

The κ -good small object argument. / MM (version 2)

Assumptions: $\mathcal{A}$: cocomplete category, κ is a regular cardinal, I is a set of arrows in $\mathcal{A}$ such that for all $i \in I$, $\text{dom}(i) \in \mathcal{A}_\kappa$.

Theorem Any arrow f in $\mathcal{A}$ can be factored as $f = h \circ g$, where

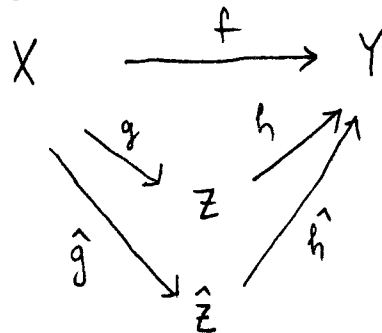
$$g \in \text{Gd}_{\kappa} \text{dir}_{\kappa}(\text{Po}(I)) \cap \text{Cell}(\text{Po}(I))$$

and

h has the right lifting property w.r. to I .

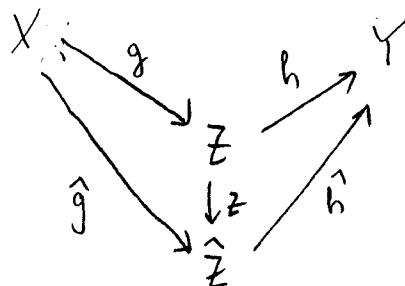
Terminology. Let $X \xrightarrow{f} Y$ be an arrow in $\mathcal{A}$.

If (g, h) and $(\hat{g}, \hat{h})$ are two factorizations of f , $f = h \circ g = \hat{h} \circ \hat{g}$



we say that $(\hat{g}, \hat{h})$ is a continuation of (g, h) along $z: Z \rightarrow \hat{Z}$, or $(\hat{g}, \hat{h})$ is a z -continuation of (g, h)

if



is commutative, $g^1 = z \circ g$ and $h = \hat{h} \circ z$.

A test for an arrow $h: Z \rightarrow Y$ to have the right lifting property (RLP) w.r. to I , or simply, an I -test for h , is a triple $t = (i, u, v)$ of arrows forming with h a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{u} & Z \\ i \downarrow & \circ & \downarrow h \\ V & \xrightarrow{v} & Y \end{array}$$

where $i \in I$. The test t is filled for h if there exists a "diagonal" $d: V \rightarrow Z$ making both triangles commute:

$$\begin{array}{ccc} U & \xrightarrow{u} & Z \\ i \downarrow & \circ & \downarrow h \\ V & \xrightarrow{v} & Y \end{array} \quad \begin{array}{c} \nearrow d \\ \circ \\ \searrow \end{array}$$

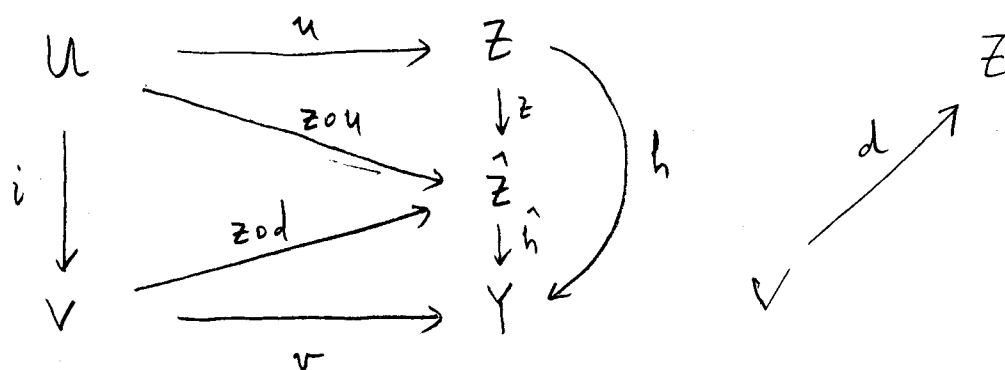
The arrow h has the RLP w.r. to I iff all I -tests for h are filled.

