correction on p14 ; insertions to p.12 (12.1, 12.2) added

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Notes on the "triple paper"

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Trivial extensions

(1.) Let $\Phi : \mathbb{P} \to \mathbb{Q}$ be a functor, $\mathbb{X}$ a cocomplete category. A diagram $B : \mathbb{Q} \to \mathbb{X}$ is a trivial extension of $A : \mathbb{P} \to \mathbb{X}$ (via Φ) if $A = B \circ \Phi$, and, for some colimit cocone:

$$\begin{tikzcd} \mathbb{Q} \arrow[r, bend left, "B"] \arrow[r, bend right, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathbb{X} \end{tikzcd}$$

(with vertex X), the restriction $\beta\Phi$:

$$\begin{tikzcd} \mathbb{P} \arrow[r, "\Phi"] & \mathbb{Q} \arrow[r, bend left, "B"] \arrow[r, bend right, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathbb{X} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{P} \arrow[r, bend left, "A = B\Phi"] \arrow[r, bend right, "\ulcorner X \urcorner \Phi"'] \arrow[r, phantom, "\Downarrow \beta\Phi"] & \mathbb{X} \end{tikzcd}$$

of β to A is a colimit cocone on A.

(1.1) Note: Suppose B is a trivial extension of A; use the above notation. Let α' be any colimit cocone α' on A with vertex some Y:

$$\begin{tikzcd} P \arrow[r, bend left, "A"] \arrow[r, bend right, "\ulcorner Y \urcorner"'] \arrow[r, phantom, "\Downarrow \alpha'"] & \mathbb{X} \end{tikzcd}.$$

Then there is a

<u>unique</u> cocone $\beta' : \mathbb{B} \to \ulcorner Y \urcorner$ extending α':

$\alpha' = \beta' \Phi$; and this β' is a colimit cocone on $\mathbb{B}$.

$\boxed{\text{Proof}}$ Let $\beta : \mathbb{B} \to \ulcorner X \urcorner$ be the colimit cocone mentioned in the definition of $\mathbb{A}$ being a trivial extension of A. The restriction $\alpha = \beta \Phi$ is a colimit cocone on A with vertex X.

Therefore, there is a unique arrow $f : X \to Y$, in fact an isomorphism, such that $\alpha' = \ulcorner f \urcorner \alpha$.

Let $\beta' = \ulcorner f \urcorner \beta : \mathbb{B} \to \ulcorner Y \urcorner$; since f is an isomorphism, β' is a colimit cocone. We have

$$\beta' \Phi = (\ulcorner f \urcorner \beta) \Phi = (\ulcorner f \urcorner \Phi)(\beta \Phi) =$$

$$= \ulcorner f \urcorner \beta \Phi = \ulcorner f \urcorner \alpha = \alpha';$$

β' extends α'.

(diagrams:

$$\begin{tikzcd} P \arrow[r, "\Phi"] & Q \arrow[r, shift left=6, "B"] \arrow[r, "\ulcorner X \urcorner" description] \arrow[r, shift right=6, "\ulcorner Y \urcorner"'] & \mathbb{X} \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[rr, shift left=6, "A = B\Phi"] \arrow[rr, "\ulcorner X \urcorner = \ulcorner X \urcorner \Phi" description] \arrow[rr, shift right=6, "\ulcorner Y \urcorner = \ulcorner Y \urcorner \Phi"'] & & \mathbb{X} \end{tikzcd}$$

)

Suppose β', β'' are both cocones $B \longrightarrow \ulcorner Y \urcorner$ extending $\alpha' : A \longrightarrow \ulcorner Y \urcorner$. They are both colimit cocones; there is an automorphism $g : Y \xrightarrow{\cong} Y$ such that $\beta'' = \ulcorner g \urcorner \beta'$.

Then $\alpha' = \ulcorner g \urcorner \alpha'$; by the uniqueness part of the universal property of the colimit cocone α' (and $\alpha' = \ulcorner id \urcorner \alpha'$), we get that $g = id_Y$; hence $\beta'' = \beta'$.

(1.2) Even more obvious is this: if $\mathbb{B}$ is a trivial extension of $\mathbb{A}$ via Φ, then the restriction of any colimit cocone on $\mathbb{B}$ to $\mathbb{A}$ is a colimit cocone on $\mathbb{A}$.

