Rearranging colimits: A categorical lemma due to Jacob Lurie

§1 Statement of the result

$\mathbf{A}$: a locally presentable category.

A *transfinite system* of length α, or briefly, an *α-system* (α : an ordinal), in $\mathbf{A}$, is a functor

$$A : [\alpha] \longrightarrow \mathbf{A}$$

where $[\alpha]$ is the ordered set of all ordinals $\beta\le\alpha$, ordered in the usual way. We write A_β for $A(\beta)$, and $A_{\gamma\beta}$, or $a_{\gamma\beta}$, for $A(\gamma\le\beta) : A_\gamma \to A_\beta$. Of course, we have

- $A_{\beta\beta} = \mathrm{id}_{A_\beta}$ $\qquad (\beta\le\alpha)$,
- $A_\delta \xrightarrow{A_{\delta\gamma}} A_\gamma \xrightarrow{A_{\gamma\beta}} A_\beta = A_\delta \xrightarrow{A_{\delta\beta}} A_\beta$ $\qquad (\delta\le\gamma\le\beta\le\alpha)$,

The system is *continuous* if, for every limit ordinal β, $\beta\le\alpha$, $(A_\gamma \xrightarrow{a_{\gamma\beta}} A_\beta)_{\gamma<\beta}$ is a *colimit* cocone on the diagram $A\restriction[<\beta]$ $(= (A_\gamma, A_\delta \xrightarrow{a_{\delta\gamma}} A_\gamma)_{\delta\le\gamma<\beta})$ (it is automatically a cocone).

Since we are interested only in continuous transfinite systems, henceforth, by "transfinite system" we mean a continuous one.

The arrows $A_\gamma \xrightarrow{a_{\gamma,\,\gamma+1}} A_{\gamma+1}$ are called the *links* of the system. The arrow $a_{0\alpha}: a_0 \longrightarrow a_\alpha$ is the *composite* of the system.

We also write $a_{\beta\top}$ for $a_{\beta\alpha} (=A_{\beta\alpha})$ and $a_{\bot\top}$ for $a_{0\top}$.

Instead of ordinals, we may use elements of any well-ordered set to index transfinite systems, without any essential change of the concept.

With I any class of arrows, $\mathcal{C}[I, \alpha]$ is the class of all composites of continuous transfinite systems of length α whose links are from the class $I\cup\{$all isomorphism arrows$\}$.

$$\mathcal{C}[I] \overset{\mathrm{DEF}}{=} \mathcal{C}[I,\infty] \overset{\mathrm{DEF}}{=} \bigcup_{\alpha\in\mathrm{Ord}} \mathcal{C}[I,\alpha] \ .$$

$$\mathcal{C}[I,<\alpha] \overset{\mathrm{DEF}}{=} \bigcup_{\beta<\alpha} \mathcal{C}[I,\beta] \ .$$

Note that $\mathcal{C}[\mathcal{C}[\mathcal{I}]] = \mathcal{C}[\mathcal{I}]$.

$\mathrm{Po}[\mathcal{I}]$ is the class of all pushouts of arrows in $\mathcal{I}$: the class of all arrows $A \to B$ for which there is a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & \square & \uparrow \\ X & \longrightarrow & Y \end{array}$$

with $(X \to Y) \in \mathcal{I}$.

Remark: These concepts will be used in two contexts: in a (fixed) category called $\mathbf{A}$, and the functor category $\mathbf{A}^{\mathbf{G}}$, with a (fixed) (small) exponent category $\mathbf{G}$. We may (but not always will) put $\mathbf{A}$, or $\mathbf{A}^{\mathbf{G}}$, in a subscript position, such as $\mathcal{C}_{\mathbf{A}}[\mathcal{I}]$, to indicate context.

The combination $\mathcal{C}[\mathrm{Po}\mathcal{I}]$ is written $\mathcal{S}[\mathcal{I}]$.

Given an α-system A in the notation above, and an arrow $A_0 \xrightarrow{p_0} \hat{A}_0$, we can take the pushout of the system A along the arrow p_0 , and get an α-system

$$\hat{A} = (\hat{A}_\beta, \hat{A}_\gamma \xrightarrow{\hat{a}_{\gamma\beta}} \hat{A}_\beta)_{\gamma \le \beta \le \alpha} :$$

for any $\beta \le \alpha$, we define the object $\hat{A}_\beta$ and the arrows $\hat{a}_{0\beta}: \hat{A}_0 \to \hat{A}_\beta$, $p_\beta: A_\beta \to \hat{A}_\beta$ by taking the pushout

$$\begin{array}{ccc} A_0 & \xrightarrow{A_{0\beta}} & A_\beta \\ {\scriptstyle p_0}\downarrow & \square & \downarrow{\scriptstyle p_\beta} \\ \hat{A}_0 & \xrightarrow[\hat{a}_{0\beta}]{} & \hat{A}_\beta \end{array} .$$

It follows that the square

$$\begin{array}{ccc} A_\gamma & \xrightarrow{A_{\gamma\beta}} & A_\beta \\ {\scriptstyle p_\gamma}\downarrow & \square & \downarrow{\scriptstyle p_\beta} \\ \hat{A}_\gamma & \xrightarrow[\hat{A}_{\gamma\beta}]{} & \hat{A}_\beta \end{array}$$

is a pushout square whenever $\gamma \le \beta \le \alpha$; from which both defining conditions for $\hat{A}$ being an element of $\mathcal{C}[\mathrm{Po}(\boldsymbol{I}),\alpha]$ (continuity at limit ordinals, and the links being pushouts of $\boldsymbol{I}$-arrows) follow.

Thus, $\mathcal{C}[\mathrm{Po}(\boldsymbol{I}),\alpha]$ is closed under pushouts, $\mathrm{Po}(\mathcal{C}[\mathrm{Po}(\boldsymbol{I}),\alpha]) = \mathcal{C}[\mathrm{Po}(\boldsymbol{I}),\alpha]$. In fact,

$$\mathcal{S}[\mathcal{S}[\boldsymbol{I}]] = \mathcal{S}[\boldsymbol{I}] \ .$$

A class $\boldsymbol{X}$ of arrows in category $\boldsymbol{A}$ is *$\mathcal{S}$-closed* if $\mathcal{S}[\boldsymbol{X}]=\boldsymbol{X}$. An $\mathcal{S}$-closed class $\boldsymbol{X}$ is *small-$\mathcal{S}$-generated* if there is a small set $\boldsymbol{I}$ such that $\mathcal{S}[\boldsymbol{I}]=\boldsymbol{X}$; a small-$\mathcal{S}$-generated class is, in particular, $\mathcal{S}$-closed.

Consider a small-$\mathcal{S}$-generated class $\boldsymbol{X}$ of arrows in the category $\boldsymbol{A}$, and let $\boldsymbol{G}$ any small category. The class $\langle \boldsymbol{X}, \boldsymbol{G}\rangle$ of arrows in $\boldsymbol{A}^{\boldsymbol{G}}$ is defined to be the class of all arrows $F: U \longrightarrow V$ (natural transformations) in $\boldsymbol{A}^{\boldsymbol{G}}$ such that every component $F_G: U(G) \longrightarrow V(U)$ ($G \in \mathrm{Ob}(\boldsymbol{G})$) belongs to $\boldsymbol{X}$. It is obvious that if $\boldsymbol{X}$ is $\mathcal{S}$-closed (in $\boldsymbol{A}$), then $\langle \boldsymbol{X}, \boldsymbol{G}\rangle$ is $\mathcal{S}$-closed (in $\boldsymbol{A}^{\boldsymbol{G}}$) as well.

Lemma (J. Lurie) Assume $\boldsymbol{A}$ is a locally presentable category, $\boldsymbol{G}$ a small category. If $\boldsymbol{X}$ is a small-$\mathcal{S}$-generated class of arrows in $\boldsymbol{A}$, then so is the class $\langle \boldsymbol{X}, \boldsymbol{G}\rangle$ in $\boldsymbol{A}^{\boldsymbol{G}}$.

I have found this interesting, since the proof seems to require something unexpected: systematical rearranging of colimits, in particular, transfinite composites, into other types of colimits.

§2 Rearranging colimits in general

CAT : the super-large category of possibly large categories (so that Set , the category of small sets, in in CAT). Let $\boldsymbol{A}$ be "normal" category (in CAT and locally small; later: locally presentable). Form CAT/$\boldsymbol{A}$, the comma-category (whose objects are $\begin{matrix}\boldsymbol{X}\\ \downarrow\\ \boldsymbol{A}\end{matrix}$, and whose arrows are commutative triangles $\begin{matrix}\boldsymbol{X} & \longrightarrow & \boldsymbol{Y}\\ & \searrow \ \circ \ \swarrow & \\ & \boldsymbol{A} & \end{matrix}$). We have a pair of (partial) adjoints (D is total, L is partial)

$$\boldsymbol{A} \underset{L}{\overset{D}{\rightleftarrows}} \mathrm{CAT}/\boldsymbol{A} \qquad\qquad L \dashv D$$

$$D:\ A \mapsto \begin{array}{c}\boldsymbol{A}/A\\ \downarrow\\ \boldsymbol{A}\end{array}$$

$$\boldsymbol{A} \underset{L\ =\ \mathrm{colim}}{\overset{D}{\rightleftarrows}} \mathrm{CAT}/\boldsymbol{A}\ ;$$

In more detail: D is defined by:

$$\boldsymbol{A} \longrightarrow \mathrm{CAT}/\boldsymbol{A}$$

$$A \longmapsto \begin{array}{c}\boldsymbol{A}/A\\ \downarrow \mathrm{d}\\ \boldsymbol{A}\end{array}$$

$$A \xrightarrow{a} A' \longmapsto \begin{array}{c}\boldsymbol{A}/A \xrightarrow{\Sigma_a} \boldsymbol{A}/A'\\ \mathrm{d}\searrow \ \circ\ \swarrow \mathrm{d}\\ \boldsymbol{A}\end{array}$$

mistake! not bold

where d stands for "domain" $\left(\begin{array}{c}\boldsymbol{X}\\ \downarrow\\ \boldsymbol{A}\end{array} \xrightarrow{\mathrm{d}} \boldsymbol{X}\right)$, and $(X \xrightarrow{x} A) \xmapsto{\Sigma_a} (X \xrightarrow{ax} A')$.

also: ⊢

Indeed, the adjunction bijection

$$\frac{L\left(\begin{array}{c}\boldsymbol{G}\\ \downarrow\Gamma\\ \boldsymbol{A}\end{array}\right) \xrightarrow{f} A}{\begin{array}{c}\boldsymbol{G}\\ \downarrow\Gamma\\ \boldsymbol{A}\end{array} \xrightarrow{\gamma} D(A)} = \frac{\mathrm{colim}\Gamma \xrightarrow{f} A}{\begin{array}{c}\boldsymbol{G} \xrightarrow{\gamma} \boldsymbol{A}/A\\ \Gamma\searrow \ \circ\ \swarrow \mathrm{d}\\ \boldsymbol{A}\end{array}}$$

expresses that cocones with vertex A on diagram Γ are in a bijective correspondence with arrows $f\colon \mathrm{colim}\Gamma \longrightarrow A$.

To say that $\boldsymbol{A}$ is small-cocomplete, is to say that $L\left(\begin{array}{c}\boldsymbol{G}\\ \downarrow\\ \boldsymbol{A}\end{array}\right)$ is defined for all small $\boldsymbol{G}$.

Let us apply the fact that (partial) adjoints preserve colimits.

(For easier reading, we will write Colimit with a capital C when it is used in CAT or CAT/$\boldsymbol{A}$.)

What that means is that if in CAT/$\boldsymbol{A}$ we have a diagram Λ of objects at each of which L is defined, and $\mathrm{Colim}(\Lambda)$ exists in CAT/$\boldsymbol{A}$, then, in $\boldsymbol{A}$, $L(\mathrm{Colim}(\Lambda))$ is defined if and only if colim$(L\circ\Lambda)$ exists, and they are the same:

Fact 1

$$L(\text{Colim}(\Lambda)) \simeq \text{colim}(L\circ\Lambda)$$

[$\simeq$: Kleene's "complete equality" (IM p. 327). Here it means that if either side is defined, so is the other, and the two values are *isomorphic* (rather than equal as in Kleene)].

In the applications of this fact, we also use that Colimits in CAT/$\mathbf{A}$ are computed as in CAT : the forgetful functor CAT/$\mathbf{A}\longrightarrow$CAT preserves (in fact, creates) Colimits.

We will apply the above in the following "rearrangement" form.

Let $A:P\longrightarrow\mathbf{A}$ be a diagram, and assume that A_T=colim A exists.

Suppose

$$Q:\mathbf{Q}\longrightarrow\text{Cat} \qquad (x\in Q\mapsto Q_x\,,\ x\xrightarrow{f}y\mapsto Q_x\xrightarrow{q_f}Q_y)$$

is a diagram in Cat ($\subset$CAT) , and P=Colim Q in Cat , with coprojections $q_x:Q_x\longrightarrow P$. Let $A:P\longrightarrow\mathbf{A}$ be a diagram in $\mathbf{A}$, and let $\Gamma\circ q_x:Q_x\longrightarrow\mathbf{A}$ be the restriction of Γ ($x\in Q$). Assume that B_x = colim$(\Gamma\circ q_x)$ exists for all $x\in Q$; let, for $u\in Q_x$, $b_{uT}:\Gamma q_x u\longrightarrow B_x$ be the coprojection. We have a canonical arrow $b_{x\xrightarrow{f}y}:B_x\rightarrow B_y$, defined by the property

$$\begin{array}{ccc} & \Gamma q_x u \;=\; \Gamma q_y q_f u & \\ {\scriptstyle b_{uT}}\swarrow & \circ & \searrow{\scriptstyle b_{q_f u}} \\ B_x & \xrightarrow[\;b_{x\xrightarrow{f}y}\;]{} & B_y \end{array}$$

for all $u\in Q_x$. We have a diagram $B:Q\longrightarrow\mathbf{A}$, $B(x)=B_x$, $B(x\xrightarrow{f}y)=b_{x\xrightarrow{f}y}$; this is what we call a *rearrangement* of the original diagram A.

Fact 2 The rearrangement has the same colimit as the original diagram, and in fact, one colimit exists iff the other one does:

$$\text{colim}\,B \simeq \text{colim}\,A$$

This is an application of Fact 1. $\Lambda:\mathbf{Q}\longrightarrow$CAT/$\mathbf{A}$ is given by

$$\Lambda : \mathbf{Q} \longrightarrow \mathrm{CAT}/\mathbf{A}$$

$$x \longmapsto \begin{array}{c} Q_X \\ \downarrow \Gamma\circ q_X \\ \mathbf{A} \end{array}$$

$$x \xrightarrow{f} y \qquad \begin{array}{ccc} Q_X & \xrightarrow{q_f} & Q_Y \\ & \searrow \ \circ \ \swarrow & \\ & \mathbf{A} & \end{array}$$

We have that $\mathrm{Colim}\Lambda = \begin{array}{c} P \\ A\downarrow \\ \mathbf{A} \end{array}$, the original diagram; $L\circ\Lambda$ is the rearrangement $\begin{array}{c} Q \\ B\downarrow \\ \mathbf{A} \end{array}$; the colimit (L) of the first is the same as the colimit of the second.

We will use Fact 2 in a certain special kind of situation.

Suppose $A: P\to\mathbf{A}$ is a diagram on the *poset* P. Let Q be a collection of subsets of P, and consider Q to be the poset ordered by containment ($\subseteq$). Define the diagram $\Phi: Q\to\mathrm{CAT}$ as "identity": $\Phi(X)=X$ (more precisely, the poset X with the order induced by that of P on X), with $\Phi(X\subseteq Y) = \mathrm{inclusion}: X\to Y$.

Assume that $\mathrm{Colim}\,\Phi$ is P; more precisely, assume that the family of inclusions

$$\langle X \xrightarrow{\mathrm{incl}} P\rangle_{X\in Q} \text{ is a colimit cocone in } \mathrm{CAT}. \qquad (*)$$

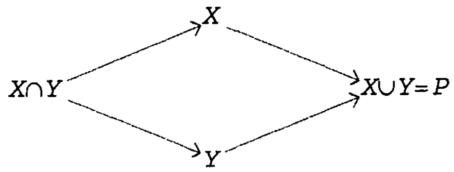
Note that it is *not enough* to have a colimit in POSET ; $\mathrm{POSET}\xrightarrow{\mathrm{incl}}\mathrm{CAT}$ does not preserve all Colimits.

One case when (*) holds is when we have 1) and 2) as follows:

1) $\bigcup Q = \bigcup_{X\in Q} X = P$;

2) Q is directed under the subset-ordering: if $X, Y\in Q$, then there is $Z\in Q$ such that $X\subseteq Z$ and $Y\subseteq Z$.

Another case when (*) holds is this. We have subsets X and Y of P such that $X\cup Y=P$, and for $x\in X-(X\cap Y)$, $y\in Y-(X\cap Y)$, x and y are incomparable in the order on P; we take $Q=\{X\cap Y, X, Y\}$. In this case, with the posets meant to be the induced subposets of P, the colimit $\mathrm{Colim}\,\Phi$ is given by the diagram



with all arrows inclusions, a Pushout in CAT .

Although the just stated case of rearrangement is one that is important for us, it might be noted that it falls under the more general conditions, ensuring (*), which are 1) above, the condition that each $X\in Q$ is an initial segment of P, and the condition that $X, Y\in Q$ imply $X\cap Y\in Q$.

A *third* case when (*) holds is when $R\subseteq P$ is an initial segment of P $(x\le y\in R \Longrightarrow x\in R)$, and, with $x{\downarrow}=\{y\in P: y\le x\}$, we have $Q=\{R\}\dot\cup\{R\cup x{\downarrow}: x\in P-R\}$. Note that, for $x, y\in P-R$, $R\cup x{\downarrow}\subseteq R\cup y{\downarrow}$ iff $x\le y$; and R is the bottom element of Q.

Of course, in all three cases, the verification of (*) is a routine check.

A collection $Q\subseteq\mathcal{P}(P)$ satisfying the assumption (*) gives rise to a *rearrangement* of the diagram A as follows.

Let, for each $X\in Q$, A_X denote a (choice of) $\mathrm{colim}(A{\restriction}X)$, with $A{\restriction}X: X\to\mathbf{A}$ the restriction of A to X, with the ordering on X induced by that on P. Let $\langle a_{xX}: A_x\to A_X\rangle_{x\in X}$ be the corresponding colimit cocone.

We stipulate that when X has a top (maximum) element w (that is, $w\in X$ and for all $x\in X$, we have $x\le w$), then $A_X \overset{\mathrm{DEF}}{=} A_w$ and $a_{xX}=a_{xw}$. (as cocone of choice for colim(A↾X))

Whenever $X\subseteq Y$, $X, Y\in Q$, we have the canonical map $a_{XY}: A_X\to A_Y$ for which $a_{XY}\circ a_{xX}=a_{xY}$ $(x\in X)$. It is easy to see that we have a diagram $A[Q]: Q\to\mathbf{A}$, $A[Q]=(A_X, a_{XY})_{X,\ Y\in Q,\ X\subseteq Y}$.

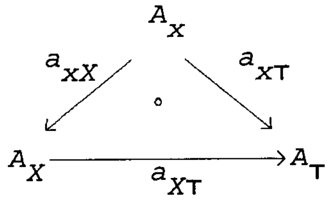
The assertion is that, under the foregoing conditions,

Fact 3 $\mathrm{colim}(A[Q])\simeq\mathrm{colim}(A)$.

This is a special case of Fact 2: the diagram B of Fact 2 is $A[Q]$.

We make two, essentially equivalent, detailed assertions out of Fact 3, the "direct" and the "converse" versions. The direct version says (in a detailed manner) that if $\mathrm{colim}(A)$ exists, then so does $\mathrm{colim}(A[Q])$; the "converse" version says the converse.

