ having been handled.

Assume $U\neq\emptyset$, and let $u\in\mu U=\underline{B}$. Let us shift u "down", to get the cut $\tilde{C}$. Then, with

$S_1=(S_u^{-\infty})^{C}=(S_u^{-\infty})^{\tilde{C}}$, $S_2=(S_{\infty}^{u})^{C}=(S_{\infty}^{u})^{\tilde{C}}$, we have

$$[\![C]\!]=(S_{\infty}^{-\infty})^{C} = S_1\cdot\mathrm{c}u\cdot S_2\ ,$$

$$[\![\tilde{C}]\!]=(S_{\infty}^{-\infty})^{\tilde{C}} = S_1\cdot\mathrm{d}u\cdot S_2\ .$$

We know that $\mathrm{s}[\![\tilde{C}]\!] \subseteq \mathrm{s}(\mathrm{d}\Gamma) \cup \bigcup_{v\in\tilde{U}=U-\{u\}} \mathrm{s}^{\circ}(\mathrm{c}v)$. Therefore, $\mathrm{s}_1(\mathrm{c}u)\subseteq\mathrm{s}^{\circ}(\mathrm{c}u)$ is disjoint from both $\mathrm{s}_1(S_1)$ and $\mathrm{s}_1(S_2)$. By the induction hypothesis, $[\![\tilde{C}]\!]$ is 1-separated; hence, both S_1 and S_2 are 1-separated, and $\mathrm{s}_1(S_1)$, $\mathrm{s}_1(S_2)$ are disjoint. From the expression for $[\![C]\!]$, it now follows that $[\![C]\!]$ is 1-separated.

(ii) The proof for (i) is to be repeated, with the following addition: since $[\![\tilde{C}]\!]$ is separated, and $\mathrm{d}u$ is not an identity, we have that $\mathrm{dd}u\neq\mathrm{cc}u$. This is enough to conclude that, given that $[\![\tilde{C}]\!]$ is separated, so is $[\![C]\!]$.

10.8 Let $\rho=\rho[u]$, $\sigma=\sigma[v]$ be 1-atoms, and assume that $\rho\cdot\sigma$ is well-defined. Assume that $\mathrm{c}u$, $\mathrm{d}v$ are proper (not identities), and that $\rho\wedge\sigma \overset{\mathrm{DEF}}{=} \mathrm{c}\rho=\mathrm{d}\sigma$ is 1-separated.

Then $\rho\longleftrightarrow\sigma$ (see section 2, after item 2.1) if and only if

$$\mathrm{s}_1(\mathrm{c}u)\cap\mathrm{s}_1(\mathrm{d}v)=\varnothing\,. \qquad (14)$$

Proof "If": Assume (14), to show $\rho\longleftrightarrow\sigma$.

Let $\rho\wedge\sigma$ be displayed in (13); the r_i's are distinct. Let $\rho=b\cdot u\cdot e$, $\sigma=\hat{b}\cdot v\cdot\hat{e}$.

We have $\rho\wedge\sigma=\mathrm{c}\rho=b\cdot\mathrm{c}u\cdot e$; since $\mathrm{c}u$ is proper, there are $1\le i\le j\le N$ such that $\mathrm{c}u=r_i\cdot\ldots\cdot r_j$. Similarly, we have $1\le k\le\ell\le N$ such that $\mathrm{d}v=r_k\cdot\ldots\cdot r_\ell$. Clearly, (14) iff either $j<k$ (case 1), or $\ell<i$ (case 2). In the first case, let $S=r_{j+1}\cdot\ldots\cdot r_{k-1}$ (when $k=j+1$, $S=\mathrm{id}_{X_k}$); we have now the condition 4.1(i) with S as given; thus, $\rho\longrightarrow\sigma$. The second case gives $\sigma\longrightarrow\rho$.

"Only if": Clear from the "moreover" part of 4.1(i).

a, b denote 1-pd's.

The 1-pd a is a *part of* the 1-pd b if $b=b_1\cdot a\cdot b_2$ for suitable b_1 and b_2; notation: $a\subseteq b$. If b is separated and $a\subseteq b$, then a is separated. If both a and b are separated, then $a\subseteq b$ iff $\mathrm{s}(a)\subseteq\mathrm{s}(b)$. If b is separated, $a\subseteq b$ and $\mathrm{d}b,\mathrm{c}b\in\mathrm{s}(a)$, then $a=b$.

