$\boxed{\text{The } \kappa\text{-good small object argument.}}$ / MM

version 2

Assumptions: $\mathbb{A}$: cocomplete category, κ is a regular cardinal, I is a set of arrows in $\mathbb{A}$ such that for all $i \in I$, $\mathrm{dom}(i) \in \mathbb{A}_\kappa$.

Theorem Any arrow f in $\mathbb{A}$ can be factored as $f = h \circ g$, where

$$g \in \mathrm{Gd}_\kappa \mathrm{dir}_\kappa(\mathrm{Po}(I)) \cap \mathrm{Cell}(\mathrm{Po}(I))$$

and

h has the right lifting property w.r.t. to I.

Terminology. Let $X \xrightarrow{f} Y$ be an arrow in $\mathbb{A}$.

If (g, h) and $(\hat{g}, \hat{h})$ are two factorizations of f, $f = h \circ g = \hat{h} \circ \hat{g}$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, "\hat{g}"'] & & Y \\
& Z \arrow[ur, "h"] & \\
& \hat{Z} \arrow[uur, "\hat{h}"'] &
\end{tikzcd}$$

we say that $(\hat{g}, \hat{h})$ is a continuation of (g, h) along $z: Z \to \hat{Z}$, or $(\hat{g}, \hat{h})$ is a z-continuation of (g, h) if

$$\begin{tikzcd}
X \arrow[dr, "g"] \arrow[ddr, "\hat{g}"'] & & Y \\
& Z \arrow[ur, "h"] \arrow[d, "z"] & \\
& \hat{Z} \arrow[uur, "\hat{h}"'] &
\end{tikzcd}$$

is commutative, $\hat{g} = z \circ g$ and $h = \hat{h} \circ z$.

A test for an arrow $h: Z \longrightarrow Y$ to have the right lifting property (RLP) w.r. to I, or simply, an I-test for h, is a triple $t = (i, u, v)$ of arrows forming with h a commutative diagram

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "i"'] \arrow[dr, phantom, "\circ"] & Z \arrow[d, "h"] \\ V \arrow[r, "v"'] & Y \end{tikzcd}$$

where $i \in I$. The test t is filled for h if there exists a "diagonal" $d: V \to Z$ making both triangles commute:

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "i"'] & Z \arrow[d, "h"] \\ V \arrow[ur, "d"] \arrow[r, "v"'] & Y . \end{tikzcd}$$

The arrow h has the RLP w.r. to I iff all I-tests for h are filled.

Given $Z \xrightarrow{z} \hat{Z}$ and the commutative triangle

$$\begin{tikzcd} Z \arrow[dr, "h"] \arrow[d, "z"'] & \\ \hat{Z} \arrow[r, "\hat{h}"'] & Y \end{tikzcd}$$

and $t = (i, u, v)$ is an I-test for h, then the z-continuation of t is the I-test

$\hat{t} = (i,\ z\circ u,\ v)$ for $\hat{h}$. If t is filled for h by d, say, then its z-continuation $\hat{t}$ continues to be filled, namely by $z\circ d$:

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "u"] \arrow[d, "i"'] \arrow[drr, "z\circ u" near start] & & Z \arrow[d, "z"] \arrow[dd, bend left=60, "h"] \\
 & & \hat{Z} \arrow[d, "\hat{h}"] \\
V \arrow[rr, "v"'] \arrow[urr, "z\circ d"] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \\
V \arrow[ur, "d"] &
\end{tikzcd}$$

<u>Lemma</u> Suppose
$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\
 & Z \arrow[ur, "h"'] &
\end{tikzcd}$$
is a factorization of f, and A is a κ-good, κ-directed diagram $A : P \longrightarrow \mathbb{A}$ such that $\langle A\rangle = g$.