(i)(direct) Let $\langle a_{xT}:A_x\longrightarrow A_T\rangle$ be a colimit cocone on A, and define, for $X\in Q$, $a_{XT}:A_X\longrightarrow A_T$ by



for all $x\in X$. Then $\langle a_{XT}\rangle_{X\in Q}$ is a colimit cocone on the diagram $A[Q]:Q\to\mathbf{A}$.

(ii)(converse) Let $\langle a_{XT}:A_X\longrightarrow A_T\rangle_{X\in Q}$ be a colimit cocone on the diagram $A[Q]:Q\to\mathbf{A}$, and define, for $x\in P$, $a_{xT}:A_x\longrightarrow A_T$ as $a_{xT}=a_{XT}\circ a_{xX}$ with some/any $X\in Q$ such that $x\in X$. (By 1), there is such X; and by the directedness axiom 2), one sees that a_{xT} so defined is independent of the choice of X.) Then $\langle a_{xT}:A_x\longrightarrow A_T\rangle_{x\in P}$ is a colimit cocone.

§3 Good diagrams

κ is an infinite regular cardinal.

In this section, we only assume of the category $\mathbf{A}$ that it is (locally small and small-)cocomplete. Of course, it still make sense to say of an object that it is κ-presentable.

Let $P=(P,\leq)$ be a partial order. $<$ is the irreflexive version of $\leq$. $u, v, w, x, \ldots$ range over P. $x\!\downarrow=\{y:y\leq x\}$; $x\!\downarrow\downarrow=\{y:y<x\}$.

We make two assumptions on P:

1) P has a least element $\bot$ (for which $\bot\leq x$ for all x).
2) $<$ is well-founded (no decreasing infinite sequence $x_0>x_1>x_2>\ldots$).

Let $x\in P$. If $x\!\downarrow\downarrow$ has a top (maxim*um*) element x^- (such that $y<x\Longleftrightarrow y\leq x^-$), we call x *isolated*; x^- is the *predecessor* of x. Note that the notion of "successor" is not well-defined: it may happen that different points have the same predecessor (unlike in the linearly ordered case).

A point x which is not isolated, and which is unequal to $\bot$, is called a *limit point.* (We could call a limit point "colimit point", in view of the role this notion is going to have.)

P is of κ-*good* if for all $x\in P$, $\#(x\!\downarrow)<\kappa$.

A *good* diagram in $\mathbf{A}$,

$$A=\langle A_x,\ A_x \xrightarrow{a_{xy}} A_y \rangle_{x,y\in P;\ x\le y} : P \longrightarrow \mathbf{A}$$

is a functor from a good poset P to the category $\mathbf{A}$ such that, for every limit point x, the subdiagram $A\upharpoonright(x\downarrow)$ is a *colimit* diagram: the family $\gamma=\langle a_{yx}:A_y\to A_x\rangle_{y<x}$ is a *colimit* cocone on the diagram $A\upharpoonright(x\downarrow\downarrow)$ (the fact that γ is a cocone on $A\upharpoonright(x\downarrow\downarrow)$ is already given).

Note that, in the notation introduced for Fact 3, the goodness condition can be expressed by saying that the arrow $a_{x\downarrow\downarrow,\,x}:A_{x\downarrow\downarrow}\longrightarrow A_x$ is an isomorphism whenever x is a limit point of P.

The good diagram is κ-*good* if the underlying poset is κ-good.

Let us denote the colimit $\mathrm{colim}(A)$ by $A_\top$. We write $a_{x\top}:A_x\to A_\top$ for the colimit coprojection ($x\in P$). The *composite* of the good diagram A is the coprojection $a_{\bot\top}:A_\bot\longrightarrow A_\top$; the composite of the good diagram A is sometimes denoted $\langle A\rangle:A_\bot\longrightarrow A_\top$.

Clearly, the notion of good diagram essentially generalizes that of transfinite system. The only "difference" in the two concepts is that, in "transfinite system", we have included the composite itself as data. More precisely, if A is a transfinite system of length $\alpha\ge 1$, then $A\upharpoonright\alpha = A\upharpoonright\{\beta<\alpha\}$ is a good diagram, and its composite is isomorphic to the composite of A (in the original sense of "composite" of a transfinite system).

The *links* of a good diagram are (using the notation above) the arrows $a_{x^{-}x}$ for the isolated points x.

Let $\mathcal{G}[\mathcal{J}]$ be the class of composites of (small) good diagrams all whose links are in $\mathcal{J}$. We let $\mathcal{G}[\mathcal{J},\ (<\kappa)]$ denote the class of composites of κ-good diagrams all whose links are in $\mathcal{J}$.

For λ a cardinal number, $\mathcal{G}[\mathcal{J},\ \lambda]$ will denote the (set of) composites $\langle A\rangle$ of good diagrams $A:P\to\mathbf{A}$ such that $\#(P-\{\bot\})=\lambda$.

$$\mathcal{G}[\mathcal{J},\ \le\lambda] \overset{\mathrm{DEF}}{=} \bigcup_{\mu\le\lambda}\mathcal{G}[\mathcal{J},\ \mu]\text{ , where } \mu \text{ ranges over cardinal numbers } \le\lambda .$$

We use $\mathcal{G}[\mathcal{J},(<\kappa),\lambda]$ in the sense $\mathcal{G}[\mathcal{J},\ (<\kappa)]\cap\mathcal{G}[\mathcal{J},\ \lambda]$; and similarly with $\le\lambda$ replacing λ; etc.

The first fact, 1. Proposition, that justifies the passage to the more general concept of good diagram is that it shares the main property of transfinite systems. As a matter of fact, however, 1. Proposition will not be used for our technical purposes.

Let ℓ and r be two arrows in the category $\mathbf{A}$. We say that ℓ is *left-orthogonal* to r, equivalently, r is *right-orthogonal* to ℓ, in notation $\ell\perp r$, if

This is probably WRONG! You must mean that the diagram involved has both properties: $(<\kappa)$ and λ

for every $\begin{array}{ccc} \cdot & \xrightarrow{g} & \cdot \\ \ell\downarrow & \circ & \downarrow r \\ \cdot & \xrightarrow[h]{} & \cdot \end{array}$, there is $\begin{array}{ccc} \cdot & \xrightarrow{g} & \cdot \\ \ell\downarrow & \circ \nearrow^{k} \circ & \downarrow r \\ \cdot & \xrightarrow[h]{} & \cdot \end{array}$.

If R is a set of arrows, then $\ell\perp R$ means $\forall r\in R.\ \ell\perp r$; and similarly for other combinations.

1. Proposition Let $r\in \mathrm{Arr}(\mathbf{A})$. Suppose that A is a good diagram and for every link ℓ in A, $\ell\perp r$. Then $\langle A\rangle\perp r$.

The proof is "the same" as for transfinite systems. For completeness, we outline it.

Let A be a good diagram; we use the notation above.

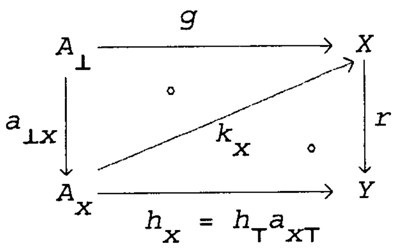
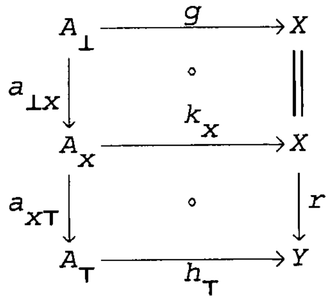
Let $r:X\to Y$. Let us fix $g:A_\perp\longrightarrow X$ and $h=h_\top:A_\top\longrightarrow Y$ such that

$$\begin{array}{ccc} A_\perp & \xrightarrow{g} & X \\ \langle A\rangle\downarrow & \circ & \downarrow r \\ A_\top & \xrightarrow[h_\top]{} & Y \end{array} ;$$

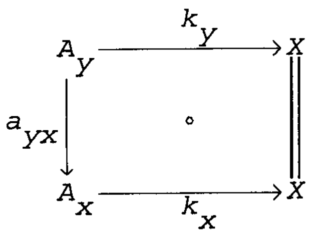
we seek $k_\top:A_\top\longrightarrow X$ such that

$$\begin{array}{ccc} A_\perp & \xrightarrow{g} & X \\ \langle A\rangle\downarrow & \circ \nearrow^{k_\top} \circ & \downarrow r \\ A_\top & \xrightarrow[h_\top]{} & Y \end{array} \qquad (1^-)$$

By recursion on the well-founded relation $<$ (the order on P) (!), we define $k_x:A_x\to Y$ such that

 , that is,  (1)

(here, $a_{\perp x}$ is the structure map in the diagram A ; $a_{x\top}$ is the colimit coprojection), with the additional condition that the k_x are compatible: every time $y<x$, we have

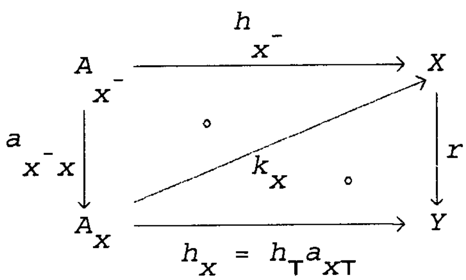
 : (2)

in other words, $\langle k_x : A_x \to X\rangle_{x\in P}$ is a cocone with vertex X on the diagram A.

For $x=\perp$, $k_\perp \overset{\text{DEF}}{=} g$; the assumption ensures that we are in the right for (1); (2) is vacuous.

For x limit, use the cocone $\langle k_y : A_y \to X\rangle_{y<x}$ on $A\upharpoonright(x\downarrow\downarrow)$, to get the unique map $k_x : A_x (=\mathrm{colim}(A\upharpoonright(x\downarrow\downarrow))) \longrightarrow X$ that makes $\langle k_y : A_y \to X\rangle_{y\le x}$ into a cocone on the diagram $A\upharpoonright(x\downarrow)$; we have ensured (2). (1) will be true because: the upper commutativity is the cone property of $\langle k_y : A_y \to X\rangle_{y\le x}$, tested with $y_1=\perp<y_2=x$, since $g=h_\perp$; and for the lower commutativity, the two maps $A_x \rightrightarrows Y$ that are to be shown equal are equal when composed with $a_{yx} : A_y \to A_x$ $(y<x)$, and x is the colimit of $A\upharpoonright(x\downarrow\downarrow)$.

For x isolated: use the assumption that $a_{x^- x} \perp r$, to obtain a_x such that



You will see that both (1) and (2) will follow for the present x ; in case of (2), for $y=x^-$ first, and then for all $y<x$.

(1^-) is shown as the inductive case for x limit (although " $x=\top$ " we cannot say).

This completes the proof.

Below, we will see that, in fact, the two operations $\mathcal{C}[\mathrm{Po}[-]]$ and $\mathcal{G}[\mathrm{Po}[-]]$ coincide (in particular, if one accepts as known that 1. Prop. holds for $\mathcal{C}$ instead of $\mathcal{G}$, then 1. Prop. itself becomes superfluous). The point of the new $\mathcal{G}$-operation lies in the parameter κ, in the specific version $\mathcal{G}[-, (<\kappa)]$, which has no direct counterpart for the $\mathcal{C}$-operation.

For posets P and Q, we write $P\sqsubseteq Q$ if P is a non-empty *initial segment* of Q : P is an induced subposet of Q (for $x, y\in P$, $x\le^{(P)}y \iff x\le^{(Q)}y$), and P is closed downward in Q : $x\in P$ and $y\le^{(Q)}x$ imply that $y\in P$.

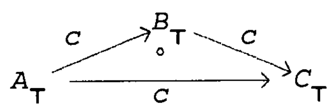
(For a subset X of a poset P, $X\sqsubseteq P$ means that X is a non-empty initial segment of the poset P in the usual sense (X is closed downward); for two subsets X, Y of P, we write $X\sqsubseteq Y$ in the obvious appropriate sense.)

If Q is a good poset, and $P\neq\emptyset$, $P\sqsubseteq Q$, then P is good as well; if Q is κ-good, so is P.

Let Q be a good poset. Let $P\sqsubseteq Q$. Then for every $x\in P$, $(x{\downarrow}{\downarrow})^{(P)}=(x{\downarrow}{\downarrow})^{(Q)}$. If $x^-=\max((x{\downarrow}{\downarrow})^{(Q)})$ exists, then $x^-\le x$, thus $x^-\in P$, and $\max((x{\downarrow}{\downarrow})^{(P)}) = \max((x{\downarrow}{\downarrow})^{(Q)})$. We see that for $x\in P$, the concepts of "isolated point", "predecessor", "limit point", are the same in P as in Q.

Therefore, if $B:Q\to\mathbf{A}$ is a good diagram, $P\neq\emptyset$, $P\sqsubseteq Q$, then $B\restriction P$ is a good diagram as well; if B is κ-good, $B\restriction P$ is κ-good as well.

For good diagrams $A:P\to\mathbf{A}$ and $B:Q\to\mathbf{A}$, we write $A\sqsubseteq B$ if $P\sqsubseteq Q$ and $A=B\restriction P$. Note that, in this case, we have the canonical arrow $c=c[A, B]$ $=c[P\sqsubseteq Q]:A_\top\to B_\top$ for which $c\circ a_{x\top}=b_{x\top}$ for all $x\in P$. If $A\sqsubseteq B\sqsubseteq C$, then



.

Let $A:P\to\mathbf{A}$ be a κ-good diagram. Consider the family Q of all nonempty $X\sqsubseteq P$ such that $\#X<\kappa$. Q is ordered by containment $\subseteq$; and Q is κ-directed: any union of initial segments of P is an initial segment, and the union of κ-many ones is of cardinality $<\kappa$.

Fact 3, *case one*, is applicable: 1) holds by P being κ-good; 2) has been checked. In this situation, we will use the "direct version" of Fact 3 , in the situation stated under 1) and 2) there.

Let's record this special case as

Fact 4 Let $A:P \to \mathbf{A}$ be a κ-good diagram. Let $\hat{P}$ be the collection of all initial segments of P cardinality less than κ.

1) $(\hat{P}, \subseteq)$ is a $<\kappa$-directed (κ-filtered) poset

2) $\hat{P}$ gives rise to a rearrangement $\hat{A}=A[\hat{P}]$ of A (in the sense of Fact 3).

Note that $A:P \to \mathbf{A}$ can be treated as a subdiagram of $\hat{A}:\hat{P} \to \mathbf{A}$, by identifying P with a subposets of $\hat{P}$, under the identification of a with $a\downarrow$; recall that we had $A_{x\downarrow} = A_x$.

Although its use is less essential, it is convenient to use $\tilde{P}$ for the poset of *all* non-empty initial segments of P, ordered by inclusion, and $\tilde{A}$ the rearrangement diagram for A on $\tilde{P}$.

The concept of "end-segment" and the facts about it, to be treated next, are obvious in the case of transfinite systems; they are not hard, and they are also important, in the general case.

Let $A:P \to \mathbf{A}$ be a good diagram, $R \sqsubseteq P$ a non-empty initial segment of P; $P|R \overset{\text{DEF}}{=} \{\bot\} \dot{\cup} (P-R)$ (here $\bot$ is the bottom element of P); $P|R$ also denotes the induced subposet of P.

The diagram $A|R:Q \to \mathbf{A}$ is defined thus:

$$A|R:\ P|R \longrightarrow \mathbf{A}$$

$$x \longmapsto A_{R\cup x\downarrow}$$

$$(x \le y) \longmapsto a_{R\cup x\downarrow,\ R\cup y\downarrow}$$

Fact 5 **1)** $A|R$ is good, κ-good if A is, its links are pushouts of links of P, and its composite $\langle A|R\rangle$ equals the arrow $a_{RT}:A_R \to A_T$.

2) In particular, $a_{RT} \in \mathcal{G}[\mathrm{Po}[\mathcal{I}], \lambda]$ for $\lambda = \#(P-R)$, $\mathcal{I}$=the set of links of P.

More generally, for $R \sqsubseteq S \sqsubseteq P$, by the foregoing applied to $A \restriction S$, we have $a_{RS} \in \mathcal{G}[\mathrm{Po}[\mathcal{I}], \lambda]$ for $\lambda = \#(S-R)$.

3) $(P|R)^{\wedge} \cong P \| R \overset{\text{DEF}}{=} \{X \in \tilde{P} : R \sqsubseteq X \ \&\ \#(X-R) < \kappa\}$, by the map $Y \mapsto R \cup Y$. The diagram $(A|R)^{\wedge}$ is thereby identified with $\tilde{A} \restriction (P\|R)$, the restriction of the diagram $\tilde{A}$ to the subposet $P\|R$ of $\tilde{P}$.

Proof of 1) The last fact about the composite is the "third case" of Fact 3 stated above.

Temporarily, we write Q for $P|R$, and B for $A|R$.

More generally, for any non-empty initial segment X of Q, by using Fact 3 for $R\dot{\cup}X$ in place of P, we can, *and do*, take $B_X\,(=\mathrm{colim}\, B_X)$ to be $A_{R\dot{\cup}X}$, and $b_{XY}: B_X \longrightarrow B_Y$ to be $a_{R\dot{\cup}X,\, R\dot{\cup}Y}: A_{R\dot{\cup}X} \longrightarrow A_{R\dot{\cup}Y}$ ($\bot\in X\sqsubseteq Y\sqsubseteq Q$).

For any $x\in Q-\{\bot\}$, $(x\downarrow\downarrow)^{(Q)} = \{\bot\}\dot{\cup}(x\downarrow\downarrow-R)$. Therefore, if x isolated for Q , then the corresponding B-link

$$B_{(x^-)^{(Q)}} \xrightarrow{\;b_{(x^-)^{(Q)},\,x}\;} B_x$$

equals the arrow

$$A_{R\cup x^-\downarrow} = A_{R\cup x\downarrow\downarrow} \xrightarrow{\;a_{R\cup x\downarrow\downarrow,\; R\cup x\downarrow}\;} A_{R\cup x\downarrow} .$$

Let $x\in Q$ be isolated in P. Then x is isolated in Q as well: *either* $x\downarrow\downarrow-R\neq\emptyset$, in which case x^- , the P-predecessor of x, is in $x\downarrow\downarrow-R$ (otherwise $x^-\in R$, and since R is closed downward, $x^-\downarrow=x\downarrow\downarrow\subseteq R$, and $x\downarrow\downarrow-R=\emptyset$), and $(x^-)^{(Q)}=x^-$; *or* $x\downarrow\downarrow-R=\emptyset$ and $(x^-)^{(Q)}=\bot$.

Let $x\in P-R$, and consider the diagram of inclusions:

$$\begin{array}{ccc} R\cup x\downarrow\downarrow & \longrightarrow & R\cup x\downarrow \\ \uparrow & \square & \uparrow \\ x\downarrow\downarrow & \longrightarrow & x\downarrow \end{array}$$

This is a Pushout in CAT , as in "case two" of Fact 3. Therefore,

$$\begin{array}{ccc}
A_{R\cup x\downarrow\downarrow} & \xrightarrow{\;a\;} & A_{R\cup x\downarrow} \\
a\uparrow & \square & \uparrow a \\
A_{x\downarrow\downarrow} & \xrightarrow[a]{} & A_{x\downarrow}
\end{array}$$

with each a being the corresponding " a-arrow", is a pushout in $\mathbf{A}$.

If x is a limit point of P, the lower horizontal a is an isomorphism; if x is isolated in P, the same is a link of A . Thus, the upper horizontal is always in $\mathrm{Po}[\mathcal{I}]$. Therefore, if $x\in Q$ is an isolated point of Q, the corresponding B-link, being the upper horizontal in the last diagram, is in $\mathrm{Po}[\mathcal{I}]$.