Given $Z \xrightarrow{z} \hat{Z}$ and the commutative triangle

$$\begin{array}{ccc} Z & \xrightarrow{h} & Y \\ z \downarrow & \circ & \nearrow \hat{h} \\ \hat{Z} & & \end{array}$$

and $t = (i, u, v)$ is an I -test for h , then the z -continuation of t is the I -test

$\hat{t} = (i, zou, v)$ for $\hat{h}$. If t is filled for h by d , say, then its z -continuation $\hat{t}$ continues to be filled, namely by zod :

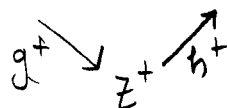


Lemma Suppose $X \xrightarrow{f} Y$ is a factorization of f , and A is a κ -good, κ -directed diagram $A: P \rightarrow \mathbb{A}$ such that $\langle A \rangle = g$.

Then there are:

1) a κ -good, κ -directed diagram $A^+: P^+ \rightarrow \mathbb{A}$ end-extending A (P is an initial segment of P^+) such that every link in A^+ is either a link already in A , or it is an arrow in $Po(\mathbb{I})$;

2) a factorization $X \xrightarrow{f} Y$ of f



where $Z^+ = A^+_T$ and $g^+ = \langle A^+ \rangle$

such that

3) (g^+, h^+) is a z -continuation of (g, h)

for the canonical arrow $z \stackrel{\text{def}}{=} \text{can}_{A, A^+} : A_T \rightarrow A_T^+$

(determined by the commutativity of all triangles

$$\begin{array}{ccc}
 & a_{xT} \nearrow & A_T \\
 A_x = A_x^+ & & \downarrow \\
 & a_{xT}^+ \searrow & A_T^+
 \end{array}, \quad x \in P \subseteq P^+$$

and

4) the z -continuation of every I -test for h is filled for h^+ .

Proof. We first perform the construction for a single I -test $t = (i, u, v)$ for h : we construct $\hat{A}, \hat{g}, \hat{h}$ having the properties stated for A^+, g^+, h^+ but without A^+ required to be κ -directed, and 4) required only for the single I -test t .

Since A is κ -directed, and U is κ -presentable, there are $x \in P$ and $\hat{u} : U \rightarrow A_x$ such that

$$\begin{array}{ccc}
 U & \xrightarrow{u} & Z (= A_T) \\
 \searrow \hat{u} & & \nearrow a_{xT} \\
 & A_x &
 \end{array}$$

commutes.

Let $\hat{P}$ be the poset for which $P \sqsubseteq \hat{P}$ (P an initial segment of $\hat{P}$), $\hat{P}$ has exactly one element, x^+ , not in P , and if $y \in P$, then $y < x^+$ iff $y \leq x$. Clearly, $\hat{P}$ is κ -good, x^+ is isolated in $\hat{P}$, $(x^+)^- = x$.

Define the object R and the arrows a and $\hat{v}$ to form a pushout diagram

$$\begin{array}{ccc} U & \xrightarrow{\hat{u}} & A_x \\ i \downarrow & & \downarrow a \\ V & \xrightarrow{\hat{v}} & R \end{array},$$

Define $\hat{A}: \hat{P} \longrightarrow A$ by the conditions that $\hat{A}$ extends A , $\hat{A}_{x^+} = R$ and $\hat{a}_{xx^+} = a$.