From now on, *we assume that* $\underline{\Gamma}=(\boldsymbol{X},\Gamma)$ *is positive and separated*. All indets and pd's are in $\boldsymbol{X}$ -- unless stated otherwise. With $\boldsymbol{Y}=\boldsymbol{X}\restriction\le 1$, $\Theta=(\boldsymbol{Y},\mathbf{N},\prec,\Theta\langle S\rangle)_S$ denotes the complete PPS displaying $\underline{\Gamma}$; we know Θ is planar.

10.9 Let $C=(U, L)$ be any cut in $(\mathbf{N}, \prec)$, B its border, $x\in\mathbf{N}$. Then $x\in\underline{B}$ iff $\mathrm{c}x\subseteq[\![C]\!]$, and $x\in\bar{B}$ iff $\mathrm{d}x\subseteq[\![C]\!]$.

Proof The "only if" assertions are obvious.

We show that if $u\in U\text{-}\underline{B}$, then $\neg(\mathrm{c}x\subseteq[\![C]\!])$.

Assume $u\in U\text{-}\underline{B}$. There is $v\in U$ such that $u\prec!v$. By 4.3(v), let Φ be such that $u\prec_\Phi!v$ and $\neg(u\longleftrightarrow_\Phi v)$; that is, for $\rho=\varphi^\Phi[u]$, $\sigma=\varphi^\Phi[v]$, we have $\neg(\rho\longleftrightarrow\sigma)$. By 10.8, we have $\mathrm{s}_1(\mathrm{c}u)\cap\mathrm{s}_1(\mathrm{d}v)\neq\emptyset$.

Let $f\in\mathrm{s}_1(\mathrm{c}u)\cap\mathrm{s}_1(\mathrm{d}v)$. Let the cut $D=(V, M)$ be defined by $V=\{w:w\underline{\succ}v\}$. We have $u, v\in V$ and $D<C$.

I claim that $f\notin s([\![D]\!])$. Namely, for $\tilde{D}$ obtained from D by shifting v *down*, $[\![\tilde{D}]\!]=d_1\cdot dv\cdot d_2$; and $[\![D]\!]=d_1\cdot cv\cdot d_2$ with the same d_1 and d_2. But $[\![\tilde{D}]\!]$ is separated (10.4); in particular, $\mathrm{s}_1(\mathrm{d}v)$ is disjoint from $\mathrm{s}_1(d_1)\cup\mathrm{s}_1(d_2)$. Therefore, since $f\in\mathrm{s}(\mathrm{d}v)$, we have $f\notin\mathrm{s}(d_1)\cup\mathrm{s}(d_2)$. Also, $\mathrm{s}_1(\mathrm{c}v)$ is disjoint from $\mathrm{s}_1(\mathrm{d}v)$. It follows that $f\notin\mathrm{s}([\![D]\!])$.

Since $D<C$, we have the relation (12') (10.5'). For all $w\in V\text{-}U$, $w\neq u$ (since $u\in V$) and so $\mathrm{s}^\circ(\mathrm{c}w)$, $\mathrm{s}^\circ(\mathrm{c}u)$ are disjoint; $\mathrm{f}\in\mathrm{s}(\mathrm{c}u)$ and $f\notin\mathrm{s}(\mathrm{c}w)$. It follows that $f\notin[\![C]\!]$. We have shown that $\neg(\mathrm{c}x\subseteq[\![C]\!])$.

Similarly, $v\in L\text{-}\bar{B}$ implies that $\neg(\mathrm{d}x\subseteq[\![C]\!])$.

Assume that $\mathrm{c}u\subseteq[\![C]\!]$. $u\in L$ would imply $\mathrm{s}_1(\mathrm{c}u)\cap[\![C]\!]=0$ (see 10.5), which, since $\mathrm{s}_1(\mathrm{c}u)\neq\emptyset$, contradicts $\mathrm{c}u\subseteq[\![C]\!]$. Thus, $u\in U$. But then $u\in\underline{B}$, since $u\in U\text{-}\underline{B}$ would imply $\neg(\mathrm{c}u\subseteq[\![C]\!])$.

10.10 Every 1-pd is a part of a 1-cut.

Proof By induction of $N=\#\mathbf{N}$, the number of 2-indets in Γ.

The assertion is clear when $N=0$.