Finally, if x is a limit point of Q, then (as we saw above) x is a limit point of P as well, $B_{(x\downarrow\downarrow)}(Q) \xrightarrow{\;b\;} B_x$ is $A_{R\cup x\downarrow\downarrow} \xrightarrow{\;a\;} A_{R\cup x\downarrow}$, and the latter arrow, being a pushout of the isomorphism $A_{x\downarrow\downarrow} \xrightarrow{\;a\;} A_{x\downarrow}$, is an isomorphism itself.

This completes the (overly fussy?) proof of Fact 5.

2. Proposition

(i) $\mathcal{G}[\mathcal{J}] \subseteq \mathcal{C}[\mathrm{Po}[\mathcal{J}]]$. In fact, $\mathcal{G}[\mathcal{J}, \lambda] \subseteq \mathcal{C}[\mathrm{Po}[\mathcal{J}], \lambda]$ for any infinite cardinal λ .

(ii) For any cardinal number (=initial ordinal number) $\lambda\geq\kappa$,

$$\mathcal{G}[\mathcal{J}, (<\kappa), \leq\lambda] \subseteq \mathcal{C}[\mathrm{Po}[\mathcal{J}], \lambda] .$$

[In other words: if $A: P\to\mathbf{A}$ is a κ-good diagram such that $\lambda=\#P\geq\kappa$, and the links of A are in the class $\mathcal{J}$, then there is a transfinite sequence $B: [\lambda]\to\mathbf{A}$ of length the initial ordinal λ , whose links are in $\mathrm{Po}[\mathcal{J}]$, such that $\langle A\rangle=\langle B\rangle$.]

Proof We remind the reader of a well-known fact: for any well-founded partial order $(P, <)$, there is a well-ordering (well-founded total ordering) $\prec$ of P extending $<$ ($x<y$ implies $x\prec y$).

[The proof is by Zorn's lemma. Let $\mathcal{X}$ consist of all well-ordered sets $(X, \prec_X)$ such that

$$X\subseteq P \;\;\&\;\; \forall x, y\in X(x<y \Rightarrow x\prec_X y) \;\;\&\;\; \forall x, y\in X(y<x\Rightarrow y\in X \;\;(\&x\prec_X y))$$

($\prec_X$ extends $<\restriction X$; and X is an initial segment of $(P, <)$);

and let $\prec\!\prec$ be the partial ordering of "initial segment" on $\mathcal{X}$:

$$(X, \prec_X) \prec\prec (Y, \prec_Y) \iff$$
$$X \subseteq Y \ \& \ \forall x_1 x_2 (x_1 \prec_X x_2 \Rightarrow x_1 \prec_Y x_2) \ \& \ \forall y \in Y. \forall x \in X. (y \prec_Y x \Rightarrow y \in X) \ .$$

Clearly, the union of any $\prec\prec$-chain in $\mathcal{X}$ is again a member of $\mathcal{X}$. Let $(X, \prec_X)$ be in $(\mathcal{X}, \prec\prec)$, and let $u \in P - X$. Define $Y = X \dot{\cup} \{u\}$, and $\prec_Y$ on Y such that $\prec_Y$ extends $\prec_Y$, and $x \prec_Y u$ for all $x \in X$. Then $(Y, \prec_Y)$ belongs to $\mathcal{X}$; note that $\prec_Y$ extends $< \restriction Y$, because $u < x$ with $x \in X$ is impossible, since $u \notin X$ and X is an initial segment of $(P, <)$. Thus, with $(X, \prec_X)$ maximal, $X = P$.]

To prove part (i), let $A : P \longrightarrow \mathbf{A}$ be a good diagram; let $\prec$ be a (total) well-ordering of the set P extending the given well-founded partial order $<$ on P.

Note that $\bot$, the bottom element for $<$, is necessarily the least element for $\prec$ as well.

For any $x \in P$, let $[x) = \{y \in P : y \prec x\}$ and $[x] = \{y \in P : y \preceq x\}$. Define the subclass Q of $\mathcal{P}(P)$ as

$$Q = \{ [x] : x \in P \} \ \dot{\cup} \ \{ [x) : x \in \mathrm{Lim}_{\prec}(P) \}$$

($\mathrm{Lim}_{\prec}(P)$ is the set of points that are limit points with respect to the well-ordering $\prec$).

Clearly, $Q \subseteq \mathcal{P}(P)$ is suitable for a rearrangement of the diagram A in the sense of Fact 3. In addition, Q is well-ordered by $\subset$ (strict subset relation). The limit points of $(Q, \subset)$ are the sets $[x)$ for x a $\prec$-limit point. $[x]$ is a successor unless $x = \bot$: the $\subset$-predecessor of $[x]$ is $[x)$.

In fact, the order type of $(Q, \subset)$ *is equal to that of* $(P, \prec)$ *if the latter is a limit ordinal; and one more if the latter is a successor ordinal.*

We have, in the notation of Fact 3, the diagram $A[Q] : Q \rightarrow \mathbf{A}$ such that $\mathrm{colim}\, A[Q] = \mathrm{colim}\, A$, and $\langle A[Q] \rangle = \langle A \rangle$; adding a top element to $A[Q]$ gives us a transfinite system. The continuity of the transfinite system is an application of rearrangement (Fact 3, first situation), coming from the fact that, for a $\prec$-limit point x, $[x)$ is the directed union

$$[x) = \bigcup_{y \prec x} [y] \ \cup \bigcup_{\substack{y \prec x \\ y \ \prec\text{-}limit}} [y) \ .$$

As to the links of $A[Q] : Q \rightarrow \mathbf{A}$, the typical $A[Q]$-link $a_{[x),\, [x]}$ appears in the pushout diagram

$$
\begin{array}{ccc}
A_{[x)} & \xrightarrow{\;a_{[x),\,[x]}\;} & A_{[x]} \\
\uparrow{\scriptstyle a_{x\downarrow\downarrow,\,[x)}} & & \uparrow{\scriptstyle a_{x,\,[x]}} \\
A_{x\downarrow\downarrow} & \xrightarrow[\;a_{x\downarrow\downarrow,\,x}\;]{} & A_{x}
\end{array}
$$

which is the result of "Fact 3, second type" rearrangement, according to the CAT-Pushout

$$
\begin{array}{ccc}
[x) & \longrightarrow & [x] \\
\uparrow & & \uparrow \\
x\downarrow\downarrow & \longrightarrow & x\downarrow
\end{array}
$$

of induced subposets [we have $[x)\cap x\downarrow = x\downarrow\downarrow$, $[x)\cup x\downarrow = [x]$, and any $y\in [x)-x\downarrow\downarrow$ is not $\le$-comparable to any $z\in x\downarrow - x\downarrow\downarrow$ (*otherwise*: $z=x$; $z=x\le y$ would imply $x\underline{\prec} y$, false since $y\in [x)$; so $y<x$, contradicting $y\notin x\downarrow\downarrow$)].

By the assumptions on A, the lower horizontal $A_{x\downarrow\downarrow}\xrightarrow{a_{x\downarrow\downarrow,\,x}} A_x$ is an isomorphism when x is a $<$-limit, and an element of $\mathcal{J}$ when x is isolated (in this case $A_{x\downarrow\downarrow}=A_{x^-}$).

Thus, the $A[Q]$-links are all in $Po[\mathcal{J}]$.

This proves part (i).

To see part (ii), assume the hypotheses of (ii) on A. An elementary argument shows that now the well-ordering $\prec$ of the above proof can be chosen so that the order-type of $(P,\prec)$ equals the (initial) ordinal λ.

[In the next few lines, each of the symbols $<$, $\le$ is used in two different senses. They are used in the standard senses in contexts like $\beta<\alpha$, $\beta\le\alpha$ for ordinals α,β; and they are used in the sense of the given κ-good partial order on P, in contexts like $x<y$, $x\le y$ for $x, y\in P$.]

[Let $(\alpha\mapsto x_\alpha)$ be a bijection $[\lambda)\longrightarrow P$. Keeping with the notation $x\downarrow=\{y\in P: y\le x\}$, with the original κ-good ordering $<$ of P, let

$$X_\alpha = x_\alpha\downarrow - \bigcup_{\beta<\alpha} x_\beta\downarrow .$$

P is the disjoint union $P=\dot{\bigcup}_{\alpha<\lambda} X_\alpha$, and $Y_\alpha=\dot{\bigcup}_{\beta\le\alpha} X_\beta = x_\alpha\downarrow$ is closed downward

I think, I had this starting [, to watch the choice on [illegible] page at *

($z<y\in Y_\alpha \Rightarrow y\in Y_\alpha$) . Let $\alpha[x]$ denote the ordinal α for which $x\in X_{\alpha[x]}$. Choose a well-ordering $\prec_\alpha$ of X_α (which set, of course, may be empty) such that $\prec_\alpha$ extends $<\upharpoonright X_\alpha$, by the opening general fact above. Define the relation $\prec$ on P by

$$x\prec y \iff \textit{either}\ \alpha[x]<\beta[x]\ \textit{or}\ (\alpha[x]=\beta[x]=\alpha \text{ and } x\prec_\alpha y).$$

$\prec$ is a well-ordering of P in order-type the *ordinal sum*

$$\sum_{\alpha<\lambda}\delta_\alpha ,$$

where δ_α is the order-type of $(X_\alpha, \prec_\alpha)$.

$\prec$ extends the partial order $<$ on P: let $x, y\in P$, $\alpha=\alpha[x]$, $\beta=\alpha[y]$, and assume $x<y$, to prove $x\prec y$. Since $\dot{\bigcup}_{\gamma\leq\beta} X_\gamma$ is closed downward for $<$ on P, we have $x\in \dot{\bigcup}_{\gamma\leq\beta} X_\gamma$, and thus $\alpha\leq\beta$. Then either $\alpha<\beta$, in which case $x\prec y$ as desired; or $\alpha=\beta$, in which case $x<y$ implies $x\prec_\alpha y$ implies $x\prec y$ as desired.

Since P is κ-good, each set $x\downarrow$ is of cardinality $<\kappa$; hence each set $X_\alpha\subseteq x_\alpha\downarrow$ is of cardinality less than κ; hence, since κ is regular, $\delta_\alpha<\kappa$. Therefore, since $\kappa\leq\lambda$ and κ is regular, we have $\sum_{\alpha<\lambda}\delta_\alpha\leq\lambda$. Of course, as the order-type of the set P of cardinality λ , $\sum_{\alpha<\lambda}\delta_\alpha\geq\lambda$. Therefore, $\sum_{\alpha<\lambda}\delta_\alpha$, the order-type of $(P, \prec)$, equals the initial ordinal λ .] ⚹

As the italicised sentence above says, the well-ordered set $(Q, \sqsubset)$, constructed in the proof above in part (i), has order-type that of $(P, \prec)$, the latter being the limit ordinal λ ; therefore, the transfinite system $A[Q]$ constructed in the proof of part (i) is of length equal to the initial ordinal λ as desired.

The converse of 2. Prop., 3. Prop. below, is somewhat more difficult. In preparation for 3. Prop., we introduce some constructions.

Directed union of good diagrams

Here is the first construction on good diagrams, *directed union*, that we will need.

Suppose $A^i : P^i \to \mathbf{A}$ is a good diagram for $i\in I$, where $I=(I, \leq)$ is a *non-empty* directed poset ($(i, j\in I) \Longrightarrow \exists k\in I.\, i\leq k \,\&\, j\leq k$); and $A^i \sqsubseteq A^j$ whenever $i\leq j$. Then we can define the union $A = \bigcup^{\uparrow}_{i\in I} A^i$, a good diagram, such that $A^i\sqsubseteq A$ for all $i\in I$.

Namely, we let $P = \bigcup^{\uparrow}_{i\in I} P^i$, the poset whose underlying set is the union of the underlying sets of the posets P^i, and for which $x\le^{(P)} y \iff x\le^{(P^i)} y$ for some, equivalently any, $i\in I$ such that x, y are both elements of P^i. Clearly, $P^i\sqsubseteq P$ $(i\in I)$. P is a good poset: check that each of the conditions 1), 2) follows from its truth for (some/)all P^i. Similarly, if each P^i is κ-good, so is P.

The diagrams A^i, A^j must agree on their common domain, since we have some $k\ge i, j$ and A^k extends both A^i and A^j. Thus, it is meaningful to define $A: P\to\mathbf{A}$ by the condition that $A\restriction P^i = A^i$. A so defined is a good diagram;. κ-good if each A^i is κ-good.

Let us apply Fact 3, "converse" version (ii), to the collection $Q=\{P_i : i\in I\}$ of initial segments of P. We are allowed to do that since P is the directed union of the members of Q. Let's repeat, in a suitable notation, what we get now.

Let $A^i_\top = \mathrm{colim}(A^i)$ with coprojections $a^i_{x\top}: A^i_x \longrightarrow A^i_\top$. For $i\le j$, let $a^{ij}=c[A^i, A^j]: A^i_\top \longrightarrow A^j_\top$, the canonical arrow. These data form a diagram $\hat{A}: I\longrightarrow\mathbf{A}$. Let $A_\top = \mathrm{colim}(\hat{A}) = \underset{i\in I}{\mathrm{colim}}(A^i_\top)$ with coprojection $a^{i\top}: A^i_\top \longrightarrow A_\top$.

For $x\in P$, $a^i_{x\top} \overset{\mathrm{DEF}}{=} a^{i\top}\circ a^i_{x\top}: A^i_x \longrightarrow A_\top$ for some/any i such that $x\in P_i$.

We have that the $a_{x\top}$ for $x\in P = \bigcup^{\uparrow}_{i\in I} P^i$ form a cocone on the diagram $A = \bigcup^{\uparrow}_{i\in I} A^i$, and in fact, this is a colimit cocone.

Here is the second construction we will need, actually two similar constructions, both adjoining a new link to a diagram.

Adjoining a link

Given a good diagram $A: P\to\mathbf{A}$, and an initial segment $X\sqsubseteq P$ of P. Let A_X denote $\mathrm{colim}(A\restriction X)$; when $w=\max(X)$ exists, we put $A_X = A_w$. Let $a_{yX}: A_y\to A_X$ be the colimit coprojection ($y\in X$), and $a_{X\top} = c[X, P]: A_X\to A_\top$ the canonical map; when $u=\max(X)$ exists, $a_{yX} = a_{yw}$ $(y\in X)$ (in particular, $a_{wX} = \mathrm{id}_{A_w}$), and $a_{X\top} = a_{w\top}$.

Suppose also given an arrow $f: A_X\to\hat{B}$ (thus, the domain of f is the given object A_X, its codomain $\hat{B}$ is arbitrary). We construct the new diagram $B = A[f/X]$, the result of "adjoining f to A at X", as follows.

We define the new poset Q by adjoining two new elements, x and x^+, to P. We let

$$Q = P\dot{\cup}\{x\}\dot{\cup}\{x^+\} ;$$

$$u <^{(Q)} v \Longleftrightarrow$$

$$(u,\ v\in P \ \&\ u <^{(P)} v) \vee (u\in X \ \&\ (v=x \vee v=x^+)) \vee (u=x \ \&\ v=x^+) .$$

P is an initial segment of Q. We have that $(x\downarrow\downarrow)^{(Q)} = X$.

If $\max(X)$ does not exist in P, x is a limit point in Q; if w=max(X) does exist in P, then x is isolated in Q and $x^- = w$.

x^+ is isolated: $x^{+-} = x$. Q is good if P is good; Q is κ-good if P is κ-good and $\#X<\kappa$.

We define the diagram $B:Q\longrightarrow\mathbf{A}$ by stipulating that

$$B\upharpoonright P = A ,$$
$$B_x = A_X ,$$
$$B_{x^+} = \hat{B} ,$$

$$b_{yx} = a_{yX} \qquad (y\in X)$$
$$b_{xx^+} = f .$$

If $\max(X)$ does not exist, the construction ensures that the continuity condition at the new limit point x holds true; in this case, there is just one new link, $b_{xx^+} = f$. In case w=max(X) does exist, there is no new limit point, and there are two new links, $b_{wx}=\mathrm{id}_{A_X}$ and $b_{xx^+} = f$.

If A is good, then so is $A[f/X]$. If, in addition, $\#X<\kappa$, and A is κ-good, then $A[f/X]$ is κ-good.

The important fact about this construction is that the diagram

$$
\begin{array}{ccc}
A_{\mathsf{T}} & \xrightarrow{\;c\,[P\sqsubseteq Q]\;} & B_{\mathsf{T}} \\
{\scriptstyle a_{X\mathsf{T}}}\uparrow & \square & \uparrow{\scriptstyle b_{x^+\mathsf{T}}} \\
B_X = A_X & \xrightarrow[\;f\;]{} & \hat{B}
\end{array}
$$

is a pushout; in other words, the canonical arrow from the colimit A_{T} of the original diagram A to that of the extension, $A[f/X]_{\mathsf{T}}$, is a pushout of the adjoined link f.

Conversely, if we define the items $\tilde{B}$, c and b by the pushout

$$
\begin{array}{ccc}
A_{\mathsf{T}} & \xrightarrow{\;c\;} & \tilde{B} \\
{\scriptstyle a_{X\mathsf{T}}}\uparrow & \square & \uparrow{\scriptstyle b} \\
B_X = A_X & \xrightarrow[\;f\;]{} & \hat{B}
\end{array}
$$

then, for the diagram $B=A[f/X]$, we can take $\mathrm{colim}\, B = B_{\mathsf{T}}$ to be $B_{\mathsf{T}} = \tilde{B}$, with colimit cocone $\langle b_{y\mathsf{T}} : B_y \to \tilde{B} \rangle_{y\in Q}$ given as $b_{u\mathsf{T}} = c\circ a_{u\mathsf{T}}$ $(u\in P)$, $b_{X\mathsf{T}} = b\circ f = c\circ a_{X\mathsf{T}}$ and $b_{x^+\mathsf{T}} = b$.

These two facts are the direct and converse aspects of the rearrangement of the diagram $A[f/X]$, according to Fact 3, in the second case mentioned there, with the roles of X, Y, $X\cap Y$ and $P=X\cup Y$ played by the sets $X\dot{\cup}\{x, x^+\}$, P, X and Q, respectively.

3. Proposition Assume the domain of each arrow in the set I of arrows is κ-presentable. Then

(i) $\mathcal{C}[\mathrm{Po}[I]] \subseteq \mathcal{G}[\mathrm{Po}[I], \kappa]$;

Is this right? no! Stupid!

and more specifically:

(ii) $\mathcal{C}[\mathrm{Po}[I], <\lambda^+] \subseteq \mathcal{G}[\mathrm{Po}[I], (<\kappa), \leq\lambda]$

The following are immediate consequences of 3. and 2. Prop's.

4. Corollary Assume $\mathbf{A}$ is a (small-)cocomplete category, and $\mathcal{I}$ is any class of arrows in $\mathbf{A}$. Then

(i) $\mathcal{C}[\mathrm{Po}[\mathcal{I}]] = \mathcal{G}[\mathrm{Po}[\mathcal{I}]]$.

If, in addition, the domain of each arrow in the class $\mathcal{I}$ is κ-presentable, then

(ii) $\mathcal{C}[\mathrm{Po}[\mathcal{I}]] = \mathcal{G}[\mathrm{Po}[\mathcal{I}], <\kappa]$;

and, for any initial ordinal (=cardinal) $\lambda \geq \kappa$,

(iii) $\mathcal{C}[\mathrm{Po}[\mathcal{I}], <\lambda^{+}] = \mathcal{C}[\mathrm{Po}[\mathcal{I}], \leq\lambda]$.

(ii) (probably!)

Proof of 3. Prop., part (i) Assume A is a transfinite system of length α all whose links are from $\mathcal{I}$. By transfinite recursion, we define, for each $\beta \leq \alpha$, the following items:

1) A κ-good poset P^{β} such that, for $\gamma \leq \beta \leq \alpha$, P^{γ} is an initial segment of P^{β} ($P^{\gamma} \sqsubseteq P^{\beta}$).