(1.3) We have the following neat "2 out of 3" property: suppose we have:

$$\Theta = \Psi \circ \Phi \qquad
\begin{tikzcd}
P \arrow[d, "\Phi"'] \arrow[drr, "A"] \arrow[dd, bend right=60, "\circ"'] & & \\
Q \arrow[rr, "B"] \arrow[d, "\Psi"'] \arrow[r, phantom, "1", near start, shift left=2] \arrow[r, phantom, "2", near start, shift right=2] & & \mathbb{X} \\
R \arrow[urr, "C"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[dd, "\Theta"'] \arrow[dr, "A"] & \\
\arrow[r, phantom, "3"] & \mathbb{X} \\
R \arrow[ur, "C"'] &
\end{tikzcd}$$

Any two of the three extensions 1, 2, 3 being trivial ones imply that the third one is also trivial. For 1 & 2 $\Rightarrow$ 3: use (1.2).

For ② & ③ $\Rightarrow$ ①: take a colimit cocone γ on $\mathcal{C}$, and restrict to Q, to get β. By ②, β is a colimit; its restriction α along $\underline{\Phi}$ is the same as the restriction of γ along Θ, hence by ③, α is colimit. We have found a colimit on Q restricting to a colimit on P: ① is true.

For ① & ③ $\Rightarrow$ ②: let γ be a colimit on $\mathcal{C}$, β its restriction to $\mathcal{B}$, α its restriction to A; α is also the restriction of γ along Θ. By ③, α is a colimit. β restricts to α, which is a colimit. Since $\mathcal{B}$ is a trivial extension of A (i the assumption ①), by (1.1) β is colimit. And this is what we wanted.

The one-point trivial extension

(2.) Let P be any category, and Σ any *sieve* in P, i.e., Σ a set of objects in P such that if $x \in Ob(P)$ is in Σ, and $y \xrightarrow{f} x$ is any arrow, then $y (= dom(f))$ is in Σ as well.

Let $P[\Sigma]$ be the following category, the category obtained by "freely adding a cocone on Σ". Let me write Q for $P[\Sigma]$ for short. P is a full subcategory of Q. Q has exactly one object that is not in P; call it $[\Sigma]$. The only arrow in Q with domain $[\Sigma]$ is $id_{[\Sigma]}$. If $x \in Ob(P)$, and $x \notin \Sigma$, then $hom_Q(x, [\Sigma]) = \emptyset$. If $x \in \Sigma$, then $hom_Q(x, [\Sigma])$ is a singleton,

$$\{x \xrightarrow{[x]} [\Sigma]\}.$$

For the composition in Q: if $y \xrightarrow{f} x$ then the composite of $y \xrightarrow{f} x \xrightarrow{[x]} [\Sigma]$ is defined to be $y \xrightarrow{[y]} [\Sigma]$; this is legitimate since $y \in \Sigma$.

This describes $P[\Sigma]$ completely.

Let $A : P \longrightarrow \mathbb{X}$ be any diagram in a cocomplete category $\mathbb{X}$; Σ as before, we define the diagram

$$B = A[\Sigma] : Q = P[\Sigma] \longrightarrow \mathbb{X}$$

as follows: $B \restriction P = A$; $B([\Sigma])$ a chosen colimit of the diagram

$$Ai = A \restriction \Sigma : \Sigma \longrightarrow \mathbb{X} \quad (i : \Sigma \to A \text{ inclusion})$$

with a chosen colimit cocone, say $\gamma : A \restriction \Sigma \longrightarrow B([\Sigma])$. (where here Σ means the full subcategory of P on the objects in the set Σ). The definition of B is completed by requiring that

$$\left\langle B(x \xrightarrow[{[x]}]{} [\Sigma])\right\rangle_{x \in \Sigma}$$

be the chosen colimit cocone γ.

I $\boxed{\text{claim}}$ that $B = A[\Sigma]$ is a trivial extension of A via the inclusion $\Phi : P \longrightarrow Q \; (= P[\Sigma])$.