Assume $N>0$, and let $x\in\mathbf{N}$ be $\prec$-maximal (lowest): no $v\in\mathbf{N}$ such that $x\prec v$. Consider the $\prec$-cut $C=(U, L)$ for which $L=\{x\}$, $U=\mathbf{N}\text{-}\{x\}$, and consider the restrictions $\vec{\Gamma}\restriction U$, $\vec{\Gamma}\restriction L$ (see [4] in section 9). Let $\Lambda=[\![\vec{\Gamma}\restriction U]\!]$, $\varphi=\vec{\Gamma}\restriction L]\!]$; φ is an atom $\varphi[x]$, and $[\Lambda]=U$. $\vec{\Gamma}\restriction U$ is a complete PPS (see section 9); thus, by 10.4, $\vec{\Gamma}\restriction U$ is the unique planar complete PPS displaying Λ. Thus, we can apply the induction hypothesis to Λ (having one fewer 2-indets than Γ) and $\vec{\Gamma}\restriction U$ the "display" PPS given by 4.2 for Λ.

With $\boldsymbol{X}'=\mathrm{supp}(\vec{\Gamma}\restriction U)$, we have the Pd $\underline{\Lambda}=(\boldsymbol{X}', \Lambda)$ and the fact that $|\boldsymbol{X}|_{\le 1}=|\boldsymbol{X}'|_{\le 1}\dot{\cup}\,\mathrm{s}^{\circ}(\mathrm{c}x)$ (by 8.(2) and separation). Therefore,

(15) for any $X\in\mathrm{s}_0{}^{\circ}(\mathrm{c}x)$, if a 1-indet f in $\boldsymbol{X}$ is incident on X ($X=\mathrm{d}f$ or $X=\mathrm{c}f$) , then $f\in\mathrm{s}_1(\mathrm{c}x)$.

Let a be a 1-pd in $\boldsymbol{X}$. We distinguish two cases. Case 1: $\mathrm{s}_1(a)\cap\mathrm{s}_1(\mathrm{c}x)\neq\emptyset$, Case 2: otherwise.

We treat Case 1, and leave the similar Case 2 to the reader.

Assume Case 1.

It follows from (15) that there is a unique 1-pd b such that $\mathrm{s}_1(b)=\mathrm{s}_1(a)\cap\mathrm{s}_1(\mathrm{c}x)$; moreover, b is a part of both of $\mathrm{c}x$ and a . b may be denoted as $\mathrm{c}x\cap a$. b is proper (not an identity 1-cell).

Let us write $\mathrm{c}x$ in the form (13).

Consider the following four mutually exclusive and jointly exhaustive cases:

Case 1.1: b is a proper initial segment

$$X_1\xrightarrow{r_1}\ldots\xrightarrow{r_{k-1}}X_k$$

of $\mathrm{c}x$ ($1<k<m$);

Case 1.2: b is a proper end segment

$$X_i\xrightarrow{r_i}\ldots\xrightarrow{r_{m-1}}X_m$$

of $\mathrm{c}x$ ($1<i<m$);

Case 1.3: $b=\mathrm{c}x$;

Case 1.4: $r_1\notin\mathrm{s}(b)$, $r_{m-1}\notin\mathrm{s}(b)$.

In case 1.1 , we must have that $\mathrm{c}a=\mathrm{c}b=X_k$, since there cannot be any $f\in\mathrm{s}_1(a)$ with $\mathrm{d}f=X_k$. Therefore, with a' the initial segment of a ending in X_1 (a' is possibly improper), the 1-pd $\tilde{\hat{a}}=a'\hat{}\,\mathrm{d}x$ is well-formed, and it is in $\boldsymbol{X}'$. The induction hypothesis for Λ in place of Γ says that there is a cut $C'=(U', L')$ of $(\mathbf{N}', \prec')=(\mathbf{N}-\{x\}, \prec\restriction U)$ such that $\tilde{\hat{a}}$ is a part of $[\![C']\!]$.

Form the cut $\tilde{\hat{C}}=(\tilde{\hat{U}}, \tilde{\hat{L}})$ of $(\mathbf{N}, \prec)$ for which $\tilde{\hat{U}}=U'$, $\tilde{\hat{L}}=L'\dot{\cup}\{x\}$; by the choice of x , $\tilde{\hat{C}}$ is a cut. Let $\tilde{\hat{B}}$ be the border of $\tilde{\hat{C}}$.