2) A κ-good diagram $B^{\beta}: P^{\beta} \to \mathbf{A}$ with links in $\mathrm{Po}[\mathcal{I}]$ such that $B^{\beta}_{\perp} = A_0$ (as a good diagram, P^{β} has a least element $\perp_{\beta}$; on the left, $B^{\beta}_{\perp} \overset{\mathrm{DEF}}{=} (B^{\beta})_{\perp_{\beta}}$), and such that, for $\gamma \leq \beta \leq \alpha$, B^{β} is an extension of B^{γ} (in other words, $B^{\gamma} = B^{\beta} \restriction P^{\gamma}$).

3) For any $\beta \leq \alpha$, a *colimit* cocone $\langle b^{\beta}_{x\top} : B^{\beta}_{x} \to A_{\beta} \rangle_{x \in P^{\beta}}$ with vertex the given object A_{β}, on the diagram B^{β} such that $b^{\beta}_{\perp\top} = a_{0\beta}$ (note that this makes the given A_{β} the colimit of B^{β}, and the given $a_{0\beta}$ the composite of B^{β}) such that, for $\gamma \leq \beta \leq \alpha$ and $x \in P^{\gamma} \subseteq P^{\beta}$ (and so $B^{\gamma}_{x} = B^{\beta}_{x}$), we have

$$\begin{array}{ccc} B^{\gamma}_{x} & \xrightarrow{b^{\gamma}_{x\top}} & A^{\gamma} \\ \| & \circ & \downarrow a_{\gamma\beta} \\ B^{\beta}_{x} & \xrightarrow[b^{\beta}_{x\top}]{} & A^{\beta} \end{array}$$

(which makes the canonical $c[B^{\gamma}, B^{\beta}] : B^{\gamma}_{\top} \longrightarrow B^{\beta}_{\top}$ equal $a_{\gamma\beta}$).

To start, for $\beta = 0$, $P^{\beta} = \{\perp\}$, etc.

Suppose we have $\beta\leq\alpha$, and we have defined all the above items for subscripts γ with $\gamma<\beta$, with all the required compatibilities satisfied below β.

Suppose first that β is a limit ordinal. Then we can take $B^{\beta}=\bigcup\limits_{\gamma<\beta}^{\uparrow} B^{\gamma}$, according to our definition of "directed union" above. Thereby, we have fulfilled requirements 1) and 2) for all subscripts $\leq\beta$.

I claim that if, for $x\in P^{\beta}$, we define $b^{\beta}_{X\top}:B^{\beta}_{X}\to A_{\beta}$ as $b^{\beta}_{X\top}=a_{\gamma\beta}\circ b^{\gamma}_{X\top}$ with some/any $\gamma<\beta$ such that $x\in P^{\beta}$, then $b^{\beta}_{X\top}$ so defined is independent of the choice of $\gamma<\beta$, and 3) holds for $\delta\leq\gamma\leq\beta$. This follows from Fact 4, and the fact that A_{β} is the colimit of $A\restriction[<\beta]$, with coprojections $a_{\gamma\beta}:A_{\gamma}\to A_{\beta}$.

It remains to handle the case when $\beta\leq\alpha$ is a successor ordinal, $\beta=\gamma+1$.

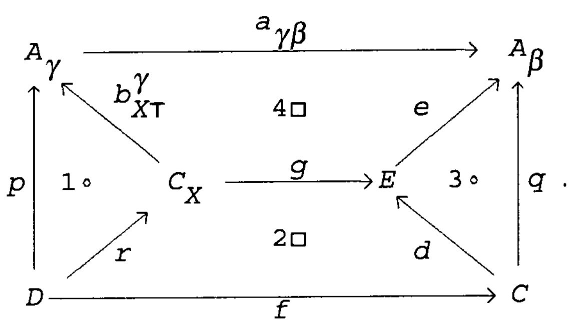
By assumption, we have a pushout diagram

$$\begin{array}{ccc} A_{\gamma} & \xrightarrow{a_{\gamma\beta}} & A_{\beta} \\ {\scriptstyle p}\uparrow & \lrcorner & \uparrow{\scriptstyle q} \\ D & \xrightarrow[f]{} & C \end{array} \qquad (1)$$

with $f\in\mathcal{I}$. We apply the induction hypothesis for γ. A_{γ} is the colimit of the κ-good diagram B^{γ}. According to Fact 4, applied to B^{γ} as A in Fact 4, A_{γ} is a the κ-directed colimit of $\langle (B^{\gamma}\restriction X)_{\top}, b_{XY}\rangle_{X\sqsubseteq Y\in\hat{P}^{\gamma}}$, with colimit coprojections $b_{X\top}:(B^{\gamma}\restriction X)_{\top}\to A_{\gamma}$, where $\hat{P}^{\gamma}$ is the poset of all $<\kappa$-size non-empty initial segments of P^{γ}.

Let us abbreviate $(B^{\gamma}\restriction X)_{\top}$ by C_{X}.

Since, by assumption, the object D is κ-presentable, there are $X\in\hat{P}^{\gamma}$ and $r:D\to C_{X}$ such that $p=b_{X\top}\circ r$. We construct the following diagram:



(2)

E, g and d are defined by making the square 2□ a pushout. e is then defined by stipulating that the triangle 3∘ and the square 4 commute. We have factored the pushout diagram (1) (the outside square in (2)) as the composite of the pushout 2□ and the commutative square 4∘ . It follows that 4□ is a pushout.

We put $B^\beta \stackrel{\text{DEF}}{=} B^\gamma[g/X]$, according to the construction "adjoining a link". Since g is a pushout of f, and f is in $\mathcal{I}$, g, the new link in B^β, is in $\text{Po}[\mathcal{I}]$ (a possible second link is in $\text{Po}[\mathcal{I}]$ since it is an identity arrow).

Concerning the data in 3):

For $u \in P^\gamma$:

$$b^\beta_{u\top} \stackrel{\text{DEF}}{=} a_{\gamma\beta} \circ b^\gamma_u \ : \ B^\gamma_u \xrightarrow{b^\gamma_u} A_\gamma \xrightarrow{a_{\gamma\beta}} A_\beta ;$$

$$b^\beta_{X\top} \stackrel{\text{DEF}}{=} a_{\gamma\beta} \circ b^\gamma_{X\top} \ : \ B_X = C_X \xrightarrow{b^\gamma_{X\top}} A_\gamma \xrightarrow{a_{\gamma\beta}} A_\beta ;$$

$$b^\beta_{X^+} \stackrel{\text{DEF}}{=} e .$$

The requirement that $\langle b^\beta_{u\top} : B^\beta_u \longrightarrow A_\beta \rangle_{u \in P^\beta}$ be a colimit cocone is ensured by the basic property of the "adjoining-a-link" construction, specifically the "converse" version: the requisite pushout now is the part 4□ of the diagram (2) .

Since $\#X < \kappa$, the diagram $B^\delta = B^\gamma[g/X]$ is κ-good.

This completes the recursive construction of the items under 1), 2) and 3) .

The diagram $P: B \to \mathbf{A}$ required for the proposition is $\bigcup^{\uparrow}_{\beta<\alpha} B^{\beta}$, a directed union of κ-good diagrams. Clearly, $\#P$ is no more than $2 \cdot \#\alpha$: the requirements of part (ii) are satisfied.

§4 Using good diagrams

As before, κ is an infinite regular cardinal, $\mathbf{A}$ is cocomplete category.

We let $\mathcal{I}$ be a class of arrows in $\mathbf{A}$ such that

both the domain and the codomain of each arrow in $\mathcal{I}$ *is* κ*-presentable.*

We define $\mathcal{X}$ be the class $\mathcal{C}(\text{Po}[\text{I}])$. From previous work, we recall that

$$\mathcal{X} = \mathcal{G}(\text{Po}[\text{I}], (<\kappa)) . \tag{1}$$

Let us start with two κ-good diagrams

$$A : P \longrightarrow \mathbf{A} , \qquad\qquad B : Q \longrightarrow \mathbf{A}$$

and the corresponding extensions

$$\hat{A} : \hat{P} \longrightarrow \mathbf{A} , \qquad\qquad \hat{B} : \hat{Q} \longrightarrow \mathbf{A} .$$

(recall that $X \in \hat{P} \iff X \sqsubseteq P$ & $\#X < \kappa$).

We use the notation we introduced before to deal with such diagrams.

Suppose given arrows r, s in $\mathbf{A}$ such that

$$\begin{array}{ccc} A_\perp & \xrightarrow{a_{\perp\top}} & A_\top \\ r\downarrow & \circ & \downarrow s \\ B_\perp & \xrightarrow[b_{\perp\top}]{} & B_\top \end{array} \tag{2}$$

DEF
A *factor* for $\rho = (r, s)$, or for (A, B, r, s) , is a triple (X, U, u) with $X \in \hat{P}$, $U \in \hat{Q}$ and an arrow u as in

$$\begin{array}{ccccc} A_\perp & \xrightarrow{a_{\perp X}} & A_X & \xrightarrow{a_{X\top}} & A_\top \\ r\downarrow & \circ & \downarrow u & \circ & \downarrow s \\ B_\perp & \xrightarrow[b_{\perp U}]{} & B_U & \xrightarrow[b_{U\top}]{} & B_\top \end{array}$$

($\circ$ indicates commutativity as usual). We say that the factor (X, U, u) *starts at* X .

Note the obvious fact that if (X, U, u) is a factor for (A, B, r, s), and (U, V, v) is one for (B, C, p, q), then $(X, V, v\circ u)$ is a factor for $(A, C, p\circ r, q\circ s)$:

$$\begin{array}{ccccc}
A_\bot & \xrightarrow{a_{\bot X}} & A_X & \xrightarrow{a_{X\top}} & A_\top \\
r\downarrow & \circ & \downarrow u & \circ & \downarrow s \\
B_\bot & \xrightarrow[b_{\bot U}]{} & B_U & \xrightarrow[b_{U\top}]{} & B_\top \\
p\downarrow & \circ & \downarrow v & \circ & \downarrow q \\
C_\bot & \xrightarrow[c_{\bot V}]{} & C_V & \xrightarrow[c_{V\top}]{} & C_\top
\end{array} .$$

Most of the time, in the definition the pair $\rho=(r, s)$ is fixed; we omit "for ρ " when ρ is understood.

If $\xi=(X, U, u)$ and $\eta=(Y, V, v)$ are factors , we write $\xi\le\eta$ if

$$\begin{array}{ccc}
A_X & \xrightarrow{a_{XY}} & A_Y \\
u\downarrow & \circ & \downarrow v \\
B_U & \xrightarrow[b_{UV}]{} & B_V
\end{array} .$$

Given a factor $\xi=(X, U, u)$, and any $V\in\hat{Q}$ such that $U\sqsubseteq V$, the triple $\hat{\xi}=(X, V, s\circ b_{UV})$ is a factor as well. $\hat{\xi}$ is referred to as a *(codomain) shift of* ξ, the V*-shift of* ξ .

5. Lemma **(i)** Given $\rho=(r, s)$ as in (2), and any $X\in\hat{P}$, there is a factor of ρ starting at X .

Moreover, if ξ is a factor starting at X, and $X\sqsubseteq Y\in\hat{P}$, then there is a factor η starting at Y such that $\xi\le\eta$.

(ii) For any two factors, ξ_1 and ξ_2, of the same ρ, there is a third one, ζ, such that $\xi_1\le\zeta$ and $\xi_2\le\zeta$.

Proof of (i) First, we show the assertion for $x\in P$, that is, for $X\in\hat{P}$ of the form $x\downarrow$.

By recursion of the well-founded order $<$ on P, we define, for all $x \in P$, a compatible family of factorizations $\xi_x = (x, U_x, u_x)$ starting at x: we have, for all $y \leq x$ in P that $\xi_y \leq \xi_x$.

Reminder: we require

$$\begin{array}{ccccc} A_\perp & \xrightarrow{a_{\perp x}} & A_x & \xrightarrow{a_{x\top}} & A_\top \\ r\big\downarrow & 1\circ & \big\downarrow u_x & 2\circ & \big\downarrow s \\ B_\perp & \xrightarrow[b_{\perp U_x}]{} & B_{U_x} & \xrightarrow[b_{U_x \top}]{} & B_\top \end{array} \qquad (3)$$

For $x=\perp$, we put $U_\perp = \{\perp_Q\}$, and $u_x = r$.

Let x be a limit point. The construction of the factor (x, U_x, u_x) is straightforward: we take the colimit of the compatible system of factorizations (y, U_y, u_y) for $y \in x{\downarrow\downarrow}$. In a bit more detail, here it goes.

We take $U_x \overset{\text{DEF}}{=} \bigcup_{y \in x\downarrow\downarrow} U_y$. The system $\langle a_{yx}: A_y \to A_x \rangle_{y \in x\downarrow\downarrow}$ is a colimit cocone on the diagram $A \restriction x{\downarrow\downarrow}$; the system

$$\langle A_y \xrightarrow{u_y} B_{U_y} \xrightarrow{b_{U_y U_x}} B_{U_x} \rangle_{y \in x\downarrow\downarrow}$$

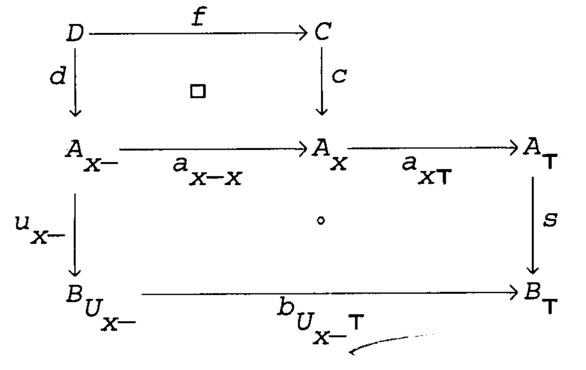
of composites is a cocone on the same diagram; therefore, we have $u_x : A_x \longrightarrow B_{U_x}$ such that

$$\begin{array}{ccc} A_y & \xrightarrow{a_{yx}} & A_x \\ u_y\big\downarrow & \circ & \big\downarrow u_x \\ B_{U_y} & \xrightarrow[b_{U_y U_x}]{} & B_{U_x} \end{array} \qquad (4)$$

commutes for every $y \in x{\downarrow\downarrow}$. It follows that (3) holds for x: $1\circ$ because $y = \perp \in x{\downarrow\downarrow}$; $2\circ$ because A_x is a colimit of $A \restriction x{\downarrow\downarrow}$, and (3) holds for all $y \in x{\downarrow\downarrow}$ in place of x.

Let x be isolated in P.

By assumption, there are $D \xrightarrow{f} C \in \mathcal{I}$ and a pushout as in the upper square in



(5)

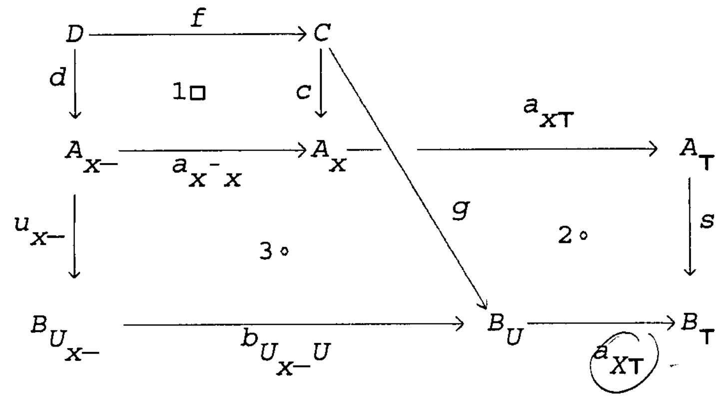
there is no Fact 4

yes, there is: p13

We use that C is κ-presentable. By Fact 4, the system

$$\langle b_{U\mathrm{T}} : B_U \longrightarrow B_{\mathrm{T}} \rangle_{U \in \hat{Q}}$$

is a colimit cocone on the κ-directed diagram $\hat{B} : \hat{Q} \to \mathbf{A}$. Therefore, the composite $s \circ a_{X\mathrm{T}} \circ c : C \to A$ factors through $b_{U\mathrm{T}} : B_U \longrightarrow B_{\mathrm{T}}$ for some $U \in \hat{Q}$, which we can take to contain U_{X-}, that is, $U_{X-} \subseteq U$. That is, we have g as in



error: b_{UT}

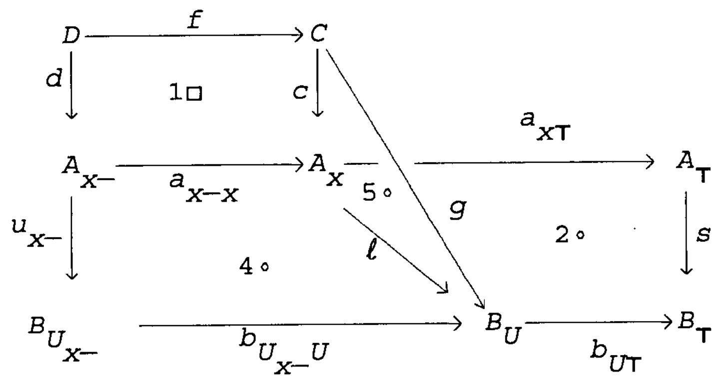
to make the commutativity $2\circ$ hold, where $2\circ$ is the equality of the two arrows from C to B_{T}.

Looking at the two parallel composites from D to B_U in the diagram, denoted $D \underset{k}{\overset{h}{\rightrightarrows}} B_U$, we see that they are coequalized by $b_{U\mathrm{T}} : B_U \longrightarrow B_{\mathrm{T}}$. Therefore, since D is κ-presentable, and B_{T} is the colimit of the κ-filtered system of the B_U's, we can choose U so that, in addition, we also have the commutativity $3\circ$; that is, $h{=}k$.

Next, since A_X is a pushout as shown, we have $\ell : A_X \to B_U$ producing the commutativities

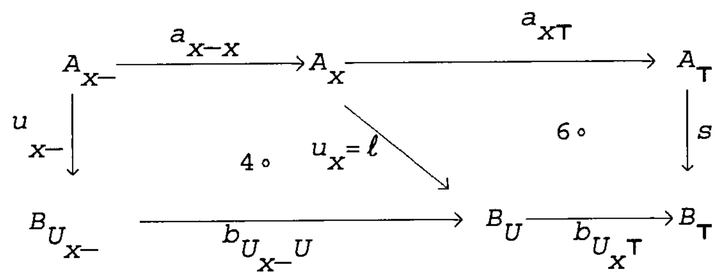
Fact 4, p13

$4\circ$ and $5\circ$, equalities of arrows $A_{x-} \rightrightarrows B_U$, respectively $C \rightrightarrows B_U$:



We put $U_x = U$ and $u_x = \ell$.

We have



;

$4\circ$ was achieved before; $6\circ$ is true since each of the pushout coprojections $c: C \to A_x$ and $A_{x-} \xrightarrow{a_{x-x}} A_x$ equalizes the two arrows $A_x \rightrightarrows B_T$.

This ensures (3), and (4) for all $y < x$ (for the latter, also because (4) holds for $x-$ in place of x).

This completes the construction

$$(x \in P) \;\longmapsto\; (U_x \in \hat{Q},\; u_x : A_x \to B_{U_x})$$

satisfying (3) and (4).

Now, let $X \in \hat{P}$ arbitrarily. In a nutshell, (X, U_X, u_X) is defined as the colimit of the compatible system of all (x, U_x, u_x) for $x \in X$.

In more detail, define $U_X = \bigcup_{x \in X} U_x$. We have that $U_X \in \hat{P}_H$ since $\#X < \kappa$, each $\#U_x < \kappa$, and κ is regular.