Then there are:

1) a κ-good, κ-directed diagram $A^+ : P^+ \longrightarrow \mathbb{A}$ end-extending A (P is an initial segment of P^+) such that every link in A^+ is either a link already in A, or it is an arrow in $Po(I)$;

2) a factorization
$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g^+"'] & & Y \\
 & Z^+ \arrow[ur, "h^+"'] &
\end{tikzcd}$$
of f

where $Z^+ = A^+_T$ and $g^+ = \langle A^+\rangle$

such that

3) (g^+, h^+) is a z-continuation of (g, h) for the canonical arrow $z \underset{def}{=} can_{A, A^+} : A_T \longrightarrow A^+_T$ (determined by the commutativity of all triangles

$$\begin{tikzcd} & A_T \arrow[dd] \\ A_x = A^+_x \arrow[ur, "a_{xT}"] \arrow[dr, "a^+_{xT}"'] & \\ & A^+_T \end{tikzcd} \quad , \quad x \in P \subseteq P^+ \;)$$

and

4) the z-continuation of every I-test for h is filled for h^+.

<u>Proof</u>. We first perform the construction for a single I-test $t = (i, u, v)$ for h: we construct $\hat{A}$, $\hat{g}$, $\hat{h}$ having the properties stated for A^+, g^+, h^+ but without A^+ required to be κ-directed, and 4) required only for the single I-test t.

Since A is κ-directed, and U is κ-presentable, there are $x \in P$ and $\hat{u} : U \longrightarrow A_x$ such that

$$\begin{tikzcd} U \arrow[rr, "u"] \arrow[dr, "\hat{u}"'] & & Z \; (= A_T) \\ & A_x \arrow[ur, "a_{xT}"'] & \end{tikzcd}$$

commutes.

Let $\hat{P}$ be the poset for which $P \sqsubset \hat{P}$ (P an initial segment of $\hat{P}$), $\hat{P}$ has exactly one element, x^+, not in P, and if $y \in P$, then $y < x^+$ iff $y \leq x$. Clearly, $\hat{P}$ is κ-good, x^+ is isolated in $\hat{P}$, $(x^+)^- = x$.

Define the object R and the arrows a and $\hat{v}$ to form a pushout diagram

$$\begin{array}{ccc} U & \xrightarrow{\hat{u}} & A_x \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle a} \\ V & \xrightarrow{\hat{v}} & R \end{array},$$

Define $\hat{A} : \hat{P} \longrightarrow \mathcal{A}$ by the conditions that $\hat{A}$ extends A, $\hat{A}_{x^+} = R$ and $\hat{a}_{xx^+} = a$.

Since $a \in Po(i)$, the conditions in 1), take-away "κ-directed", are fullfilled.

It is easy to see that the following

$$\begin{array}{ccc} A_x & \xrightarrow{a_{xT}} & Z = A_T \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle z =_{def} can_{A,\hat{A}}} \\ R & \longrightarrow & \hat{Z} = \hat{A}_T \end{array}$$

is a pushout diagram.

In the following commutative diagram:

$$
\begin{tikzcd}
U \arrow[r, "\hat{u}"] \arrow[rr, bend left, "u"] \arrow[dd, "i"'] & A_x \arrow[r, "a_{xT}"'] \arrow[d, "a"] & Z \arrow[d, "z"] \arrow[dr, "h"] & \\
 & R \arrow[r, "\hat{a}_{x+T}"] \arrow[rr, bend right, "\hat{d}"] & \hat{Z} \arrow[r, "\hat{h}"] & Y \\
V \arrow[ur, "\hat{v}"] \arrow[urrr, "v"'] & & &
\end{tikzcd}
$$

the arrow $\hat{d} : R \longrightarrow Y$ is produced by R being a pushout, and then $\hat{h} : \hat{Z} \to Y$ is produced by $\hat{Z}$ being a pushout. With $\hat{g} =_{def} \langle \hat{A} \rangle$, we have the commutative diagram

$$
\begin{tikzcd}
X \arrow[dr, "g"] \arrow[ddr, "\hat{g}"'] & & Y \\
 & Z \arrow[ur, "h"] \arrow[d, "z"] & \\
 & \hat{Z} \arrow[uur, "\hat{h}"'] &
\end{tikzcd}
$$

(for $\hat{g} = z \circ g$, by the definition of $z = can_{A,\hat{A}}$); therefore $(\hat{g}, \hat{h})$ is a factorization of f continuing (g, h) via z. The z-continuation $\hat{t} = (i, z\circ u, v)$ of the given test (i, u, v) is filled for $\hat{h}$ by the arrow $d = \hat{a}_{x+T} \circ \hat{v}$.