Since $\mathrm{d}x$ is a part of $[\![C]\!]$, we must have that $x \in \overline{\tilde{B}}$ (10.9). Therefore, we can shift x up, and form the cut $D=(V, M)$ for which $V=\tilde{U} \dot{\cup} \{x\}$, $M=\tilde{L}-\{x\} = L'$. It is clear that $a=a' \hat{\ } b$ is a part of $[\![D]\!]$.

Case 1.2 is similar to case 1.

Case 1.3: we now have that $a=a_1 \hat{\ } \mathrm{c}x \hat{\ } a_2$, with a_1 and a_2 possibly improper; we can pass to $\tilde{a}=a_1 \hat{\ } \mathrm{d}x \hat{\ } a_2$, a 1-pd in $\boldsymbol{X'}$. From here, we proceed similarly to case 1.1.

Case 1.4: In this case we must have that $a=b \subseteq \mathrm{c}x$. $a \subseteq [\![C]\!]$ for the cut $C=(U, L)$ for which $U=\mathbf{N}$, $L=\emptyset$, since $x \in \underline{B}$ for B the border of C.

The proof is complete.

A *long* 1-pd is one whose domain is $\mathrm{dd}\Gamma$, codomain is $\mathrm{cc}\Gamma$. Every 1-cut is long. As a consequence of the above, we have

10.11 The mapping $C \mapsto [\![C]\!]$ is a bijection from cuts to long 1-pd's.

For any *separated* 1-pd a, a part $b \subseteq a$, and another 1-pd $\hat{b}$ parallel, we can *substitute* $\hat{b}$ for b in a, and get the 1-pd $a[\hat{b}/b]$: if $a=a_1 \cdot b \cdot a_2$, then $a[\hat{b}/b] = a_1 \cdot \hat{b} \cdot a_2$. Note that, without the assumption of separatedness of a, the substitution notation would not be sound, since it would not necessarily be unambiguous.

This notation will be used in three ways.

On the one hand, as a simple general formula valid for any atoms $\psi[w]$,

$$\mathrm{c}\psi = (\mathrm{d}\psi)[\mathrm{c}w/\mathrm{d}w] \tag{16}$$

for all $i=1, \ldots, m$.

On the other hand, if $C <! D$ are $\prec$-cuts, and D is obtained by shifting u up, then

$$[\![D]\!] = [\![C]\!][\mathrm{c}u/\mathrm{d}u] \ . \tag{17}$$

(Write

$$[\![C]\!] = (S_u^{-\infty})^C \cdot \partial^C u \cdot (S_\infty^u)^C = (S_u^{-\infty})^C \cdot \mathrm{d}u \cdot (S_\infty^u)^C ;$$

and

$$[\![D]\!] = (S_u^{-\infty})^D \cdot \partial^D u \cdot (S_\infty^u)^D = (S_u^{-\infty})^D \cdot \mathrm{c}u \cdot (S_\infty^u)^D \ ,$$

and note that $(S_u^{-\infty})^C=(S_u^{-\infty})^D$, $(S_\infty^u)^C=(S_\infty^u)^D$; this makes (17) clear.)

The third use is an extension: it is up-substitution. Given separated 1-pd a . a part $b\subseteq a$, and a 2-pd Λ such that $\mathrm{dd}\Lambda=\mathrm{db}$, $\mathrm{cc}\Lambda=\mathrm{cb}$ (Λ is "parallel" to b), we can write $a[\Lambda/b]$ for $a_1\cdot\Lambda\cdot a_2$, where $a=a_1\cdot b\cdot a_2$.

This substitution operation has certain obvious properties, which are best mentioned if at all when they are used.

If 2-pd's Γ and Λ are in the relationship that $\Gamma=b\cdot\Lambda\cdot e$ for suitable (and obviously unique) 1-pd's b and e , we say (somewhat temporarily ...) that Λ is a *truncation* of Γ . Two 2-pd's are *truncation equivalent* if there is a third one which is a truncation of both; note that any 2-pd has a unique "smallest" truncation that has no truncation other than itself; "truncation equivalent" is the equivalence relation generated by "being a truncation of".