Define $u_X : A_X = \mathrm{colim}(A \upharpoonright X) \longrightarrow B_{U_X}$ by the condition that $u_X \circ a_{xX} = b_{U_x U_X} \circ u_x$ for all $x \in X$. The proof that (X, U_X, u_X) is a factor is similar to the proof for (x, U_x, u_x) the case of x limit above.

The moreover part of (i) follows from the main part, by applying it to the derived good diagrams $A|X : P|X \to \mathbf{A}$, $B|U : Q|U \to \mathbf{A}$, and to the situation

$$\begin{array}{ccc} A_X & \xrightarrow{\;a_{XT}\;} & A_T \\ {\scriptstyle u}\big\downarrow & \circ & \big\downarrow{\scriptstyle s} \\ B_U & \xrightarrow[\;b_{UT}\;]{} & B_T \end{array}$$

where $\xi = (X, U, u)$.

Proof of (ii) First, we prove the assertion for the special case when both ξ_1 and ξ_2 start at the same point x in P (rather than at a general $X \in \hat{P}$). More specifically, we prove, by induction on x according to the well-founded relation $<$, that factors ξ_1 and ξ_2 starting at x have a common shift.

Let $\xi_i = (x, U_i, u_i)$ $(i \in \{1, 2\})$.

The proof for $x = \bot$ is similar to that for x limit, to which we turn now.

Let x be a limit point; the argument now is a straightforward appeal to the "uniqueness property" of colimits.

In more detail:

We have $A_x = \mathrm{colim}(A \upharpoonright x{\downarrow}{\downarrow})$.

For each $y \in x{\downarrow}{\downarrow}$, and for each $i \in \{1, 2\}$, we have the factorization $\eta_y^i = (y, U_i, u_y^i)$, where $u_y^i = u_i \circ a_{yx} : A_y \to B_{U_i}$ starting at y; by induction hypothesis, there is Y_y such that the Y_y-shifts of η_y^1 and η_y^2 are equal.

Let $Y = \bigcup_{y \in x\downarrow\downarrow} Y_y$. Then the Y-shifts of η^1_y and η^2_y are equal for all $y \in x\downarrow\downarrow$. Define

$$h_i \overset{\text{DEF}}{=} b_{U_i Y} \circ u_i : A_x \longrightarrow B_{U_i} \qquad (\, i \in \{1, 2\}\,).$$

Since

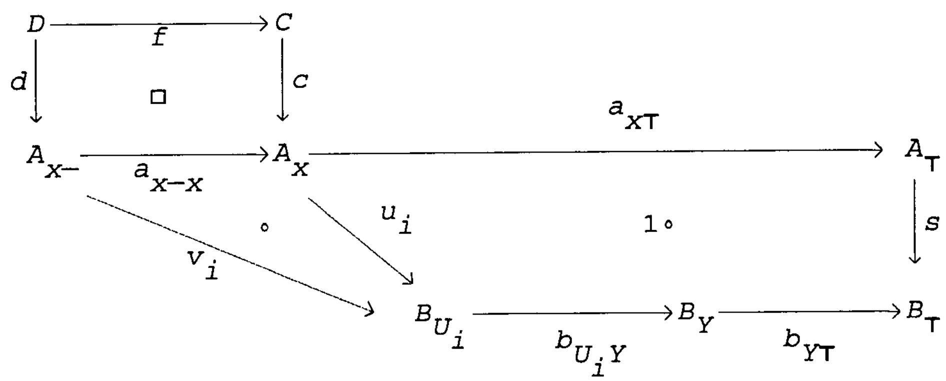
$$\langle a_{yx} : A_y \longrightarrow A_x \rangle_{y \in x\downarrow\downarrow}$$

is a colimit cocone on the diagram $A \restriction x\downarrow\downarrow$, and, for each $y \in x\downarrow\downarrow$,

$$h_1 \circ a_{yx} = b_{U_1 Y} \circ u^1_y = b_{U_2 Y} \circ u^2_y = h_2 \circ a_{yx} : A_y \longrightarrow B_Y ,$$

it follows that $h_1 = h_2$, which means that the Y-shifts of u_1 and u_2 are equal.

Let now x be isolated. Define $v_i \overset{\text{DEF}}{=} u_i \circ a_{x-x}$ $(i = 1, 2)$. We have the factorizations $(x-, U_i, v_i)$ starting at $x-$. By the induction hypothesis, their shifts are equalized at some $Y \in \hat{Q}$. For each $i = 1, 2$, consider the diagram



obtained from the fact that $a_{x-x} \in \mathrm{Po}[\mathcal{I}]$. Since (x, U_i, u_i) is a factor, we have the commutativity $1\circ$. Therefore, the two arrows $C \rightrightarrows B_T$ are equal. Since C is κ-presentable, we can choose $Y \in Q$ such that, in addition, for both $i = 1, 2$, the two arrows in the same diagram $C \rightrightarrows B_Y$ are equal.

Let $w_i = b_{U_i Y} \circ u_i$. Then, on the one hand, as we just saw

$$w_1 \circ c = w_2 \circ c \ ;$$

and on the other hand,

$$w_1 \circ a_{X-x} = v_1 \circ b_{U_1 Y} = v_2 \circ b_{U_2 Y} = w_2 \circ a_{X-x} ,$$

since, by the induction hypothesis, v_1 and v_2 are equalized at Y. Since c and a_{X-x} are pushout coprojections, it follows that $w_1 = w_2$. This means that the Y-shifts of u_1 and u_2 are equal as desired.

Finally, for the general case of (ii): let $\xi_i = (X_i, U_i, u_i)$ $(i=1, 2)$. Let $X = X_1 \cup X_2$; $X \in \hat{P}$. By the "moreover" part of part (i), we have factors ζ_1 and ζ_2, both starting at X, such that $\xi_i \le \zeta_i$. The assertion will follow for ξ_1 and ξ_2 if we can show it for ζ_1 and ζ_2. In other words, we may assume that ξ_1 and ξ_2 start at the same $X \in \hat{P}$.

This case now follows from the special case, for factors starting at points $x \in P$, proved above, by a colimit argument, exactly as above the case of x being a limit point was handled.

This completes the proof of 5.Lemma.

Next, we let $\mathbf{G}$ be a category such that $\#\mathbf{G}$ $\left(\overset{\mathrm{DEF}}{=} \#\left(\coprod_{G,\, H \in \mathrm{Ob}(\mathbf{G})} \hom_{\mathbf{G}}(G, H) \right) \right)$ is less than κ, $\#\mathbf{G} < \kappa$.

$G, H, \ldots$ range over objects of $\mathbf{G}$.

We consider the functor category $\mathbf{A}^{\mathbf{G}}$.

We recall that $\langle \mathbf{X}, \mathbf{G} \rangle$ denotes the class of all arrows $\varphi : \Phi \to \Psi$ in $\mathbf{A}^{\mathbf{G}}$ such that $\varphi_G \in \mathbf{X}$ for all G.

Let $\varphi : \Phi \longrightarrow \Psi$ be an arrow in $\langle \mathbf{X}, \mathbf{G} \rangle$; let $G \in \mathbf{G}$. By (1), we have κ-good *generating* diagram $A_G : P_G \longrightarrow \mathbf{A}$ with links in $\mathrm{Po}[\mathbf{I}]$ such that

$$\varphi_G = \langle A_G \rangle = a^G_{\bot\top} : \Phi G = A^G_{\bot} \longrightarrow A^G_{\top} = \Psi G .$$

Let us fix a system $\{A_G\}_{G \in \mathbf{G}}$ of generating diagrams A_G. Relative to the fixed system, we say that a factorization $\varphi = \sigma \circ \rho$,

$$\begin{array}{ccccc} & & \Gamma & & \\ & \overset{\rho}{\nearrow} & \circ & \overset{\sigma}{\searrow} & \\ \Phi & & \xrightarrow[\varphi]{\hspace{3em}} & & \Psi , \end{array} \qquad (6)$$

in the category $\mathbf{A}^{\mathbf{G}}$, is *good* if, for each G, the factorization $a^G_{\bot\top} = \varphi_G = \sigma_G \circ \rho_G$ is one that is given by the diagram A_G in the form

$$\rho_G = a^G_{\bot X_G} : A^G_\bot \longrightarrow A^G_{X_G} \quad \& \quad \sigma_G = a^G_{X_G\top} : A^G_{X_G} \longrightarrow A^G_\top$$

for some choice of sets $X_G \in \check{P}_G$ (X_G is an initial segment of P_G, not necessarily of cardinality $<\kappa$), one for each $G \in \boldsymbol{G}$. The sets X_G are referred to as the *carriers of* the factorization, X_G being the carrier at G.

The factorization is κ-*good* if, in addition, $X_G \in \hat{P}_G$ ($\# X_G < \kappa$).

Thus, a κ-good factorization of φ is given by a complex

$$(\langle X_G \in \hat{P}_G \rangle_{G \in \boldsymbol{G}},\ \langle \Gamma g \rangle_{G \xrightarrow{g} H \in \boldsymbol{G}}) \tag{7}$$

such that, for each $g: G \to H$, $(X_G, X_H, \Gamma g)$ is a factor for $(A_G, A_H, \Phi g, \Psi g)$:

$$\begin{array}{ccccc} A^G_\bot & \xrightarrow{a^G_{\bot X_G}} & A^G_{X_G} & \xrightarrow{a^G_{X_G\top}} & A^G_\top \\ {\scriptstyle \Phi g}\downarrow & \circ & \downarrow{\scriptstyle \Gamma g} & \circ & \downarrow{\scriptstyle \Psi g} \\ A^H_\bot & \xrightarrow[a^H_{\bot X_H}]{} & A^H_{X_H} & \xrightarrow[a^H_{X_H\top}]{} & A^H_\top \end{array} .$$

and, every time $h \circ g = k$ in $\boldsymbol{G}$, we have that $\Gamma h \circ \Gamma g = \Gamma k$.

The triple (X, U, u) being a factor involves the condition that the initial segments $X \sqsubseteq P$, $U \sqsubseteq Q$ are of cardinality $<\kappa$. If we remove this condition, the above, originally stated for κ-good factorizations, gives a characterization of a good factorizations in general. (We don't want to use "κ-good factor" for "factor", since the expression "factor" is used often in the meaning set as it is now.)

From now on, we assume that $\kappa \geq \aleph_1$.

6. Lemma For any system $\langle Y_G \rangle_{G \in \boldsymbol{G}}$ of sets $Y_G \in \hat{P}_G$, there is a κ-good factorization (7) of φ such that $Y_G \sqsubseteq X_G$ for all $G \in \boldsymbol{G}$.

Proof We define, recursively for $n\in\mathbb{N}$, sets $X^n_G\in\hat{P}_G$, one for each $G\in\boldsymbol{G}$, and arrows $u^n_g : A^G_{X^n_G}\longrightarrow A^H_{X^{n+1}_H}$, one for each $g:G\to H$, such that, for every $n\in\mathbb{N}$,

1) for each G, $X^n_G\sqsubseteq X^{n+1}_G$;

2) for each $g:G\to H$, $\xi^n_g \stackrel{\text{DEF}}{=} (X^n_G, X^{n+1}_H, u^n_g)$ is a factor for $(A_G, A_H, \Phi g, \Psi g)$;

3) for each $g:G\to H$, $\xi^n_g\le\xi^{n+1}_g$ as factors for $(A_G, A_H, \Phi g, \Psi g)$;

4) every time $k=g\circ h$, we have $u^{n+1}_k = u^{n+1}_h\circ u^n_g\circ a^G_{X^n_G X^{n+1}_G}$: in other words, with $G\xrightarrow{g}H\xrightarrow{h}K$, $G\xrightarrow{k=h\circ g}K$, we have the commutativity

$$\begin{array}{ccc} A^G_{X^n_G} & \xrightarrow{a^G_{X^n_G X^{n+1}_G}} & A^G_{X^{n+1}_G} \\ {\scriptstyle u^n_g}\big\downarrow & \circ & \big\downarrow{\scriptstyle u^{n+1}_k} \\ A^H_{X^{n+1}_H} & \xrightarrow[u^{n+1}_h]{} & A^K_{X^{n+2}_K} \end{array} .$$

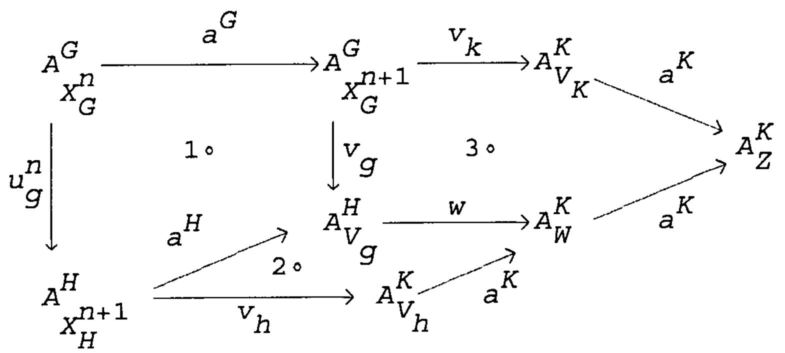
We put $X^0_G \stackrel{\text{DEF}}{=} Y_G$.

Next, for each $g:G\to H$, we use 5.(i) and let $V_g\in\hat{P}_H$ and $v_g : A^G_{X^0_G}\longrightarrow A^H_{V_g}$ such that (X^0_G, V_g, v_g) is a factor for $(A_G, A_H, \Phi g, \Psi g)$. For every $G\in\boldsymbol{G}$, we let $X^1_G\in\hat{P}_G$ be the set $X^1_G = X^0_G\cup\bigcup\{V_g : \mathrm{codom}(g)=G\}$. For every $g:G\to H$, we define, $u^0_g : A^G_{X^0_G}\longrightarrow A^H_{X^1_H}$ as $u^0_g = a^H_{V_g X^1_H}\circ v_g$. We have satisfied 2) for $n=0$.

Now suppose that $n \geq 0$ and we have defined all X_G^m for $m \leq n+1$, and all u_g^m for $m \leq n$; we'll define the X_G^{n+2} and the u_g^{n+1}.

For every $g: G \to H$, we define $V_g \in \hat{P}_H$ and $v_g: A^G_{X_G^{n+1}} \to A^H_{V_g}$ such that $X_H^{n+1} \sqsubseteq V_g$, $\zeta_g = (X_G^{n+1}, V_g, v_g)$ is a factor for $(A_G, A_H, \Phi g, \Psi g)$, and $\xi_g^n \leq \zeta_g$; for this, we use 5.(i).

Next, let us fix the triple $(g, h, k) = (G \xrightarrow{g} H \xrightarrow{h} K,\ G \xrightarrow{k=hg} K)$; we construct the following diagram:



(8)

(we have omitted the subscripts from the a-arrows).

W and w are chosen so that $\eta = (V_g, W, w)$ is a factor for $(A_H, A_K, \Phi h, \Psi k)$ such that $\zeta_h = (X_H^{n+1}, V_h, v_h) \leq \eta$; this makes $2\circ$ hold.

Since $\zeta_g = (X_G^{n+1}, V_g, v_g)$ is a factor for $(A_G, A_H, \Phi g, \Psi g)$, and $\eta = (V_g, W, w)$ is a factor for $(A_H, A_K, \Phi h, \Psi h)$, and, Φ and Ψ being functors, $\Phi k = (\Phi g) \circ (\Phi h)$ and $\Psi k = (\Psi g) \circ (\Psi h)$, it follows that $\theta = (X_G^{n+1}, W, w \circ v_g)$ is a factor for $(A_G, A_K, \Phi k, \Psi k)$. Since $\zeta_k = (X_G^{n+1}, V_k, v_k)$ is another factor for the same $(A_G, A_K, \Phi k, \Psi k)$, by 5.(ii), there is a common shift of θ and ζ_k: this is precisely the existence of Z in (7) so as to make $3\circ$ hold.

Let us re-denote the set $Z \in \hat{P}_K$ as $Z_{(g,\,h)}$ to emphasize its dependence on the composable pair (g, h) ($k = h \circ g$).

Next, consider an object K of $\mathbf{G}$, and define

$$X_K^{n+2} \overset{\mathrm{DEF}}{=} \bigcup \{Z_{(g,\,h)} : (g,\,h) \text{ composable, } \mathrm{codom}(h)=K\}\ .$$

Define, for $k:G\to H$, the arrow

$$u_k^{n+1}: A^G_{X_G^{n+1}} \longrightarrow A^K_{X_K^{n+2}}$$

so as to make $\xi_k = (X_G^{n+1},\, X_K^{n+2},\, u_k^{n+1})$ the X_K^{n+2}-shift of the factor $\zeta_k = (X_G^{n+1},\, v_k,\, v_k)$.

With $\xi_k^n \overset{\mathrm{DEF}}{=} (X_G^n,\, X_K^{n+1},\, u_k^n)$, $\xi_k^{n+1} \overset{\mathrm{DEF}}{=} \xi_k = (X_G^{n+1},\, X_K^{n+2},\, u_k^{n+1})$, we do have that $\xi_k^n \le \xi_k^{n+1}$, to satisfy condition 3) .

Returning to the a triple $(g,\, h,\, k)$ as before, and the corresponding diagram (7), with $Z=Z_{(g,\,h)}$, we have 1 ∘ by the inductive assumption $\xi_g^n \le \zeta_g$. We complete the diagram (7) with an arrow $Z_{(g,\,h)} \xrightarrow{a^K} X_K^{n+2}$, and we see that the composite arrow $A^G_{X_G^{n+1}} \to A^K_{X_K^{n+2}}$ equals u_k^{n+1} , and the one $A^G_{X_H^{n+1}} \to A^K_{X_K^{n+2}}$ equals u_h^{n+1} . We obtained that the requirement 4) holds as the commutativity of the outside of the completed diagram (7).

1) holds with $n+1$ in place of n . 2) holds as stated.

The recursive construction is complete.

We complete the proof of 6. Lemma by taking colimits.

In more detail:

For $G\in \boldsymbol{G}$, define $X_G = \bigcup_{n\in\mathbb{N}} X_G^n$. Since $\aleph_0 < \kappa$, and κ is regular, $X_G \in \hat{P}_G$.

For $G\in \boldsymbol{G}$, let $\Gamma(G) = A^G_{X_G}$. For $g:G\to H$, let

$$\Gamma(g) = \operatorname{colim} u_g^n : \Gamma(G) = \operatorname{colim} A^G_{X^n_G} \longrightarrow \operatorname{colim} A^H_{X^{n+1}_H} = \Gamma(H) \ ;$$

in other words, $\Gamma(g)$ is determined by the commutativity of

$$\begin{array}{ccc}
A^G_{X_G} & \xrightarrow{\quad\Gamma(g)\quad} & A^H_{X_H} \\
{\scriptstyle a^G_{X^n_G X_G}}\uparrow & \circ & \uparrow{\scriptstyle a^H_{X^{n+1}_H X_H}} \\
A^G_{X^n_G} & \xrightarrow[\quad u^n_g\quad]{} & A^H_{X^{n+1}_H}
\end{array}$$

for every n. Indeed, on the one hand, the left vertical arrows are the colimit coprojections of a diagram $B : \mathbb{N} \longrightarrow \mathbf{A}$; this fact is a case of rearrangement according to Fact 3, "case one". On the other hand, the composites $A^G_{X^n_G} \xrightarrow{u^n_g} A^H_{X^{n+1}_H} \xrightarrow{a^H} A^H_{X_H}$ for $n=0, 1, 2, \ldots$ form a cocone on the same diagram B, as a consequence of item 3) in the construction.