Let $Y = \mathrm{colim}\, A$, with $\alpha : A \to \ulcorner Y \urcorner$ the colimit cocone.

It suffices to construct a colimit cocone $\beta : B \longrightarrow \ulcorner Y \urcorner$ extending α: $\beta \Phi = \alpha$.

The requirement that β extend α means that we define $\beta_x : B(x) = A(x) \longrightarrow Y$ to be α_x, for all $x \in P$.

We have the colimit cocone $\gamma : Ai \to \ulcorner X \urcorner$ and the cocone $\alpha i : Ai \to \ulcorner Y \urcorner$. There is a unique arrow $f : X \longrightarrow Y$ such that $\alpha i = \ulcorner f \urcorner \gamma$.

We define

$$\beta_{[\Sigma]} : B([\Sigma]) = X \longrightarrow Y$$

as this arrow f; $\beta_{[\Sigma]} = f$.

We have to check the cocone property against all (non-identity) arrows in Q. The fact that the required property holds with respect to arrows in P is true because α is a cocone $\alpha : A \to \ulcorner Y \urcorner$. It remains to see that

$$\begin{tikzcd} B(x) \arrow[rr, "B([x])"] \arrow[dr, "\beta_x"'] & \circ & B([\Sigma]) \arrow[dl, "f (= \beta_{[\Sigma]})"] \\ & Y & \end{tikzcd}$$

commutes for $x \in \Sigma$. But this triangle is the same as

$$\begin{tikzcd} A(x) \arrow[rr, "\gamma_x"] \arrow[dr, "\alpha_x"'] & \circ & X = \mathrm{colim}(A_i) \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

which commutativity is right because of the choice of f: $\alpha i = \ulcorner f \urcorner \gamma$.

Finally: $\beta : \mathbb{B} \longrightarrow \ulcorner Y \urcorner$ is in fact a colimit cocone. To verify this, let δ be an arbitrary cocone $\delta : \mathbb{B} \longrightarrow \ulcorner Z \urcorner$. The restriction $\delta \Phi : A \longrightarrow \ulcorner Z \urcorner$, a cocone on A, factors through the colimit cocone $\alpha : A \longrightarrow \ulcorner Y \urcorner$ by a unique arrow $g : Y \to Z$: $\delta \Phi = \ulcorner g \urcorner \alpha$. We <u>must</u> show that $\delta = \ulcorner g \urcorner \beta$. The only thing missing for this is at the components at $[\Sigma] \in Q$:

$$\underline{\delta_{[\Sigma]}} \overset{?}{=} g\beta_{[\Sigma]} \underset{?}{=} \underline{gf}.$$

Let x be any object in Σ, and consider

$$\begin{tikzcd}
Y \arrow[rr, "g"] & \textcircled{4}? & Z \\
\textcircled{1} & X \arrow[ul, "f"] \arrow[ur, "\delta_{[\Sigma]}"'] & \textcircled{2} \\
& Ax \arrow[u, "B[x] = \gamma_{[x]}"] \arrow[uul, bend left, "\alpha_x"] \arrow[uur, bend right, "\delta_x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, "g"] & & Z \\
& \textcircled{3} & \\
& Ax \arrow[uul, bend left, "\alpha_x"] \arrow[uur, bend right, "\delta_x"'] &
\end{tikzcd}$$

(1) commutes by the choice of f: $\alpha i = \ulcorner f \urcorner \gamma$. (2) commutes since δ is a cocone $\delta : \mathbb{B} \longrightarrow \ulcorner Z \urcorner$. (3) commutes because of the choice of g: $\delta \Phi = \ulcorner g \urcorner \alpha$. Therefore, (4) commutes after pre-composition with $Ax \xrightarrow[\gamma_{[x]}]{} X$.

Since γ is a colimit cocone, the arrows $\gamma_{[x]}$ for $x \in \Sigma$ are jointly epimorphic. Therefore, (4) commutes as required.

(3.) $\boxed{\text{Directed colimits of trivial extensions is a trivial extension}}$

Let A be a diagram $A : \mathbb{P} \longrightarrow \mathbb{X}$ ($\mathbb{X}$ cocomplete), $\mathbb{I}$ a directed poset, $\Phi_i : \mathbb{P} \longrightarrow \mathbb{Q}_i$, $\Phi_{ij} : \mathbb{Q}_i \longrightarrow \mathbb{Q}_j$ functors for $i \in \mathbb{I}$, resp. $i \leq j$ in $\mathbb{I}$, compatibly: $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ ($i < j < k$).