Recall the "slices" of subsection [4] of section 9. A slice of a PPS Θ is another PPS, $\Theta\restriction(C_1, C_2)$, for suitable data C_1 , C_2 . Given an anchored, and in fact here positive, 2-pd Γ , a *slice of* Γ is the 2-pd which is the value of a slice of the complete PPS displaying Γ . More formally, a slice of Γ is $[\![\Theta\restriction(C_1, C_2)]\!]$, for Θ the complete planar pasting prescheme displaying Γ (10.4) and for any pair $C_1\leq C_2$ of cuts of the underlying backbone order $(\mathbf{N}, \prec)$ of Θ .

10.12 ($\underline{\Gamma}=(\boldsymbol{X}, \Gamma)$ positive separated 2-Pd). Let Λ be any 2-pd in $\boldsymbol{X}$.

(i) Λ is separated.
(ii) Λ is the truncation of a slice of $\underline{\Gamma}$.
(iii) $[\Lambda]$ (the set of 2-indets in Λ) is a $\prec$-convex subset of $\mathbf{N} = [\Gamma]$.
(iv) Every $\prec$-convex subset P of $\mathbf{N}$ is the set of 2-indets of some 2-pd in $\boldsymbol{X}$; there is a 2-pd Λ such that $[\Lambda]=P$; every convex P is "composable" (in this new sense being introduced now).
(v) If P is a horizontally full convex set, then, up to truncation equivalence, there is a unique 2-pd Λ such that $[\Lambda]=P$; every horizontally full convex set of 2--indets is "uniquely composable" (in this new and restricted sense being introduced now).

Proof of 10.12 Let Λ be any 2-pd in $\boldsymbol{X}$. Λ is represented by a molecule

$$\Psi = (\psi_1[w_1], \ldots, \psi_m[w_m]) .$$

At the moment, we are not even assuming that Ψ is top-separated, although it will soon transpire that in fact Ψ must be separated.

Consider $\mathrm{d}\Lambda$, $\mathrm{c}\Lambda$; let $X=\mathrm{dd}\Lambda$, $Y=\mathrm{cc}\Lambda$. By 10.10, there is a (non-unique) $\prec$-cut C such that

$$[\![C]\!]=b\cdot\mathrm{d}\Lambda\cdot e \tag{18}$$

for suitable 1-pd's b and e ; we fix C . As usual, we write $C=(U, L)$, B the border of C .

$[\![C]\!]$ being separated (10.7), b and e are uniquely determined by the relation (18) as the (X, Y)-*segment* of $[\![C]\!]$, where $\mathrm{c}b=X$ and $\mathrm{d}e=Y$.

As a part of the separated 1-pd $[\![C]\!]$, $\mathrm{d}\Lambda$ is separated.

$\mathrm{d}w_1$ is a part of $\mathrm{d}\Lambda=\mathrm{d}\psi_1$. We have

$$\mathrm{d}\psi_2 = \mathrm{c}\psi_1 = (\mathrm{d}\psi_1)[\mathrm{c}w_1/\mathrm{d}w_1] . \qquad (19)$$

Being a part of $\mathrm{d}\Lambda$, $\mathrm{d}w_1$ is a part of $[\![C]\!]$. By 10.9, $w_1\in\bar{B}$. By shifting w_1, we obtain from $C=C_1$ the cut C_2. We have

$$[\![C_2]\!] = [\![C_1]\!][\mathrm{c}w_1/\mathrm{d}w_1] . \qquad (20)$$

Since Γ is separated, $\mathrm{s}^\circ(\mathrm{c}w_1)$ is disjoint from $\mathrm{s}[\![C_1]\!]$ (10.5"), and *a fortiori*, from $\mathrm{s}^\circ(\mathrm{d}\Lambda)=\mathrm{s}^\circ(\mathrm{d}\psi_1)$.

(18) says

$$[\![C_1]\!] = b\cdot\mathrm{d}\psi_1\cdot e \qquad (21)$$

By (21), (20) and $\mathrm{c}\psi_1=\mathrm{d}\psi_2$,

$$[\![C_2]\!] = b\cdot\mathrm{d}\psi_2\cdot e \qquad (22) .$$

What we have seen so far is the beginning of an obvious induction. By induction, we prove that

(**) there are cuts $C_1, \ldots, C_{m+1}$, such that, for all $i=1, \ldots, m$, C_{i+1} is obtained from C_i by shifting w_i up .