Let $n \in \mathbb{N}$, and let $G \xrightarrow{g} H \xrightarrow{h} K$, $k = h \circ g$, and consider the diagram

$$\begin{array}{ccccc}
A^G_{X_G} & \xrightarrow{\quad\Gamma(g)\quad} & A^H_{X_H} & \xrightarrow{\quad\Gamma(h)\quad} & A^K_{X_K} \\
 & 1\circ & {\scriptstyle a^H}\uparrow & 2\circ & \uparrow{\scriptstyle a^K} \\
{\scriptstyle a^G}\uparrow & \overset{u^n_g}{\nearrow} & A^H_{X^{n+1}_H} & \xrightarrow{\quad u^{n+1}_h\quad} & A^H_{X^{n+1}_H} \\
 & & 3\circ & \overset{a^K}{\nearrow} & \\
A^G_{X^n_G} & \xrightarrow[\quad u^n_k\quad]{} & A^K_{X^{n+1}_K} & &
\end{array} \quad .$$

$1\circ$ and $2\circ$ hold by the definitions of $\Gamma(g)$ and $\Gamma(h)$. $3\circ$ is item 4) of the construction. The resulting outside commutativity says that

$$\begin{array}{ccc}
A^G_{X_G} & \xrightarrow{\quad\Gamma(h)\circ\Gamma(g)\quad} & A^K_{X_K} \\
{\scriptstyle a^G}\uparrow & \circ & \uparrow{\scriptstyle a^H} \\
A^G_{X^n_G} & \xrightarrow[\quad u^n_k\quad]{} & A^K_{X^{n+1}_K}
\end{array}$$

commutes for all n -- which says that $\Gamma(h)\circ\Gamma(g)$ answers the description of, and therefore equals to, $\Gamma(k)$.

This completes the proof of 6. Lemma.

7. Proposition Assume:

κ is a regular cardinal, $\kappa\geq\aleph_1$;

$\boldsymbol{G}$ is a category such that $\#\boldsymbol{G}<\kappa$;

$\boldsymbol{A}$ is a cocomplete category;

$\boldsymbol{I}$ is class (set) of arrows in $\boldsymbol{A}$ such that for every $f\in\boldsymbol{I}$, both $\mathrm{dom}(f)$ and $\mathrm{codom}(f)$ are κ-presentable.

Then

$$\langle C^{(\boldsymbol{A})}[\mathrm{Po}[\boldsymbol{I}]],\boldsymbol{G}\rangle = C^{(\boldsymbol{A}^{\boldsymbol{G}})}[\langle C^{(\boldsymbol{A})}[\mathrm{Po}[\boldsymbol{I}],<\kappa],\boldsymbol{G}\rangle] \quad .$$

Proof The fact that the class on the right-hand side is contained in the one on the left-hand side is obvious.

Let $\varphi:\Phi\to\Psi$ be an arrow in the class on left-hand side. By the conclusion (1) stated at the start of this section, drawn from the work in previous sections, we have, for each $G\in\boldsymbol{G}$, a κ-good diagram $A_G:P_G\to\boldsymbol{A}$ with links in the class $\mathrm{Po}[\boldsymbol{I}]$, and such that $\langle A_G\rangle=\varphi_G:\Phi G\to\Psi G$.

Recall the notion of *good* (not just κ-*good*) factorization.

With a limit ordinal α , let $\langle x_\beta\rangle_{\beta<\alpha}$ be an indexing by ordinals of the set $\bigsqcup_{G\in\boldsymbol{G}} P_G$.

We are going to construct, by transfinite recursion, a transfinite system

$$\langle\Gamma_\beta\in\boldsymbol{A}^{\boldsymbol{G}},\ \Phi\xrightarrow{\rho_\beta}\Gamma_\beta\xrightarrow{\sigma_\beta}\Psi\rangle_{\beta<\alpha}$$

of good factorizations of ψ, with carriers X_G^β $(G\in\boldsymbol{G},\ \beta<\alpha)$, "carrying" the arrows

$$\Gamma_\beta(g)\ :\ \Gamma_\beta G=A^G_{X_G^\beta}\longrightarrow A^H_{X_H^\beta}=\Gamma_\beta H \qquad (\ g:G\to H\ )\ , \qquad (9)$$

and with the following additional properties:

1) For a fixed $G\in\boldsymbol{G}$, the carriers X_G^β form an increasing continuous system whose union is P_G:

a) $\gamma\le\beta<\alpha \Longrightarrow X_G^{\gamma} \sqsubseteq X_G^{\beta}$;

b) for β limit ordinal, $\bigcup_{\gamma<\beta} X_G^{\gamma} = X_G^{\beta}$;

c) for G, β and $x\in P_G$ if $\iota(x)=x_{\gamma}$, for ι the coprojection $\iota: P_G \to \coprod_{G\in\mathbf{G}} P_G$ (briefly, if $x_{\gamma}\in P_G$), we have $x\in X_G^{\gamma+1}$.

2) Whenever $\beta+1<\alpha$, $\#(X_G^{\beta+1}-X_G^{\beta}) < \kappa$ ("small increments").

3) For any $g: G\to H$ in $\mathbf{G}$, and $\gamma\le\beta<\alpha$, the diagram

$$\begin{array}{ccc} A^G_{X_G^{\gamma}} & \xrightarrow{\;a^G_{X_G^{\gamma}X_G^{\beta}}\;} & A^G_{X_G^{\beta}} \\ {\scriptstyle \Gamma_{\gamma}g}\Big\downarrow & \circ & \Big\downarrow{\scriptstyle \Gamma_{\beta}g} \\ A^H_{X_H^{\gamma}} & \xrightarrow[\;a^H_{X_H^{\gamma}X_H^{\beta}}\;]{} & A^H_{X_H^{\beta}} \end{array}$$

commutes.

Before we carry out the construction, we want to elaborate on the (rather obvious) fact that the construction proves the proposition. For this purpose, one repeatedly uses the fact that colimits in a functor category are computed componentwise.

Suppose we have 1) to 3) done.

We have natural transformations $\mu_{\gamma\beta}:\Gamma_{\gamma}\to\Gamma_{\beta}$ $(\gamma\le\beta)$ for which $(\mu_{\gamma\beta})_G = a_{X_G^{\gamma}X_G^{\beta}}$: the naturality of $\mu_{\gamma\beta}$ is 3).

We have a functor $\Gamma: [\alpha)=\{\beta:\beta<\alpha\} \longrightarrow \mathbf{A}^{\mathbf{G}}$ for which $\Gamma(\beta)=\Gamma_{\beta}$, and $\Gamma(\gamma\le\beta)=\mu_{\gamma\beta}$; for $G\in\mathbf{G}$, the G-component $[\alpha) \xrightarrow{\Gamma} \mathbf{A}^{\mathbf{G}} \xrightarrow{\mathrm{ev}_G} \mathbf{A}$ of Γ is the functor $[\alpha) \longrightarrow \mathbf{A}$ defined as

$$\beta \longmapsto A^G_{X_G^{\beta}}, \qquad (\gamma\le\beta) \longmapsto a^G_{X_G^{\gamma}X_G^{\beta}} .$$

Γ is a continuous transfinite system $\Gamma: [\alpha) \to \mathbf{A}^{\mathbf{G}}$: for β a limit ordinal $<\alpha$,

$\langle \mu_{\gamma\beta}:\Gamma_\gamma \to \Gamma_\beta \rangle_{\gamma<\beta}$ is a colimit cocone on the diagram $\Gamma \upharpoonright [\beta)$, because this holds after evaluation at every $G \in \mathbf{G}$:

$$\langle A^G_{X^\gamma_G} \xrightarrow{a_{X^\gamma_G X^\beta_G}} A^G_{X^\beta_G} \rangle_{\gamma<\beta}$$

is a colimit cocone on the diagram

$$\langle A^G_{X^\delta_G} \xrightarrow{a_{X^\delta_G X^\gamma_G}} A^G_{X^\gamma_G} \rangle_{\delta\le\gamma} ,$$

this fact being a case of "rearrangement" of the colimit $A^G_X = \mathrm{colim}(A^G \upharpoonright X)$ according to "case one" of Fact 3.

In fact, we have an extension of Γ , also denoted Γ , to Γ: $[\alpha] \to \mathbf{A}^{\mathbf{G}}$ (that is, $\Gamma(\alpha)$ is defined) as follows.

The system $\langle \sigma_\beta : \Gamma_\beta \to \Psi \rangle_{\beta<\alpha}$ of natural transformations σ_β is a colimit cocone on the diagram Γ , since after evaluation at every $G \in \mathbf{G}$, this is true:

$$\langle A^G_{X^\beta_G} \xrightarrow{a_{X^\gamma_G \top}} A^G_\top \rangle_{\beta<\alpha}$$

is a colimit cocone on the diagram

$$\langle A^G_{X^\gamma_G} \xrightarrow{a_{X^\gamma_G X^\beta_G}} A^G_{X^\beta_G} \rangle_{\gamma\le\beta<\alpha} ,$$

this fact being a case of "rearrangement" of the colimit $A^G_\top = \mathrm{colim}(A^G)$ according to "case one" of Fact 3 *because of* $\bigcup_{\beta<\alpha} X^\beta_G = P_G$, which holds by 1)c).

At the same time, we see that the composite $\langle \Gamma \rangle$ of Γ is φ , since this fact is true after evaluation at each $G \in \mathbf{G}$.

(but)

For $G \in \mathbf{G}$, the G-components of the links $\mu_{\gamma,\ \gamma+1}$ of the $\mathbf{A}^{\mathbf{G}}$-system Γ , which are the arrows $a^G_{X^\gamma_G X^{\gamma+1}_G}$, are in $\mathcal{G}[Po[\mathcal{I}], <\kappa]$ by condition 2) and Fact 5, part 2). Therefore, they are in $\mathcal{C}[\mathrm{Po}[\mathcal{I}],<\kappa]$, by 2. Prop., part (i) ("in fact,..."). In other words, the links $\mu_{\gamma,\ \gamma+1}$ themselves are in $\langle \mathcal{C}[\mathrm{Po}[\mathcal{I}], <\kappa], \mathbf{G}\rangle$.

We have shown that φ belongs to $\mathcal{C}^{(\mathbf{A}^{\mathbf{G}})}(\langle \mathcal{C}[\mathrm{Po}[\mathcal{I}], <\kappa], \mathbf{G}\rangle)$ as desired.

In turn, we carry out the construction 1) to 3).

For $\beta=0$, we let $\Gamma_0=\Lambda$, $\rho_0=id_\Lambda$, $X^0_G=B^G_\perp$.

Let $\beta<\alpha$, and suppose all items for smaller ordinals have been defined. The new items to be defined are the ones displayed in (9).

If β is a limit ordinal, the new items are uniquely determined by conditions 1)b) (defining the sets X^β_G $(G\in\mathbf{G})$) and 3). Given $g:G\to H$, the facts of 1)a) and 3) being true, for all pairs (δ,γ) such that $\delta\le\gamma<\beta$ in place of $(\gamma,\ \beta)$, ensure that there is a unique arrow $\Gamma_\beta g$ satisfying 3) with all $\gamma<\beta$, *because*

$$\langle A^G_{X^\gamma_G}\xrightarrow{a_{X^\gamma_G X^\beta_G}}A^G_{X^\beta_G}\rangle_{\gamma<\beta}$$

is a colimit cocone on the diagram

$$\langle A^G_{X^\delta_G}\xrightarrow{a_{X^\delta_G X^\gamma_G}}A^G_{X^\gamma_G}\rangle_{\delta\le\gamma},$$

this fact being a case of "rearrangement" of the colimit $A^G_X=\mathrm{colim}(A^G\restriction X)$ according to "case one" of Fact 3.

(One is tempted to dismiss the issue by putting

$$\Gamma^\beta \overset{\mathrm{DEF}}{\cong} \operatorname*{colim}_{\gamma<\beta}\Gamma^\gamma ,$$

but this is of course not enough: we are not defining things here merely up to isomorphism).

It remains to handle the case when β is a successor ordinal, $\beta=\gamma+1$.

Recall the construction of the "end-segment" diagram $A\,|\,R$ and the facts about it from Fact 5.

We will apply 6. Lemma to the the arrow $\sigma_\gamma:\Gamma_\gamma\longrightarrow\Psi$ in place of $\varphi:\Phi\longrightarrow\Psi$, the diagram $A_G|X_G^\gamma: P_G|X_G^\gamma\longrightarrow\mathbf{A}$ in place of $A_G: P_G\longrightarrow\mathbf{A}$ (one for each $G\in\mathbf{G}$).

Note that $\Gamma_\gamma(G) = A^G_{X_G^\gamma} = (A_G|X_G^\gamma)_\perp$. Note that, by Fact 5, the κ-good diagram $A_G|X_G^\gamma$ generates as its composite the arrow

$$(\sigma_\gamma)_G = a^G_{X_G^\gamma\perp} : A^G_{X_G^\gamma}\longrightarrow A^G_\top .$$

We have $x_\beta\in \coprod_{G\in\mathbf{G}} P_G$, picked out by the ordinal β at hand. Let $G_0\in\mathbf{G}$ be the object for which $x_\gamma\in P_{G_0}$. For an application of 6. Lemmma, we put $Y_{G_0}=x_\gamma\!\downarrow$, and $Y_G=\emptyset$ for $G\neq G_0$.

By Fact 5, $(P_G|X_G^\gamma)^\wedge$ is identified with $P_G\|X_G^\gamma = \{Z\in\tilde{P}_G : X_G^\gamma\sqsubseteq Z\ \&\ \#(Z-X_G^\gamma)<\kappa\}$, and the diagram $(A_G|X_G^\gamma)^\wedge$ with $\tilde{A}_G$ restricted to $P_G\|X_G^\gamma$. Thus, 6. Lemma gives us, (·) (for p 39 question)

a functor $\Gamma_\beta:\mathbf{G}\rightarrow\mathbf{A}$ (as Γ of 6.Lemma),

natural transformations $\Gamma_\gamma\xrightarrow{\ \tau\ }\Gamma_\beta\xrightarrow{\ \sigma_\beta\ }\Psi$ such that $\sigma_\beta\circ\tau=\sigma_\gamma$,

and

for each G, a set, denoted X_G^β (as X_G) in $\tilde{P}_G$ such that $X_G^\gamma\sqsubseteq X_G^\beta$, $\#(X_G^\beta-X_G^\gamma)<\kappa$, and

$$\Gamma_\beta(G) = A^G_{X_G^\beta},\qquad \tau_G=a^G_{X_G^\gamma X_G^\beta}:A^G_{X_G^\gamma}\longrightarrow A^G_{X_G^\beta},\qquad (\sigma_\beta)_G=a^G_{X_G^\beta\top}:A^G_{X_G^\beta}\longrightarrow A^G_\top$$

.

This is sufficient.

§5 The final part of the proof, *not* using good diagrams

The next proposition is completely independent from the work done so far.

8. Proposition Assume that

κ is a regular cardinal;
$\mathbf{A}$ is a locally κ-presentable category;
$\mathcal{I}$ is a set of arrows in $\mathbf{A}_\kappa$.

Then $\mathcal{C}[\mathrm{Po}^{(\mathbf{A})}[\mathcal{I}], <\kappa] = \mathrm{Po}[\mathcal{C}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa]]$.

Proof The fact that the right-hand side is contained in the left-hand side is obvious.

Let me use the notation $\mathcal{T}[\mathcal{J}, \alpha]$ for the class of all continuous transfinite systems $A: [\alpha] \to \mathbf{A}$ of length α whose links are in the class $\mathcal{J}$. Thus, $f \in \mathcal{C}[\mathcal{J}, \alpha] \iff \exists A \in \mathcal{T}[\mathcal{J}.\alpha] . f = \langle A \rangle$.

$\mathcal{T}[\alpha]$ denotes the class of all continuous transfinite systems $A: [\alpha] \to \mathbf{A}$ of length α, without anything being said on links.

Let α be an ordinal $<\kappa$, and let $A \in \mathcal{T}[\mathrm{Po}[\mathcal{I}], \alpha]$. For every $\beta<\alpha$, we have, and we fix, a pushout diagram

$$\begin{array}{ccc} A_\beta & \xrightarrow{a_{\beta,\beta+1}} & A_{\beta+1} \\ {\scriptstyle p_\beta}\uparrow & \square & \uparrow{\scriptstyle q_\beta} \\ D_\beta & \xrightarrow[f_\beta]{} & C_\beta \end{array}$$

with $f_\beta \in \mathcal{I}$.

We are going to construct $B \in \mathcal{T}[\mathrm{Po}[\mathcal{I}], \alpha]$, $B: [\alpha] \to \mathbf{A}_\kappa$ (!) such that the given diagram A is a pushout of B (see §1), and, in particular, $\langle A \rangle$ is a pushout of $\langle B \rangle$. The construction will be a recursive one; we will construct the restriction $B \restriction [\beta] : [\beta] \to \mathbf{A}_\kappa$ by recursion on the ordinal $\beta \le \alpha$. Simultaneously, we have to produce other items, to keep the recursion going.

More fully stated, we propose to construct an "augmented triangular matrix of objects and arrows" in $\mathbf{A}$ (mostly in $\mathbf{A}_\kappa$):

objects B^μ_β ($\beta\le\alpha$, $\beta\le\mu\le\alpha$) in $\mathbf{A}_\kappa$;

"horizontal" arrows $B^\mu_\gamma \xrightarrow{b^\mu_{\gamma\beta}} B^\mu_\beta$ ($\gamma\le\beta\le\mu\le\alpha$);

"vertical" arrows $B^\nu_\beta \xrightarrow{b^{\nu\mu}_\beta} B^\mu_\beta$ ($\beta\le\nu\le\mu\le\alpha$);

"upper augmentation" arrows $B^\mu_\beta \xrightarrow{b^{\mu\top}_\beta} A_\beta$ ($\beta\le\mu\le\alpha$);

"lower augmentation" arrows

$$D_\beta \xrightarrow{\hat{p}_\beta} B^\beta_\beta, \qquad C_\beta \xrightarrow{\hat{q}_{\beta+1}} B^{\beta+1}_{\beta+1} \quad (\beta<\alpha);$$

all subject to the following 1) to 4):

1) For any $\mu\le\alpha$, the "horizontal" diagram $B^\mu : [\mu] \to \mathbf{A}_\kappa$ is a continuous transfinite system.

[Explanation: B^μ is defined by

$$B^\mu(\beta) \overset{\text{DEF}}{=} B^\mu_\beta, \qquad B^\mu(\gamma\le\beta) \overset{\text{DEF}}{=} b^\mu_{\gamma\beta};$$

thus, we require

$$b^\mu_{\beta\beta} = \text{id}, \qquad b^\mu_{\gamma\beta}\circ b^\mu_{\delta\gamma} = b^\mu_{\delta\beta} \qquad (\delta\le\gamma\le\beta\le\mu) ;$$

and

for $\beta\le\mu$ a limit ordinal,

$$\langle b^\mu_{\gamma\beta} : B^\mu_\gamma \longrightarrow B^\mu_\beta \rangle_{\gamma<\beta}$$ is a colimit cocone on the diagram $B^\mu \restriction [\beta)$.]

2.1) For any $\beta\le\alpha$, the "vertical" diagram $B_\beta : [\beta, \alpha] \to \mathbf{A}_\kappa$ is a (*not necessarily* continuous) transfinite system.

[Explanation: $[\beta, \alpha] = \{\mu : \beta\le\mu\le\alpha\}$,

$$B_\beta(\mu) \overset{\text{DEF}}{=} B_\beta^\mu , \qquad B_\beta(\nu\le\mu) \overset{\text{DEF}}{=} b_{\gamma\beta}^\mu ;$$

thus, we require

$$b_\beta^{\mu\mu} = \mathrm{id} , \qquad b_\beta^{\nu\mu} \circ b_\beta^{\rho\nu} = b_\beta^{\rho\mu} \qquad (\beta\le\rho\le\nu\le\mu) \ ;$$

Moreover,

2.2) $\langle b_\beta^{\mu\mathrm{T}} : B_\beta^\mu \longrightarrow A_\beta \rangle_{\beta\le\mu<\alpha}$ is a cocone on the diagram B_β .