This completes the one-test construction.

The proof of the Lemma is a straightforward transfinite iteration of the one-test construction just described. Let $t_\alpha = (i_\alpha, u_\alpha, v_\alpha)$ for $\alpha \leq \lambda$, λ a limit ordinal, be a transfinite enumeration of the <u>set</u> of all I-tests for the arrow h.

Recursively, we define the κ-good diagram $A^\alpha : P_\alpha \longrightarrow \mathbb{A}$ and the factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\langle A^\alpha \rangle = g_\alpha"'] & & Y \\ & Z_\alpha = A^\alpha_T \arrow[ur, "h_\alpha"'] & \end{tikzcd}$$

for $\alpha \leq \lambda$ such that, when $\beta < \alpha \leq \lambda$, $A^\beta \sqsubseteq A^\alpha$ and (g_α, h_α) continues (g_β, h_β) via

$$z_{\beta\alpha} = \mathrm{can}_{A^\beta, A^\alpha} : Z_\beta \longrightarrow Z_\alpha .$$

We let $A^0 = A$, $g_0 = g$, $h_0 = h$.

Given $A^\alpha, g_\alpha, h_\alpha$, to get $A^{\alpha+1}$, we take t_α, the α^{th} I-test for h, and let t be the $z_{0\alpha}$-continuation of t_α, an I-test for h_α.

Let $A^{\alpha+1}$, $g_{\alpha+1}$, $h_{\alpha+1}$ be constructed as $\hat{A}, \hat{g}, \hat{h}$ are in the one-test construction, with A there being A^α here, and $g = g_\alpha$, $h = h_\alpha$.

For a limit ordinal $\alpha \le \lambda$, we let

$$A^\alpha = \bigcup_{\beta<\alpha} A^\beta, \quad g_\alpha = \langle A^\alpha \rangle, \quad \text{and } h_\alpha : Z_\alpha = A^\alpha_T \to Y$$

be defined uniquely by the condition that

$$\begin{tikzcd} Z_\beta \arrow[d, "z_{\beta\alpha}"'] \arrow[dr, "h_\beta"] & \\ Z_\alpha \arrow[r, "h_\alpha"'] & Y \end{tikzcd}$$

commutes (the arrows $z_{\beta\alpha} = \mathrm{can}_{A^\beta, A^\alpha}$, for $\beta<\alpha$, form a colimit cocone on the diagram Z_β $(\beta<\alpha)$, $z_{\gamma\beta} : Z_\gamma \to Z_\beta$ $(\gamma \le \beta < \alpha)$).

This completes the transfinite recursive construction of the items A^α, g_α and h_α for $\alpha \le \lambda$.

Given the fact that a once-filled test remains filled in all continuations, it is clear that A^λ, g_λ, h_λ satisfy all the conditions in the lemma except the requirement of κ-directedness.

We let A^+ be a <u>trivial</u> end-extension of A, with the same composite as A, which is κ-good, κ-directed, and has no new link not already in A; such A^+ exists.

We define $g^+ = g_\lambda$, $h^+ = h_\lambda$.
This completes the proof of the lemma.