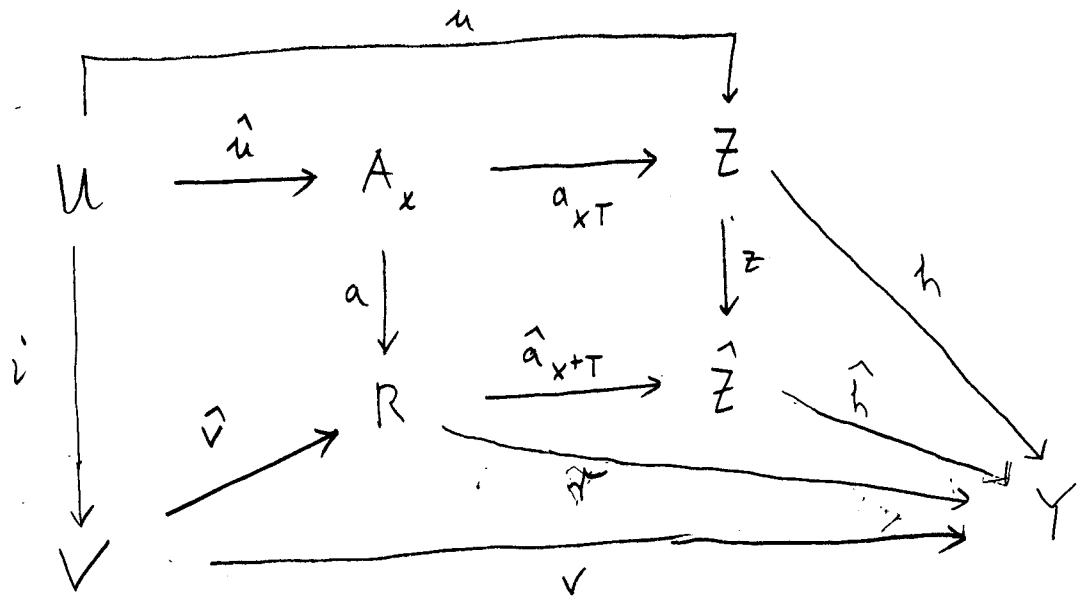
Since $a \in \text{Po}(i)$, the conditions in 1), take-away " κ -directed", are fulfilled.

It is easy to see that the following

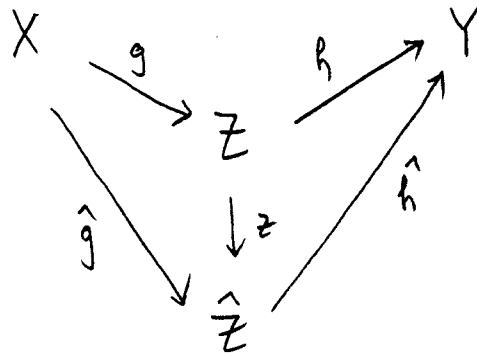
$$\begin{array}{ccc} A_x & \xrightarrow{a_{xT}} & Z = A_T \\ a \downarrow & & \downarrow z \stackrel{\text{def}}{=} \text{can}_{A, \hat{A}} \\ R & \longrightarrow & \hat{Z} = \hat{A}_T \end{array}$$

is a pushout diagram.

In the following commutative diagram:



the arrow $\hat{v}: R \longrightarrow Y$ is produced by R being a pushout, and then $\hat{h}: \hat{Z} \rightarrow Y$ is produced by $\hat{Z}$ being a pushout. With $\hat{g} \stackrel{\text{def}}{=} \langle \hat{A} \rangle$, we have the commutative diagram



(for $\hat{g} = z \circ g$, by the definition of $z = \text{can}_{\Lambda, \hat{A}}$); therefore $(\hat{g}, \hat{h})$ is a factorization of f . continuing (g, h) via z . The z -continuation $\hat{f} = (i, z \circ u, v)$ of the given test (i, u, v) is filled for $\hat{h}$ by the arrow $d = \hat{a}_{x+T} \circ \hat{v}$.

This completes the one-test construction.

The proof of the Lemma is a straight forward transfinite iteration of the one-test construction just described. Let $t_\alpha = (i_\alpha, u_\alpha, v_\alpha)$ for $\alpha \leq \lambda$, λ a limit ordinal, be a transfinite enumeration of the set of all I-tests for the arrow h .