[that is:

$$b_\beta^{\mu\mathrm{T}} \circ b_\beta^{\nu\mu} = b_\beta^{\nu\mathrm{T}} \qquad (\beta \le \nu \le \mu \le \alpha) \qquad]$$

3) The following are pushouts:

3.1)

$$\begin{array}{ccc} B_\gamma^\mu & \xrightarrow{\ b_{\gamma\beta}^\mu\ } & B_\beta^\mu \\ {\scriptstyle b_\gamma^{\nu\mu}}\uparrow & \square & \uparrow{\scriptstyle b_\beta^{\nu\mu}} \\ B_\gamma^\nu & \xrightarrow[\ b_{\gamma\beta}^\nu\ ]{} & B_\beta^\nu \end{array} \qquad (\gamma\le\beta\le\nu\le\mu\le\alpha)$$

3.2)

$$\begin{array}{ccc} A_\gamma & \xrightarrow{\ a_{\gamma\beta}\ } & A_\beta \\ {\scriptstyle b_\gamma^{\mu\mathrm{T}}}\uparrow & \square & \uparrow{\scriptstyle b_\beta^{\mu\mathrm{T}}} \\ B_\gamma^\mu & \xrightarrow[\ b_{\gamma\beta}^\mu\ ]{} & B_\beta^\mu \end{array} \qquad (\gamma\le\beta\le\le\mu\le\alpha)$$

3.3)

$$\begin{array}{ccc} B^{\beta+1}_{\beta} & \xrightarrow{\;b^{\beta+1}_{\beta,\,\beta+1}\;} & B^{\beta+1}_{\beta+1} \\ {\scriptstyle b^{\beta,\,\beta+1}_{\beta}}\uparrow & & \\ B^{\beta}_{\beta} & \square & \uparrow{\scriptstyle \hat{q}_{\beta+1}} \\ {\scriptstyle \hat{p}_{\beta}}\uparrow & & \\ D_{\beta} & \xrightarrow{\;f_{\beta}\;} & C_{\beta} \end{array} \qquad (\ \beta<\alpha\ )$$

4) For all $\beta\leq\alpha$, we have

$$\begin{array}{ccc} A_{\beta} & \xleftarrow{\;b^{\beta\mathsf{T}}_{\beta}\;} & B^{\beta}_{\beta} \\ {\scriptstyle p_{\beta}}\uparrow & \circ & \nearrow{\scriptstyle \hat{p}_{\beta}} \\ D_{\beta} & & \end{array}$$

The desired entity, denoted $B\colon[\alpha]\to\mathbf{A}_K$ above, is going to be the "horizontal" diagram B^{α} mentioned in 1) for $\mu=\alpha$. 3.3) says that the links of B^{α} are in $\mathrm{Po}[I]$; 3.2) says, for $\gamma=0$, $\beta=\mu=\alpha$, that $\langle A\rangle$ is a pushout of $\langle B\rangle$. Thus, the construction, once it is carried out, will certainly prove the proposition.

Suppose $\beta\leq\alpha$, and the construction of entities marked as sub- or superscripts by ordinals $\gamma<\beta$ has been carried out.

For $\beta=0$, we only need to say that $B^{0}_{0}=D_{0}$, $\hat{p}_{0}=id_{D_0}$, and $b^{0\mathsf{T}}_{0}=p_{0}$.

Let $\beta>0$ be a limit ordinal, $\beta\leq\alpha$.

The construction of the entities marked by β (*and*, possibly, by smaller ordinals) is carried out in two steps. In the first step, we construct entities *like* the required ones *except* the arrow $\hat{p}_{\beta}\colon D_{\beta}\longrightarrow B^{\beta}_{\beta}$ of which we don't obtain a version. In order to be able to get the last arrow, in the second step, we modify, by an appeal to Fact 6, the system we got in the first step.

The first step of the construction is a straightforward "taking-limits" action.

The entities gotten in the first step are denoted by the letters E (when they are objects) and e

(when they are arrows); their sub- and superscripts will exactly follow their, eventually desired, B- , respectively b-versions.

We define, for $\gamma<\beta$,

$$E^{\beta}_{\gamma} \overset{\mathrm{DEF}}{=} \operatorname*{colim}_{\nu<\beta} B^{\nu}_{\gamma} = \operatorname{colim}\,(B_{\gamma}\upharpoonright[\gamma,\beta))\ ,$$

with $B_{\gamma}\upharpoonright[\gamma,\beta)$ the "vertical" diagram B_{γ} (see 2)) restricted to the set $[\gamma,\beta)=\{\nu:\gamma\le\nu<\beta\}$.

Since $\gamma<\beta\le\alpha<\kappa$, the definition makes E^{β}_{β} κ-presentable.

The vertical arrow $e^{\nu\beta}_{\gamma}:B^{\nu}_{\gamma}\to E^{\beta}_{\gamma}$ is a colimit coprojection ($\nu<\beta$).

The horizontal arrow $e^{\beta}_{\delta\gamma}:E^{\beta}_{\delta}\to E^{\beta}_{\gamma}$ is defined, through E^{β}_{δ} being a colimit, by the requirement that, for all $\nu<\beta$, the following commute:

$$\begin{array}{ccc} E^{\beta}_{\delta} & \xrightarrow{\ e^{\beta}_{\delta\gamma}\ } & E^{\beta}_{\gamma} \\ {\scriptstyle e^{\nu\beta}_{\delta}}\uparrow & \circ & \uparrow{\scriptstyle e^{\nu\beta}_{\gamma}} \\ B^{\nu}_{\delta} & \xrightarrow[\ b^{\nu}_{\delta\gamma}\ ]{} & B^{\nu}_{\gamma} \end{array}\ . \qquad (1)$$

One notes that 3.1) for ordinals $<\beta$ implies that (1) is, in fact, a pushout square as we want to fulfill 3.1) for the new cases.

For $\gamma<\beta$, the vertical arrow $e^{\beta\mathrm{T}}_{\gamma}:E^{\beta}_{\gamma}\longrightarrow A_{\gamma}$ is defined, again by E^{β}_{γ} being a colimit, by the condition that, for all $\nu<\beta$, $e^{\beta\mathrm{T}}_{\gamma}\circ e^{\nu\beta}_{\gamma}=e^{\nu\mathrm{T}}_{\gamma}$.

Using, for $\delta\le\gamma<\beta$, the (already known) pushouts

$$\begin{array}{ccc} A_{\delta} & \xrightarrow{\ a_{\delta\gamma}\ } & A_{\gamma} \\ {\scriptstyle b^{\mu\mathrm{T}}_{\delta}}\uparrow & \square & \uparrow{\scriptstyle b^{\mu\mathrm{T}}_{\gamma}} \\ B^{\mu}_{\delta} & \xrightarrow[\ b^{\mu}_{\delta\gamma}\ ]{} & B^{\mu}_{\gamma} \end{array}$$

we see that the squares

$$\begin{array}{ccc} A_\delta & \xrightarrow{a_{\delta\gamma}} & A_\gamma \\ {\scriptstyle e^{\beta\mathrm{T}}_\delta}\uparrow & \square & \uparrow{\scriptstyle e^{\beta\mathrm{T}}_\gamma} \\ E^\beta_\delta & \xrightarrow[e^\beta_{\delta\gamma}]{} & E^\beta_\gamma \end{array} \qquad (2)$$

are pushouts.

$$E^\beta_\beta \overset{\mathrm{DEF}}{=} \operatorname*{colim}_{\gamma<\beta} E^\beta_\gamma = \operatorname{colim} E^\beta \upharpoonright [\beta) \ ,$$

$b^\beta_{\gamma\beta}: B^\beta_\gamma \longrightarrow B^\beta_\beta$: colimit coprojections.

Since $\beta\le\alpha$, and $\alpha<\kappa$, E^β_β is κ-presentable.

Note that there are no new pushout requirements according to 3.1) involving the object E^β_β. However, we need, and do have, the pushout

$$\begin{array}{ccc} A_\gamma & \xrightarrow{a_{\gamma\beta}} & A_\beta \\ {\scriptstyle e^{\beta\mathrm{T}}_\gamma}\uparrow & \square & \uparrow{\scriptstyle e^{\beta\mathrm{T}}_\beta} \\ E^\beta_\gamma & \xrightarrow[e^\beta_{\gamma\beta}]{} & E^\beta_\beta \end{array} \qquad (3)$$

as a "horizontal" colimit of the pushouts (2).

$\mathbf{A}$ is locally κ-presentable; since $E^\beta_0\in\mathbf{A}_\kappa$, the comma category $\mathcal{X}=E^\beta_0\downarrow\mathbf{A}$ is also locally κ-presentable, and its κ-presentable objects are those in $\mathcal{X}_\kappa=E^\beta_0\downarrow\mathbf{A}_\kappa$. For the object

$$X = \begin{array}{c} A_0 \\ {\scriptstyle e^{\beta\mathrm{T}}_0}\uparrow \\ E^\beta_0 \end{array}$$

of $\mathcal{X}$, the comma category $\mathcal{X}_\kappa\downarrow X$ is κ-filtered, and the forgetful functor

$$F\quad:\quad \mathcal{X}_\kappa{\downarrow}X \longrightarrow \mathcal{X}$$

$$(Y,\,y,\,x)\;=\;\begin{array}{ccc} Y & \xrightarrow{\quad x\quad} & A_0 \\ & {\scriptstyle y}\nwarrow \quad \circ \quad \nearrow {\scriptstyle e_0^{\beta\mathrm{T}}} & \\ & E_0^\beta & \end{array} \quad\longmapsto\quad \begin{array}{c} Y \\ {\scriptstyle y}\uparrow \\ E_0^\beta \end{array}$$

has X as its colimit, via the cocone $(Y,\,y,\,x)\mapsto x$.

Let us write $\hat{\mathcal{X}}$ for the comma category $\hat{\mathcal{X}}=E_\beta^\beta{\downarrow}\mathbf{A}$, and take the functor "pushout along $e_{0\beta}^\beta : E_0^\beta \longrightarrow E_\beta^\beta$":

$$G:\mathcal{X} \longrightarrow \hat{\mathcal{X}}$$

$$\begin{array}{ccc} Y & \xrightarrow{\;u\;} & Z \\ {\scriptstyle y}\uparrow & \nearrow {\scriptstyle z} & \\ E_0^\beta & & \end{array} \quad\longmapsto\quad \begin{array}{ccc} \hat{Y} & \xrightarrow{\;\tilde{u}\;} & \hat{Z} \\ {\scriptstyle \hat{y}}\uparrow & \nearrow {\scriptstyle \hat{z}} & \\ E_\beta^\beta & & \end{array}$$

By (3) for $\gamma=0$, the value of G at X is (can be taken to be) $\hat{X} = \begin{array}{c} A_\beta \\ {\scriptstyle e_\beta^{\beta\mathrm{T}}}\uparrow \\ E_\beta^\beta \end{array}$. G preserves colimits; it preserves the colimit of F described above. Combine this fact with the fact that the forgetful functor $\hat{\mathcal{X}}\to\mathbf{A}$ preserves connected colimits, in particular, the colimit of $G\circ F:\mathcal{X}_\kappa{\downarrow}X\to\hat{\mathcal{X}}$.

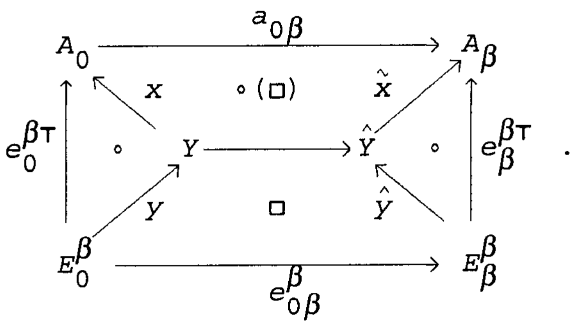
It follows that the system $\langle \hat{Y}\xrightarrow{\tilde{x}}A_\beta\rangle_{(Y,\,y,\,x)}$, indexed by the commutative triangles

$$(Y,\,y,\,x)\;=\;\begin{array}{ccc} Y & \xrightarrow{\quad x\quad} & A_0 \\ & {\scriptstyle y}\nwarrow \quad \circ \quad \nearrow {\scriptstyle e_0^{\beta\mathrm{T}}} & \\ & E_0^\beta & \end{array}$$

, is a colimit cocone on the κ-filtered diagram

$$\begin{array}{ccc} \mathcal{X}_\kappa{\downarrow}X & \longrightarrow & \mathbf{A} \\ (Y,\,y,\,x) & \longmapsto & \hat{Y} \end{array}; \qquad (4)$$

here, $\hat{Y}$ and $\hat{Y} \xrightarrow{\tilde{x}} A_\beta$ are defined, for a variable triple $(Y, y, x) \in \mathrm{Ob}(\mathcal{X}_\kappa \downarrow X)$, by the diagram

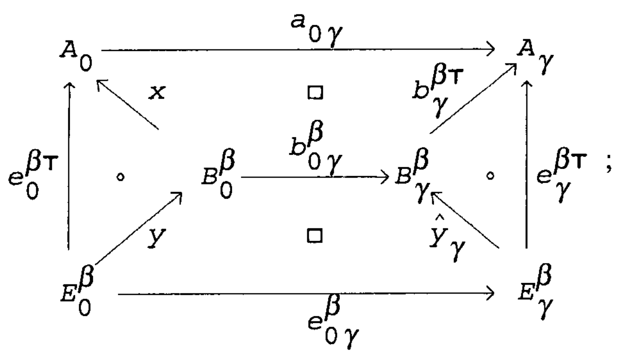


We now apply the fact that the object D_β is a κ-presentable. Therefore, the arrow $p_\beta : D_\beta \to A_\beta$ factors through an object of the diagram (4); that is, there are, and we fix such, triple $(Y, y, x) \in \mathrm{Ob}(\mathcal{X}_\kappa \downarrow X)$, and an arrow $u : D_\beta \to \hat{Y}$ such that $\tilde{x} \circ u = e_\beta^{\beta\mathsf{T}}$.

We define the objects B_0^β as $B_0^\beta = Y$, B_β^β as $B_\beta^\beta = \hat{Y}$, and the arrows $b_\beta^{\beta\mathsf{T}} : B_\beta^\beta \to A_\beta$ as $b_\beta^{\beta\mathsf{T}} = \tilde{x}$, $\hat{p}_\beta : D_\beta \to B_\beta^\beta$ as $\hat{p}_\beta = u$.

In between, the transfinite system $B^\beta : [\beta] \to \mathbf{A}_\kappa$ is defined as the pushout of E^β along $y : E_0^\beta \to Y = B_0^\beta$ (see §1 for this notion); this matches the definition of B_β^β given before.

For $\gamma \le \beta$, we have the arrows $b_\gamma^{\beta\mathsf{T}} : B_\gamma^\beta \to A_\gamma$ $(\gamma \le \beta)$ and $\hat{y}_\gamma : E_\gamma^\beta \to B_\gamma^\beta$, defined by the diagram



this matches the previous determination of $b_\beta^{\beta\mathsf{T}}$; also, $b_0^{\beta\mathsf{T}} = x$ and $\hat{y}_\gamma = y$.

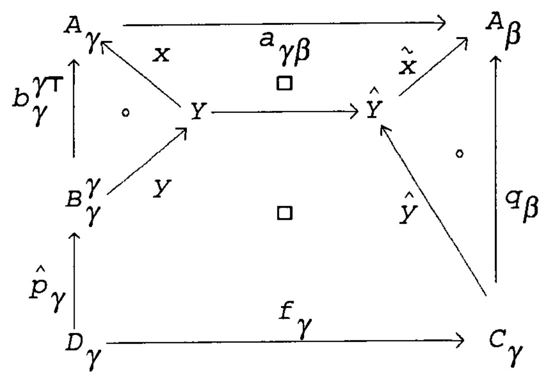
We complete the definition of the remaining arrows for the stage β by putting $b_{\gamma}^{\nu\beta} = \hat{y}_{\gamma} \circ e_{\gamma}^{\nu\beta}$ $(\gamma \leq \nu \leq \beta)$.

By what we have, the conditions under 3) for β hold true.

This completes the recursive step for $\beta \leq \alpha$ a limit ordinal.

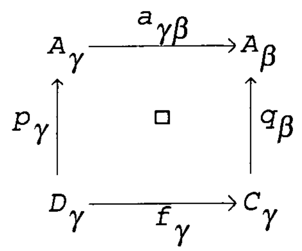
Finally, let $\beta \leq \alpha$ be a successor ordinal, $\beta = \gamma + 1$. We have the items that are marked by γ and lesser ordinals.

The outside square of the following diagram:



(5)

is, via 4), the pushout



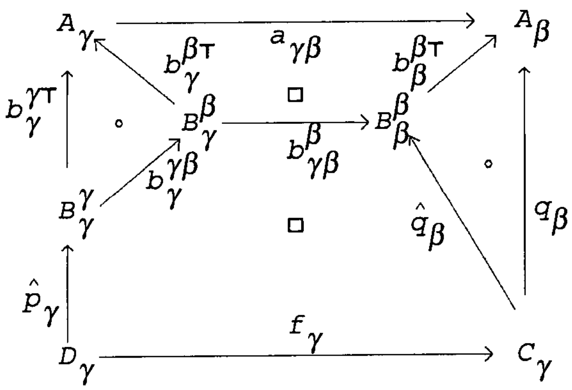
.

From this, we obtain (as before), for an arbitrary factorization (Y, y, x) , $x \circ y = b_{\gamma}^{\gamma T}$, of the arrow $b_{\gamma}^{\gamma T}$, by pushout, the diagram (5), involving a factorization $(\hat{Y}, \hat{y}, \hat{x})$, $\hat{x} \circ \hat{y} = q_{\beta} (= q_{\gamma+1})$, of the arrow q_{β} .

Since A_{γ} is the colimit of the objects $Y \in \mathbf{A}_{\kappa}$, arranged in the κ-filtered diagram $[(Y \xrightarrow{x} A_{\gamma}) \mapsto Y] : \mathbf{A}_{\kappa} \downarrow A \longrightarrow \mathbf{A}$, we have that A_{β} is the colimit of the objects $\hat{Y}$, via the diagram $[(Y \xrightarrow{x} A_\gamma) \mapsto \hat{Y}] : \mathbf{A}_\kappa \downarrow A \longrightarrow \mathbf{A}$. [This argument was done more carefully in an essentially identical case above.]

Since $D_\beta (=D_{\gamma+1}) \in \mathbf{A}_\kappa$, it follows that there is (Y, y, x) , $x \circ y = b_\gamma^{\gamma\top}$, such that $p_\beta : D_\beta \to A_\beta$ (not in the (5)) factors through $\hat{Y} \xrightarrow{\tilde{x}} A_\beta$: we have an arrow $u : D_\beta \xrightarrow{u} \hat{Y}$ such that $\tilde{x} \circ u = p_\beta$.

We define β-indexed items by appropriately (re-)naming things in (5):



We define $\hat{p}_\beta = u$. Furthermore, we define

$$B^\beta \upharpoonright [\gamma) \overset{\text{DEF}}{=} B^\gamma , \qquad b_\delta^{\beta\top} \overset{\text{DEF}}{=} b_\delta^{\gamma\top} \text{ for } \delta < \gamma , \qquad b_{\delta\beta}^\beta \overset{\text{DEF}}{=} b_{\gamma\beta}^\beta \circ b_{\delta\gamma}^\gamma .$$

This completes the proof of 8. Lemma.

Although the following 9. Lemma has nothing to do with transfinite composites or good diagrams, its proof is quite similar to that 6. Lemma. The situation for 9. Lemma is a more elementary one. Both lemmas lift facts for the category $\mathbf{A}$ to the functor category $\mathbf{A}^{\mathbf{G}}$. There may be a common generalization that would spare us the repetitions involved.