Moreover, let, for each $i \in I$, $B_i : Q_i \to \mathbb{X}$

be a diagram such that $B_i \Phi_i = A$,

$B_j \Phi_{ij} = B_i \quad (i < j)$

In other words, we have a directed

system of $\langle B_i \rangle_{i \in I}$ extensions of the diagram A.

We can consider the colimit of the system:

$Q = \operatorname{colim}_{i \in I} Q_i$ (the colimit taken

in the category of categories).

Let me adjoin a new least element $\perp$ to

the poset I and get $I^\perp$; define $Q_\perp = P$, $B_\perp = A$,

$\Phi_{\perp i} = \Phi_i$; we have a diagram

$$Q^\perp : I^\perp \to \mathbf{Cat}$$

$$Q^\perp(i) = Q_i, \quad Q^\perp(ij) = \Phi_{ij}$$

and a cocone $B^\perp : Q^\perp \to \ulcorner \mathbb{X} \urcorner$

$$\begin{tikzcd} I^{\perp} \arrow[r, bend left, "Q^{\perp}"] \arrow[r, bend right, "\ulcorner \mathbb{X} \urcorner"'] \arrow[r, phantom, "\Downarrow B^{\perp}"] & \mathrm{Cat} \end{tikzcd}$$

$\left(B^{\perp}(i) = B_i \ , \ \text{for } i \in I^{\perp} \right).$

Q is the colimit of $Q^{\perp}$; $Q = \mathrm{colim}\, Q^{\perp}$

Therefore, I have $\mathbb{B} : Q \longrightarrow \mathbb{X}$

such that $\mathbb{B} \circ \Psi_i = B_i \quad (i \in I^{\perp})$;

with $\Psi_i : Q_i \to Q \ (i \in I^{\perp})$ the

colimit coprojections.

The $\boxed{\text{claim}}$ is: if for each $i \in I$,

B_i is a trivial extension of A via Φ_i,

then $\mathbb{B}$ is also a trivial extension of A

via $\Psi_{\perp} : P \longrightarrow Q$.

Note that by "① & ③ $\Rightarrow$ ②"

part of the 'two out of three' property, the assumption implies that for every $i<j$, B_j is a trivial extension of B_i via Φ_{ij}.

The claim is a typical "rearrangement-of-colimits" fact (see §2 of "Rearranging ...."), but I presume, it can be checked directly. The proof should use the characterization of 'trivial extension' given in 1.1.

see: insertion on p. 12.1

4. Let κ be a regular cardinal, $A: P \to \mathbb{X}$ a κ-good diagram.

The $\boxed{\text{claim}}$ is that there are:

a κ-good poset Q, $\Phi: P \to Q$ an initial inclusion of P in Q (P is an initial segment of Q), such that:

1) all elements in $Q-P$ are limit elements (x is a limit element if $\{y : y<x\}$ has no top (largest) element)

2) Q is κ-directed;

(insert to p.12, lin 8)

In fact, the proof is quite trivial, at least in the special case when each $\Phi_i : P \to Q_i$, $\Phi_{ij} : Q_i \to Q_j$ is an inclusion ($\Phi : P \to Q$, a map (functor) of posets is an inclusion if it acts as the identity on objects (elements), and full as a functor: for $x, y \in P$, $x \le_P y \iff x \le_Q y$). In this case (using that the poset I is directed), we have $Q = \bigcup_{i \in I} Q_i$ and $B : Q \to \mathbb{X}$ uniquely defined by the condition that $Q_i \xrightarrow[\text{incl}]{} Q \xrightarrow{B} \mathbb{X}$ equals B_i (we do not need to check that $Q = \bigcup_{i \in I} Q_i$ is in fact a colimit). We also have the poset $I^{\perp}$ as on p 10; $A : P \to \mathbb{A}$, and inclusions $\Phi_i : P \to Q_i$ such that B_i is a trivial extension of A via Φ_i.