Recursively, we define the κ -good diagram

$A^\alpha: P_\alpha \rightarrow A$ and the factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \scriptstyle \langle A^\alpha \rangle = g_\alpha & \nearrow \scriptstyle h_\alpha \\ & Z_\alpha = A^\alpha_T & \end{array}$$

for $\alpha \leq \lambda$ such that, when $\beta < \alpha \leq \lambda$, $A^\beta \sqsubseteq A^\alpha$ and (g_α, h_α) continues (g_β, h_β) via

$$z_{\beta\alpha} = \text{can}_{A^\beta, A^\alpha}: Z_\beta \longrightarrow Z_\alpha.$$

We let $A^0 = A$, $g_0 = g$, $h_0 = h$.

Given $A^\alpha, g_\alpha, h_\alpha$, to get $A^{\alpha+1}$, we take t_α , the α th I-test for h , and let t be the $z_{0\alpha}$ -continuation of t_α , an I-test for h_α .

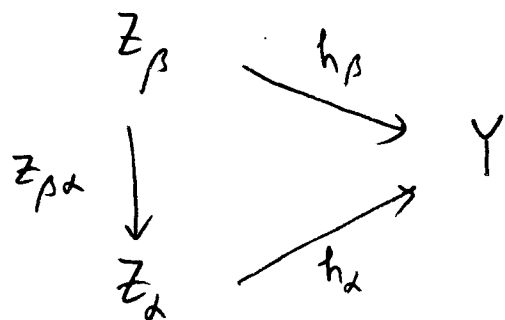
Let $A^{\alpha+1}, g_{\alpha+1}, h_{\alpha+1}$ be constructed as $\hat{A}, \hat{g}, \hat{h}$ are in the one-test construction, with A there

being A^α here, and $g = g_\alpha$, $h = h_\alpha$.

For a limit ordinal $\alpha \leq \lambda$, we let

$$A^\alpha = \bigcup_{\beta < \alpha} A^\beta, \quad g_\alpha = \langle A^\alpha \rangle, \quad \text{and } h_\alpha: Z_\alpha = A_T^\alpha \rightarrow Y$$

be defined uniquely by the condition that



commutes (the arrows $z_{\beta\alpha} = \text{can}_{A^\beta, A^\alpha}$, for $\beta < \alpha$, form a colimit cocone on the diagram Z_β ($\beta < \alpha$), $z_{\gamma\beta}: Z_\gamma \rightarrow Z_\beta$ ($\gamma \leq \beta < \alpha$)).

This completes the transfinite recursive construction of the items A^α , g_α and h_α for $\alpha \leq \lambda$.

Given the fact that a once-filled test remains filled in all continuations, it is clear that A^λ , g_λ , h_λ satisfy all the conditions in the lemma except the requirement of κ -directedness.

We let A^+ be a trivial end-extension of A , with the same composite as A , which is κ -good, κ -directed, and (has no new link not already in A ; such A^+ exists.

We define $g^+ = g_\lambda$, $h^+ = h_\lambda$.

This completes the proof of the lemma.

Proof of the Theorem

Let A^0 be the one-object diagram $A^0: \{1\} \rightarrow \mathcal{A}$

for which $A^0_1 = X$; $g_0 \stackrel{\text{def}}{=} \langle A^0 \rangle = 1_X$, $h_0 \stackrel{\text{def}}{=} f$.

We define A^α , g_α , h_α for $\alpha \leq \kappa$ by a simple iteration of the +- construction of the Lemma:

$$A^{\alpha+1} = (A^\alpha)^+, \quad g_{\alpha+1} = (g_\alpha)^+, \quad h_{\alpha+1} = (h_\alpha)^+$$

and, for $\alpha \leq \kappa$ limit,

$$A^\alpha = \bigcup_{\beta < \alpha} A^\beta, \quad g_\alpha = \langle A^\alpha \rangle, \quad h_\alpha \text{ satisfying}$$

$$\begin{array}{ccc} Z_\beta = A^\beta_T & \xrightarrow{h_\beta} & Y \\ \downarrow \scriptstyle Z_{\beta\alpha} = \text{can}_{A^\beta, A^\alpha} & \circ & \\ Z_\alpha & \xrightarrow{h_\alpha} & \end{array}$$

We claim that $g = g_\kappa$, $h = h_\kappa$ will satisfy the requirements in the Theorem.