Of course, there can be at most one the sequence $C_1, \ldots, C_{m+1}$ as described; only the existence is a question. We add the relation

$$[\![C_i]\!] = \begin{cases} b\cdot\mathrm{c}\psi_{i-1}\cdot e & (\ i=2,\ldots,m+1\ ) \\ b\cdot\mathrm{d}\psi_i\cdot e & (\ i=1,\ldots,m\ ) \end{cases} \qquad (23)$$

to the inductively proved properties of the C_i.

Suppose $\ell\le m$, and the above has been established for $i\le\ell$.

dw_ℓ is a part of $d\psi_\ell$, and, by (23) for $i=\ell$, $d\psi_\ell$ is a part of $[\![C_\ell]\!]$. Thus, dw_ℓ is a part of $[\![C_\ell]\!]$. It follows (10.9) that $w_\ell\in\bar{B}_\ell$. In particular, $w_\ell\neq w_i$ for $i<\ell$ since $w_i\in U_\ell$.

$w_\ell\in\bar{B}_\ell$ says that we can shift w_ℓ up and obtain the cut $C_{\ell+1}$ from C_ℓ. By (17), we have

$$[\![C_{\ell+1}]\!]=[\![C_\ell]\!][cw_\ell/dw_\ell]\ . \tag{24}$$

Therefore,

$$[\![C_{\ell+1}]\!]\underset{(24)}{=}[\![C_\ell]\!][cw_\ell/dw_\ell]\underset{(23)(i=\ell)}{=}(b\cdot d\psi_\ell\cdot e)[cw_\ell/dw_\ell]\underset{dw_\ell\subseteq d\psi_\ell}{=}b\cdot d\psi_i[cw_\ell/dw_\ell]\cdot e=$$

$$\underset{(16)}{=}b\cdot c\psi_\ell\cdot e\underset{d\psi_{\ell+1}=c\psi_\ell \text{ if } \ell<m}{=}b\cdot d\psi_{\ell+1}\cdot e\ ,$$

which is (23) for $i=\ell+1$. The induction for (**) is complete.

Let $i\in\{1,\ldots,m\}$. By 10.5", $s^\circ(cw_i)$ is disjoint from $s([\![C]\!])$, since $w_i\in L$. Since $d\Lambda$ is a part of $[\![C]\!]$, $s^\circ(cw_i)$ is disjoint from $s(d\Lambda)$. The w_i are distinct; Γ is separated; the $s^\circ(cw_i)$ are disjoint from each other. We have shown that Λ is separated (part (i) of 10.12).

At this point, we may observe that we have proved 2.13 Proposition.

Note that the set

$$P=\{w_1,\ldots,w_m\} \text{ equals } L_1\cap U_{m+1}=\mathbf{N}-U_1-L_{m+1}\ . \tag{25}$$

Consider the slicing given by the pair $C_1\leq C_{m+1}$ of cuts (see part [4] fo section 9). Let Ξ be the slice defined by the slicing: $\Xi=[\![\Theta\upharpoonright(C_1,C_2)]\!]$; Θ is the complete planar PPS displaying Γ.

By definition, $[\Xi]=P$. On the set $\mathbf{M}=\{w_1,\ldots,w_m\}$, let the order $\ll$ be defined by $w_i\ll w_j$ iff $i<j$. Since with $C_j=(U_j,L_j)$, $w_j\in L_j$ and for $i<j$, $w_i\in U_j$, $w_j\leq w_i$ is impossible; $\ll$ is compatible with $\prec$. By the completeness of $\Theta\upharpoonright(C_1,C_2)$ (9.6), there is

a molecule Φ such that $<_\Phi = \lll$ and $[\![\Phi]\!] = \Xi$. By definition, we have

$$\Phi = (\varphi_1[w_1], \varphi_2[w_2], \ldots, \varphi_m[w_m]) \quad .$$

I **claim** that

$$\varphi_i[w_i] = [\![C_i]\!][w_i/\mathrm{d}w_i] = [\![C_{i+1}]\!][w_i/\mathrm{c}w_i] \; ; \qquad (26)$$

in particular

$$[\![C_i]\!] = \begin{matrix} \mathrm{c}\varphi_{i-1} & (\, i=2,\ldots,m+1\,) & (26.1) \\ \mathrm{d}\psi_i & (\, i=1,\ldots,m\,) & (26.2) \end{matrix}$$

To prove this, we note first that $[\![C_1]\!] = \mathrm{d}\Lambda = \mathrm{d}\varphi_1$, $[\![C_{m+1}]\!] = \mathrm{c}\Lambda = \mathrm{c}\varphi_m$ by section 9.