We want to show: B is a trivial extension of A via the inclusion $\Psi_I : P \to Q$ [note: for posets P, Q, there is at most one inclusion $P \to Q$ (!)]

Take a colimit cocone
$$\begin{tikzcd} P \arrow[r, shift left=1.5, "A"] \arrow[r, shift right=1.5, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathbb{A} \end{tikzcd}$$
($\alpha : A \to \ulcorner X \urcorner$) on A.

For each i, take the unique cocone
$$\begin{tikzcd} Q_i \arrow[r, shift left=1.5, "B_i"] \arrow[r, shift right=1.5, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\beta_i\Downarrow"] & \mathbb{A} \end{tikzcd}$$
that extends α — which is a colimit [see (1.1)]. For $i < j$, β_j extends β_i, since $\beta_j \restriction Q_i : B_i \to \ulcorner X \urcorner$ extends α, hence $\beta_j \restriction Q_i$ must equal β_i.

Define the cocone
$$\begin{tikzcd} Q \arrow[r, shift left=1.5, "B"] \arrow[r, shift right=1.5, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathbb{A} \end{tikzcd}$$
, $\beta : B \to \ulcorner X \urcorner$ by the condition that $\beta \restriction Q_i : B_i \to \ulcorner X \urcorner$ extends β_i; this is made possible by the fact that I is directed. To check that β is colimit, let
$$\begin{tikzcd} Q \arrow[r, shift left=1.5, "B"] \arrow[r, shift right=1.5, "\ulcorner Y \urcorner"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathbb{A} \end{tikzcd}$$
be an arbitrary cocone. It restricts to
$$\begin{tikzcd} P \arrow[r, shift left=1.5, "A"] \arrow[r, shift right=1.5, "\ulcorner Y \urcorner"'] \arrow[r, phantom, "\Downarrow\gamma\restriction P"] & \mathbb{A} \end{tikzcd}$$

hence, there is unique $f : X \to Y$ s.t. $\gamma \upharpoonright P = \ulcorner f \urcorner \circ \alpha$.
Since B_i is trivial over A, we must have $\gamma \upharpoonright Q_i = \ulcorner f \urcorner \circ \beta_i$.
This being true for all i means that $\gamma = \ulcorner f \urcorner \circ \beta$ — and
the uniqueness of f was already clear in the first line.

and there is a K-good diagram $\mathbb{B} : \mathbb{Q} \Rightarrow \mathbb{X}$ which is a trivial extension of A via Φ.

Let us see what this gives us, before the proof. By 1.1 above, $\langle A \rangle$, the composite of A is also a composite of $\langle \mathbb{B} \rangle$.

Remember that the links of a good diagram $A : P \to \mathbb{X}$ are the $\mathbb{X}$-arrows $A(x^-) \xrightarrow{A(x^- \to x)} A(x)$ for the isolated elements x in P. Therefore, by 1) the links of the diagram $\mathbb{B}$ are exactly the links of A. Now, assume that $I \subseteq (\mathbb{X})_K^{\to}$, and the links of A are in $Po(I)$, the class of all pushouts of arrows in I. Then the links of B are in $Po(I)$ too.

Moreover, by an easy induction one proves that if a κ-good $B: Q \to \mathbb{X}_1$ has all its links in $P_0(\mathbb{X}_\kappa^{\to})$ (as is the case now), and if $B(\perp) \in \mathbb{X}_\kappa$ (in particular, if $B(\perp) = \emptyset$, the initial object, then all objects in the diagram are in $\mathbb{X}_\kappa$, $B(x) \in \mathbb{X}_\kappa$ for all $x \in Q$, hence $B: Q \to \mathbb{X}$ factors through $\mathbb{X}_\kappa \xrightarrow{\text{incl}} \mathbb{X}$.

The proof of the claim is a transfinite iteration of the one-point-trivial-extension construction in § 2 (p. 4.2), the iteration enabled by § 3 (directed colimits; p. 9)

Let $A: P \longrightarrow \mathbb{X}$ be κ-good;

pick any subset S of P of cardinality $< \kappa$.