<u>Proof of the Theorem</u>

Let A^0 be the one-object diagram $A^0 : \{\bot\} \longrightarrow \mathbb{A}$ for which $A^0_\bot = X$; $g_0 \underset{\text{def}}{=} \langle A^0 \rangle = 1_X$, $h_0 \underset{\text{def}}{=} f$.
We define A^α, g_α, h_α for $\alpha \leq \kappa$ by a simple iteration of the $+$-construction of the Lemma:

$$A^{\alpha+1} = (A^\alpha)^+, \quad g_{\alpha+1} = (g_\alpha)^+, \quad h_{\alpha+1} = (h_\alpha)^+$$

and, for $\alpha \leq \kappa$ limit,

$$A^\alpha = \bigcup_{\beta<\alpha} A^\beta, \quad g_\alpha = \langle A^\alpha \rangle, \quad h_\alpha \text{ satisfying}$$

$$\begin{tikzcd}
Z_\beta = A^\beta_T \arrow[d, "z_{\beta\alpha} = \mathrm{can}_{A^\beta, A^\alpha}"'] \arrow[dr, phantom, "\circ"] \arrow[r, "h_\beta"] & Y \\
Z_\alpha \arrow[ur, "h_\alpha"'] &
\end{tikzcd}$$

We claim that $g = g_\kappa$, $h = h_\kappa$ will satisfy the requirements in the Theorem.

$g \in \mathrm{Gd}_\kappa \mathrm{dir}_\kappa(\mathrm{Po}(\mathcal{I}))$ since $g = \langle A^\kappa \rangle$, and A^κ is a κ-good, κ-directed diagram with all links in $Po(I)$

$g \in Cell(Po(I))$, since g is the composite of the continuous transfinite system Z_α $(\alpha < \kappa)$, $z_{\beta\alpha} : Z_\beta \longrightarrow Z_\alpha$ $(\beta \le \alpha < \kappa)$ in which each link

$$z_{\alpha,\alpha+1} : Z_\alpha \to Z_{\alpha+1}$$

is of the form

$$can_{A^\alpha,(A^\alpha)^+} : A^\alpha_T \longrightarrow (A^\alpha)^+_T ,$$

which, by the $+$-construction of the Lemma, is a continuous transfinite composite with links in $Po(I)$.

Finally, h has the RLP w.r. to I. We are given an I-test $t = (i, u, v)$ for $h = h_\kappa$:

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "i"'] \arrow[dr, phantom, "\circ"] & Z_\kappa \arrow[d, "h_\kappa"] \\ V \arrow[r, "v"'] & Y \end{tikzcd}$$

U being κ-presentable, there are $\alpha < \kappa$, α successor ordinal, and $u_1 : U \longrightarrow Z_\alpha$ such that $u = z_{\alpha\kappa} \circ u_1$. Since $\alpha = \beta + 1$, $A^\alpha = (A^\beta)^+$, and thus, A^α is κ-directed. Consider $A^{\alpha+1} = (A^\alpha)^+$. We have the I-test

$t_1 = (i, u_1, v)$ for $h_\alpha = h_\kappa \circ z_{\alpha\kappa}$:

$$\begin{tikzcd}
U \arrow[r, "u_1"] \arrow[dd, "i"'] & Z_\alpha \arrow[d, "z_{\alpha\kappa}"] \arrow[dd, bend left=60, "h_\alpha"] \\
 & Z_\kappa \arrow[d, "h_\kappa"] \\
V \arrow[r, "v"'] & Y
\end{tikzcd}$$

The lemma says that the $z_{\alpha,\alpha+1}$ - continuation of t_1, $t_2 = (i,\ z_{\alpha,\alpha+1} \circ u_1, v)$, is filled for $h_{\alpha+1}$.

But our original test $t = (i, u, v)$ is the $z_{\alpha+1,\kappa}$ - continuation of t_2:

$$u = z_{\alpha+1,\kappa} \circ z_{\alpha,\alpha+1} \circ u_1$$
$$= z_{\alpha\kappa} \circ u_1$$

therefore, t is filled for $h = h_\kappa = z_{\alpha+1,\kappa} \circ h_{\alpha+1}$.

$\square$