From $\mathrm{d}(\varphi_1[\mathrm{w}_1]) = [\![C_1]\!]$, *it follows* that

$$\varphi_1[w_1] = [\![C_1]\!][w_1/\mathrm{d}w_1] \; : \qquad (27)$$

since $[\![C_1]\!]$ is separated, w_1 can be fitted to it in only one way.

Similarly, since $\mathrm{c}(\varphi_1[w_1]) = [\![C_2]\!]$,

$$\varphi_1[w_1] = [\![C_2]\!][w_1/\mathrm{c}w_1] \quad .$$

Next, we have

$$\mathrm{c}(\varphi_1[w_1]) = [\![C_1]\!][\mathrm{c}w_1/\mathrm{d}w_1] \; ;$$

this follows from (27) without any additional assumption. From (18) and (**),

$$[\![C_1]\!][\mathrm{c}w_1/\mathrm{d}w_1] = [\![C_2]\!] \quad ,$$

and, of course,

$$\mathrm{d}(\varphi_2[w_2]) = \mathrm{c}(\varphi_1[w_1]) \quad ;$$

thus, by the last three displays,

$$\mathrm{d}(\varphi_2[w_2]) = [\![C_2]\!] \quad ,$$

from which (the main step),

$$\varphi_2[w_2] = [\![C_2]\!][w_2/dw_2] \quad .$$

And so on by induction; this suffices for the **claim**.

Let us put (26) and (23) together. We obtain

$$\begin{aligned}&\varphi_i[w_i] = \\ &= [\![C_i]\!][w_i/\mathrm{d}w_i] = (b\cdot\mathrm{d}(\psi_i[w_i])\cdot e)[w_i/\mathrm{d}w_i] = \\ &= (b\cdot\psi_i[\mathrm{d}w_i]\cdot e)[w_i/\mathrm{d}w_i] = b\cdot\psi_i[\mathrm{d}w_i][w_i/\mathrm{d}w_i]\cdot e \\ &\qquad\qquad = b\cdot\psi_i\cdot e\ .\end{aligned}$$

Since $\Lambda=\psi_1\cdot\ ...\ .\cdot\psi_m$ and $\Xi=\varphi_1\cdot\ ...\ .\cdot\varphi_m$, by the distributive law (see [M]), we have $\Xi=b\cdot\Lambda\cdot e$. We have shown that Λ is a truncation of the slice Ξ.

We have proved (ii). Parts (iii), (iv) and (v) follow from section 9.

2.15 is fairly clear from 10.12. Given any 2-pd Λ in $\boldsymbol{X}$, we have $C_1\leq C_2$ as in 10.12; we write $\Gamma_1=[\![\Theta\restriction U_1]\!]$, $\Xi=[\![\Theta\restriction(C_1,C_2)]\!]=b\cdot\Lambda\cdot e$, $\Gamma_2=[\![\Theta\restriction L_2]\!]$; with $u\|\Lambda$, u new, let $\Gamma^*=\Gamma_1\cdot(b\cdot u\cdot e)\cdot\Gamma_2$; then $\Gamma=\Gamma_1\cdot\Xi\cdot\Gamma_2=\Gamma^*[\Lambda/u]$.

Part (c) of 2.15 is seen because, using the notation of the proof of 10.12, we have the fact that Γ is separated, and

$$\mathrm{s}(\Lambda) = \mathrm{s}(\mathrm{d}\Lambda)\ \dot\cup\ \bigcup_{i=1}^{m}\mathrm{s}^{\circ}(\mathrm{c}w_i)\ ,$$

$$\mathrm{s}(\Gamma^*) = \{u\}\ \dot\cup\ \mathrm{s}(\mathrm{d}\Lambda)\ \dot\cup\ \mathrm{s}^{\circ}(b)\ \dot\cup\ \mathrm{s}^{\circ}(e)\ \dot\cup\ \mathrm{s}^{\circ}(\mathrm{c}\Lambda)\cup\bigcup_{x\in U_1}\mathrm{s}^{\circ}(\mathrm{c}x)\cup\bigcup_{y\in L_2}\mathrm{s}^{\circ}(\mathrm{c}y)$$

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