Let $\Sigma = S\!\downarrow = \bigcup\limits_{x \in S} x\!\downarrow = \bigcup\limits_{x \in S} \{y : y \le x\}$.

Assume Σ (equivalently, S) has no maximum (largest) element [Otherwise; there is nothing to do!]

Since P is κ-good, $\#(x\downarrow) < \kappa$; κ regular & $\# S < \kappa$ then give us that $\# \Sigma < \kappa$.

Consider $Q = P[\Sigma]$ (see §2). It is constructed as a category; but now it is obviously a poset. Moreover, for the single new element $[\Sigma]$ of Q, $[\Sigma]\downarrow \ = \ \Sigma$, hence the κ-goodness condition is satisfied for $[\Sigma]$.

Q is well-founded, obviously, because P is.

Moreover, P is an initial segment of Q: $[\Sigma]$ is a maximal element in Q.

We have $B = A[\Sigma] : Q \longrightarrow \mathbb{X}$ constructed in §2. It is good; the condition for the limit element (!) $[\Sigma]$ is satisfied precisely by the construction of $B([\Sigma])$ as the colimit of $A \restriction \Sigma : \Sigma \to \mathbb{X}$.

In short: we have proved the following:

for any κ-good diagram $A : P \to \mathbb{X}$,

and any subset S of P of size $< \kappa$, such that S has no largest element,

⊙ there is a κ-good poset Q such that P is an initial segment of Q, and there is no new isolated point in Q, but there is an element y in Q such that $s < y$ for all $s \in S$

⊙ and there is a trivial extension $B : Q \to \mathbb{X}$ of A via the inclusion $\Phi : P \to Q$, which is a good diagram.

The construction is explicit; let us write

$$\begin{tikzcd} P \arrow[d, "\Phi"'] \arrow[dr, "A"] & \\ Q \arrow[r, "B"'] & \mathbb{X} \end{tikzcd} \quad \text{as} \quad \begin{tikzcd} P \arrow[d, "{\Phi[S]}"'] \arrow[dr, "A"] & \\ P[S] \arrow[r, "{A[S]}"'] & \mathbb{X} \end{tikzcd}$$

Let us start with a κ-good $A : P \to \mathbb{X}$.

Let $\langle S_\gamma \rangle_{\gamma < \alpha}$ be a well ordered sequence of all subsets S of P of cardinality $< \kappa$, with no top element (α a limit ordinal).

Let us define

$$B_\beta : Q_\beta \longrightarrow \mathbb{X}$$

for $\beta < \alpha$, as well as

$$\Phi_{\gamma\beta} : Q_\gamma \longrightarrow Q_\beta$$

such that, among others, $Q_0 = P$, $B_0 = A$,

$$\begin{tikzcd} Q_\gamma \arrow[dr, "B_\gamma"] \arrow[dd] & \\ & \mathbb{X} \\ Q_\beta \arrow[ur, "B_\beta"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} & Q_\gamma \arrow[dr, "\Phi_{\gamma\beta}"] \arrow[d, phantom, "\circ"] & \\ Q_\delta \arrow[ur, "\Phi_{\delta\gamma}"] \arrow[rr, bend right, "\Phi_{\delta\beta}"'] & {} & Q_\beta \end{tikzcd}$$

commute for all appropriate indices.

For $\beta = \gamma+1 < \alpha$, we put

$$Q_\beta = Q_\gamma[\underbrace{\Phi_{0\gamma}(S_\gamma)}]$$

the image of

the set $S_\gamma \subseteq P$ under

the map $\Phi_{0\gamma} : P \longrightarrow Q_\gamma$;

call it $\hat{S}_\gamma$

$$B_\beta = B_\gamma[\hat{S}_\gamma]$$

$\Phi_\beta = \Phi[\hat{S}_\gamma] : Q_\gamma \to Q_\beta$ initial inclusion

For limit $\beta < \alpha$,

$$Q_\beta = \bigcup_{\gamma<\beta} Q_\gamma$$

etc.

Notice that any (directed) union of κ-good posets

is κ-good

Put $A^+ \underset{def}{=} B_\alpha$

The final extension $\mathcal{B} : Q \to \mathcal{X}$ of P is a transfinite iteration of length κ of the $A \mapsto A^+$ construction.