$g \in \text{Gd}_\kappa \text{ dir}_\kappa (P_0(I))$ since $g = \langle A^\kappa \rangle$,
and A^κ is a κ -good, κ -directed diagram with

all links in $P_0(I)$

$g \in \text{Cell}(P_0(I))$, since g is the composite of the continuous transfinite system Z_α ($\alpha < \kappa$),
 $Z_{\beta\alpha} : Z_\beta \longrightarrow Z_\alpha$ ($\beta \leq \alpha < \kappa$) in which each link

$Z_{\alpha, \alpha+1} : Z_\alpha \longrightarrow Z_{\alpha+1}$ is of the form

$\text{can}_{A^\alpha, (A^\alpha)^+} : A_T^\alpha \longrightarrow (A^\alpha)_T^+$, which,

by the $+$ -construction of the Lemma, is a continuous transfinite composite with links in $P_0(I)$.

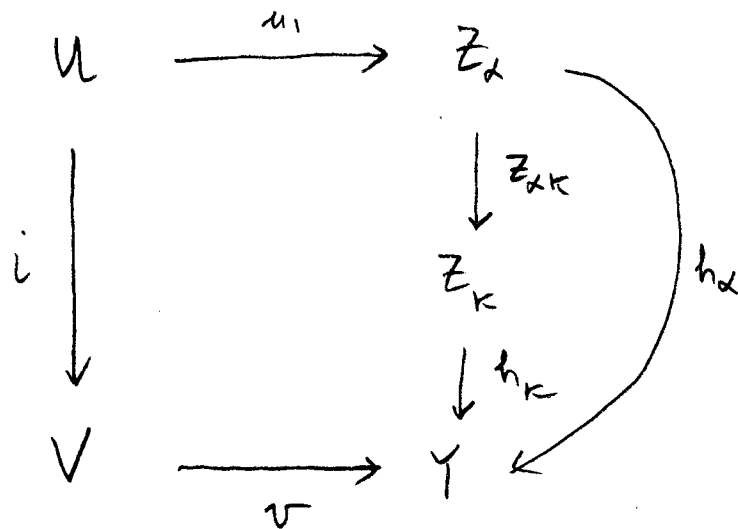
Finally, h has the RLP w.r. to I . We are given

an I -test $t = (i, u, v)$ for $h = h_\kappa$:

$$\begin{array}{ccc} U & \xrightarrow{u} & Z_\kappa \\ i \downarrow & \circ & \downarrow h_\kappa \\ V & \xrightarrow{v} & Y \end{array}$$

U being κ -presentable, there are $\alpha < \kappa$, α successor ordinal, and $u_1 : U \longrightarrow Z_\alpha$ such that $u = Z_{\alpha\kappa} \circ u_1$. Since $\kappa = \beta + 1$, $A^\alpha = (A^\beta)^+$, and thus, A^α is κ -directed. Consider $A^{\alpha+1} = (A^\alpha)^+$. We have the I -test

$$t_i = (i, u_i, v) \quad \text{for } h_\alpha = h_\kappa \circ \mathbb{Z}_{\alpha\kappa} :$$



The lemma says that the $z_{\alpha, \alpha+1}$ -continuation of t_1 , $t_2 = (i, z_{\alpha, \alpha+1} \circ u_1, v)$, is filled for $h_{\alpha+1}$.

But our original test $t = (i, n, v)$ is the $Z_{\alpha+1, k}$ -continuation of t_2 :

$$\begin{aligned} u &= z_{\alpha+1, \kappa} \circ z_{\alpha, \alpha+1} \circ u, \\ &= z_{\alpha \kappa} \circ u, \end{aligned}$$

therefore, t is filled for $h = h_\kappa = \mathbb{Z}_{\alpha+1, \kappa} \circ h_{\alpha+1}$.

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