9. Lemma Suppose given:

- κ : regular cardinal, $\kappa \geq \aleph_1$;
- $\mathbf{G}$: category of size $< \kappa$;
- $\mathbf{A}$: locally κ-presentable category;
- functors and natural transformation

$$\begin{array}{ccc} & \mathbf{G} & \\ \Phi \Big\downarrow & \xrightarrow{\varphi} & \Big\downarrow \Psi \\ & \mathbf{A} & \end{array} \qquad ;$$

- a family, indexed by objects $G \in \mathrm{Ob}(\mathbf{G})$, of pushout diagrams

$$\begin{array}{ccc} \Phi G & \xrightarrow{\varphi_G} & \Psi G \\ p_G \Big\uparrow & \square & \Big\uparrow q_G \\ D_G & \xrightarrow[f_G]{} & C_G \end{array} \tag{6}$$

(involving components of the natural transformation $\varphi : \Phi \to \Psi$, already introduced before) such that $f_G \in \mathrm{Arr}(\mathbf{A}_K)$.

Assertion: there exist

- functors and natural transformation

$$\begin{array}{ccc} & \mathbf{G} & \\ \Delta \Big\downarrow & \xrightarrow{\psi} & \Big\downarrow \Gamma \\ & \mathbf{A}_K & \end{array} \qquad ;$$

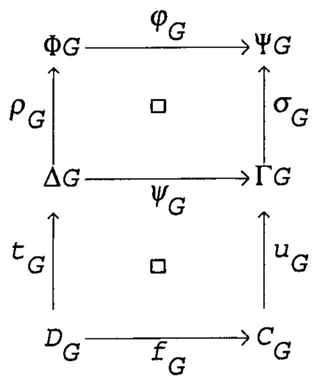
taking values in $\mathbf{A}_K$,

- a pushout diagram

$$\begin{array}{ccc} \Phi & \xrightarrow{\varphi} & \Psi \\ \rho \Big\uparrow & \square & \Big\uparrow \sigma \\ \Delta & \xrightarrow[\psi]{} & \Gamma \end{array}$$

in $\mathbf{A}^{\mathbf{G}}$ (at each G, we have a pushout diagram of the corresponding components), "interpolating" (6): there are

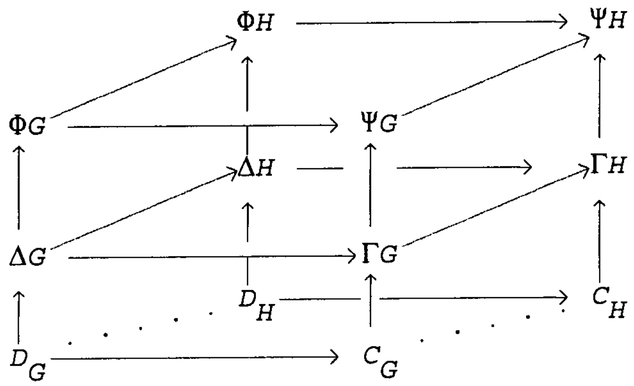
- for each G, arrows t_G and u_G as in:



such that, furthermore

∘∘ $\rho_G \circ t_G = p_G$, $\sigma_G \circ u_G = q_G$.

Proof Let $g: G \to H$. The required items should fit into the diagram



with the expected arrows, partly generated by the given g; the dots are to point to what is missing in the data from the point of view of the goal.

By recursion of n, we construct:

for every $n \in \mathbb{N}$ and $G \in \mathrm{Ob}(\boldsymbol{G})$,

- objects D_G^n, C_G^n of $\boldsymbol{A}_K$,
- arrows $r_G^n: D_G^n \to \Phi G$, $t_G^{n+1}: D_G^n \to D_G^{n+1}$,
- $s_G^n: C_G^n \to \Psi G$, $u_G^{n+1}: C_G^n \to C_G^{n+1}$,

∘ $$f^n_G : D^n_G \to C^n_G \ ,$$

and for all $n \in \mathbb{N}$ and $G \xrightarrow{g} H \in \mathrm{Arr}(\mathbf{G})$,

∘ arrows $d^n_g : D^n_G \to D^{n+1}_H$, $c^n_g : C^n_G \to C^{n+1}_H$;

satisfying the requirements displayed as follows:

∘∘
$$\begin{array}{ccc} \Phi G & \xrightarrow{\varphi_G} & \Psi G \\ r^n_G \uparrow & \square & \uparrow s^n_G \\ D^n_G & \xrightarrow[f^n_G]{} & C^n_G \end{array} , \qquad \begin{array}{ccc} D^{n+1}_G & \xrightarrow{f^{n+1}_G} & C^{n+1}_G \\ t^{n+1}_G \uparrow & \square & \uparrow u^{n+1}_G \\ D^n_G & \xrightarrow[f^n_G]{} & C^n_G \end{array} ,$$

$$\begin{array}{ccccc} & & D^{n+1}_H & \xrightarrow{f^{n+1}_H} & C^{n+1}_H \\ & {}^{d^n_g}\nearrow & \circ & \nearrow{}_{c^n_g} & \\ D^n_G & \xrightarrow[f^n_G]{} & C^n_G & & \end{array} \quad ; \qquad (7)$$

for all G :

∘∘ $f^0_G = f_G$, $r^0_G = \rho_G$, $s^0_G = \sigma_G$;

for all G and all n :

$$r^{n+1}_G \circ t^{n+1}_G = r^n_G , \qquad s^{n+1}_G \circ u^{n+1}_G = u^n_G ;$$

and, for all $G \xrightarrow{g} H \xrightarrow{h} K$, $G \xrightarrow{k=h\circ g} K$:

$$\circ\circ\quad \begin{array}{ccc} D_G^n & \xrightarrow{t_G^{n+1}} & D_G^{n+1} \\ {\scriptstyle d_g^n}\downarrow & \circ & \downarrow{\scriptstyle d_k^{n+1}} \\ D_H^{n+1} & \xrightarrow[d_h^{n+1}]{} & D_K^{n+2} \end{array} \qquad \begin{array}{ccc} C_G^n & \xrightarrow{u_G^{n+1}} & C_G^{n+1} \\ {\scriptstyle c_g^n}\downarrow & \circ & \downarrow{\scriptstyle c_k^{n+1}} \\ C_H^{n+1} & \xrightarrow[c_h^{n+1}]{} & C_K^{n+2} \end{array} \quad . \tag{8}$$

We put (as we must)

$$D_G^0 \xrightarrow{f_G^0} C_G^0 \overset{\text{DEF}}{=} D_G \xrightarrow{f_G} C_G \ ,$$

$$D_G^0 \xrightarrow{r_G^0} \Phi G \overset{\text{DEF}}{=} D_G \xrightarrow{p_G} \Phi G \ ,$$

$$C_G^0 \xrightarrow{s_G^0} \Psi G \overset{\text{DEF}}{=} C_G \xrightarrow{q_G} \Psi G \ .$$

Let $g: G \to H$.

Using the canonical κ-filtered-colimit-of-κ-presentables representation of the object ΦH, we find objects $D_g \in \mathbf{A}_\kappa$ and arrows d_g, t_g, r_g,

$$D_G \xrightarrow{d_g} D_g \ , \qquad D_H \xrightarrow{t_g} D_g \xrightarrow{r_g} \Phi H \ ,$$

such that

$$D_G \xrightarrow{\Phi g \circ p_G = r_g \circ d_g} \Phi H \ , \qquad D_H \xrightarrow{r_g \circ t_g = p_H} \Phi H \ .$$

Taking a fixed $H \in \mathrm{Ob}(\mathbf{G})$, and looking at all $g: G \to H$ with the fixed H, of which there are $<\kappa$-many only, we can make the above items dependent on H only. We have D, t and r, depending on H alone, and we have, for each (G, g) such that $G \xrightarrow{g} H$, the arrow d_g, as follows:

$$D_G \xrightarrow{d_g} D \ , \qquad D_H^0 \xrightarrow{t} \hat{D} \xrightarrow{r} \Phi H \ ,$$

such that,

$$D_G \xrightarrow{\Phi g \circ p_G = r \circ d_g} \Phi H \ , \qquad D_H \xrightarrow{r \circ t\ = p_H} \Phi H \ .$$

Using (6), we have the pushouts

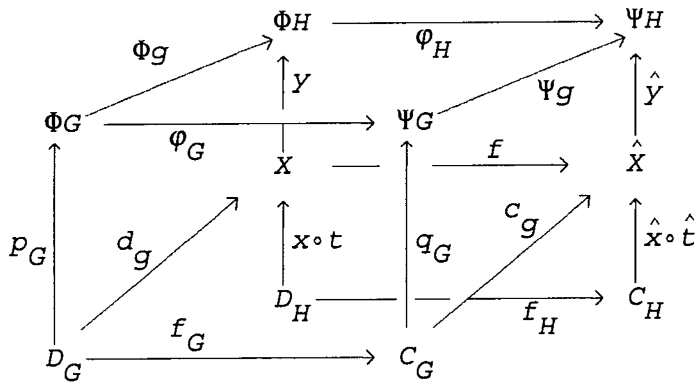
$$\begin{array}{ccc} \Phi H & \xrightarrow{\varphi_H} & \Psi H \\ {\scriptstyle r}\uparrow & \square & \uparrow{\scriptstyle \hat{r}} \\ D & \longrightarrow & \hat{D} \\ {\scriptstyle t}\uparrow & \square & \uparrow{\scriptstyle \hat{t}} \\ D_H & \xrightarrow[f_H]{} & C_H \end{array} \qquad \hat{r}\circ\hat{t}=q_H \ .$$

Consider variable triples (X, x, y) such that $X\in\mathbf{A}_\kappa$, $D\xrightarrow{x}X\xrightarrow{y}\Phi H$ and $D\xrightarrow{y\circ x=r}\Phi H$. and apply pushout to get the diagram

$$\begin{array}{ccc} \Phi H & \xrightarrow[\varphi_H]{} & \Psi H \\ {\scriptstyle y}\uparrow & \square & \uparrow{\scriptstyle \hat{y}} \\ X & \xrightarrow{f} & \hat{X} \\ {\scriptstyle x}\uparrow & \square & \uparrow{\scriptstyle \hat{x}} \\ D & \longrightarrow & \hat{D} \\ {\scriptstyle t}\uparrow & \square & \uparrow{\scriptstyle \hat{t}} \\ D_H & \xrightarrow[f_H]{} & C_H \end{array} \qquad \hat{y}\circ\hat{x}=\hat{r} \ . \qquad (9)$$

Since the object $(\Phi H, r)$ of the comma category $D\downarrow\mathbf{A}$ is a canonical κ-filtered colimit of objects (X, x) with $X\in\mathbf{A}_\kappa$, the object $(\Psi H, \hat{r})$ of the category $\hat{D}\downarrow\Psi H$ is the κ-filtered colimit of the objects $(\hat{X}, \hat{x})$, with colimit coprojections the $\hat{y}\colon (\hat{X}, \hat{x}) \to (\Psi H, \hat{r})$. Therefore, since $C_H\in\mathbf{A}_\kappa$, we can choose (X, x, y), $y\circ x=r$ so that, for every (G, g), $G\xrightarrow{g}H$, there is $c_g\colon C_G\to\hat{X}$ such that $\hat{y}\circ c_g=\Psi g\circ q_G$.

It is now worth looking at the diagram



(for f, see (9)). The "top" quadrangle is a naturality square, thus commutative. By also using the other known commutativities, we get that the two arrows $D_G \overset{f \circ d_g}{\underset{c_g \circ f_G}{\rightrightarrows}} \hat{X}$ are coequalized by the colimit coprojection $\hat{y}$. Therefore, since $D_G \in \mathbf{A}_K$. we can make the choice of (X, x, y), depending on H alone, so that, in addition to what we had before, we also have that for all $g: G \to H$, those two arrows $D_G \overset{f \circ d_g}{\underset{c_g \circ f_G}{\rightrightarrows}} \hat{X}$ are equal.

For a given object H, with the final choice of (X, x, y) and the items derived from (X, x, y), we define

$$D_H^1 \overset{\text{DEF}}{=} X, \qquad t_H^0 \overset{\text{DEF}}{=} t, \qquad r_H^1 \overset{\text{DEF}}{=} r,$$

$$C_H^1 \overset{\text{DEF}}{=} \hat{X}, \qquad u_H^0 \overset{\text{DEF}}{=} \hat{t}, \qquad s_H^1 \overset{\text{DEF}}{=} \hat{r},$$

$$f_H^1 \overset{\text{DEF}}{=} f$$ (for f, see (9));

and for $g: G \to H$,

$$d_g^0 \overset{\text{DEF}}{=} t \circ d_g, \quad c_g^0 \overset{\text{DEF}}{=} \hat{t} \circ c_g .$$

For all $G \in \mathrm{Ob}(\mathbf{G})$, and all $g \in \mathrm{Arr}(\mathbf{G})$, we have constructed

$$D_G^1,\ C_G^1,\ t_G^1,\ u_G^1,\ r_G^1,\ s_G^1,\ f_G^1,\ d_g^0,\ c_g^0,$$

and have satisfied all relevant requirements; note that the two squares (7) mentioning $n+2$ are not relevant yet.

Let $n \geq 0$, and suppose we have constructed, for all $G \in \mathrm{Ob}(\mathbf{G})$ and all $g \in \mathrm{Arr}(\mathbf{G})$, the items

$$D_G^{n+1},\ C_G^{n+1},\ t_G^{n+1},\ u_G^{n+1},\ r_G^{n+1},\ s_G^{n+1},\ f_G^{n+1},\ d_g^{n},\ c_g^{n}\ .$$

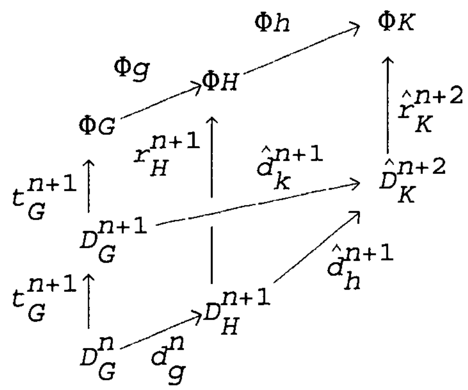
and items with lower indices.

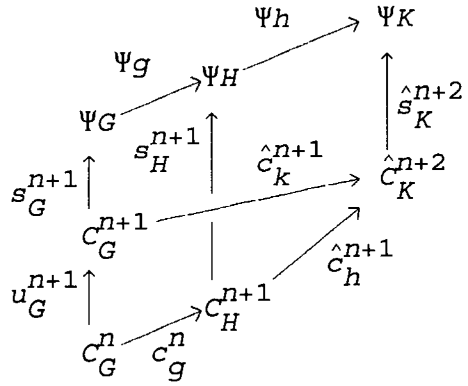
We repeat the above, and construct the items

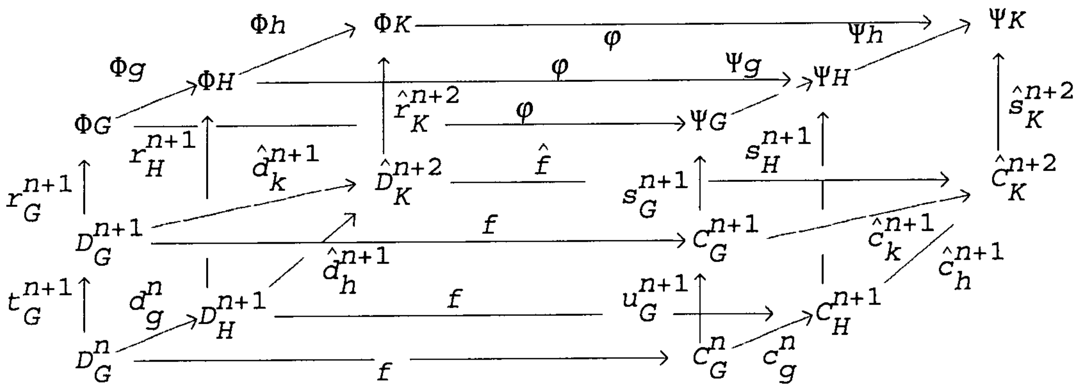
$$\hat{D}_G^{n+2},\ \hat{C}_G^{n+2},\ \hat{t}_G^{n+2},\ \hat{u}_G^{n+1},\ \hat{r}_G^{n+2},\ \hat{s}_G^{n+2},\ \hat{f}_G^{n+2},\ \hat{d}_g^{n+1},\ \hat{c}_g^{n+1}$$

that satisfy the requirements stated for the corresponding desired unhatted items, except the ones in (7).

Let $G \xrightarrow{g} H \xrightarrow{h} K$, $G \xrightarrow{k=h\circ g} K$, and consider the following diagrams, with entities already constructed:







A part of the last diagram, the rectangle

$$\begin{array}{ccc} \Phi K & \xrightarrow{\varphi_K} & \Psi K \\ \uparrow {\scriptstyle \hat{r}_K^{n+2}} & \square & \uparrow {\scriptstyle \hat{s}_K^{n+2}} \\ \hat{D}_K^{n+2} & \xrightarrow[\hat{f}_K^{n+2}]{} & \hat{C}_K^{n+2} \end{array},$$

is a pushout.

Φ and Ψ are functors, and thus $\Phi k = \Phi h \circ \Phi g$, $\Psi k = \Psi h \circ \Psi g$.

For these and other reasons, everything commutes, except the two quadrangles on the left side and the right side, the ones that correspond to the two squares in (7).

By an argument that, by now, must be familiar, for a fixed object K, we can "raise" $\hat{D}_K^{n+2}$ to D_K^{n+2}, and have corresponding arrows with all hats removed, such that, first, the left side quadrangle becomes commutative, for all situations $G \xrightarrow{g} H \xrightarrow{h} K$, $G \xrightarrow{k=h\circ g} K$, and then further so that, again for all $G \xrightarrow{g} H \xrightarrow{h} K$, $G \xrightarrow{k=h\circ g} K$, the right side quadrangle becomes commutative.

This completes the recursive construction.

To complete the proof of the lemma, we define

$$\Delta G = \operatorname*{colim}_{n\in\mathbb{N}} D_G^n, \qquad \Gamma G = \operatorname*{colim}_{n\in\mathbb{N}} C_G^n, \qquad \psi_G = \operatorname*{colim}_{n\in\mathbb{N}} f_G^n$$

$$t_G = \operatorname*{colim}_{n\in\mathbb{N}} t_G^{n+1}, \qquad u_G = \operatorname*{colim}_{n\in\mathbb{N}} u_G^{n+1}$$

more precisely, $\Delta G = \operatorname{colim} D_G$, where $D_G : \mathbb{N} \to \mathbf{A}$, $D_G(n) = D_G^n$, $D_G(n<n+1) = t_G^{n+1}$, and similarly for the others.

Since $\kappa > \aleph_0$, ΔG and ΓG are all in $\mathbf{A}_\kappa$.

Conditions (8) ensure that Δ and Γ are indeed functors, (7) ensure that ψ is a natural transformation.

This completes the proof of 9. Lemma.

10. Proposition Assume

κ is a regular cardinal, $\kappa \geq \aleph_1$;

$\mathbf{G}$ is a category of size $<\kappa$;
$\mathbf{A}$ is a locally κ-presentable category;
$\mathcal{J} \subseteq \mathrm{Arr}(\mathbf{A}_\kappa)$.

Then

$$\langle \mathrm{Po}^{(\mathbf{A})}[\mathcal{J}], \mathbf{G}\rangle = \mathrm{Po}^{(\mathbf{A}^{\mathbf{G}})}[\langle \mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{J}], \mathbf{G}\rangle] \ .$$

Proof This is direct from 9. Lemma. We take an element $\varphi:\Phi\to\Psi$ of the class on the left. We have the assumptions of the lemma; in particular, the arrows $f_G: D_G \to C_G$ from $\mathcal{J}$.

The natural transformation $\psi:\Delta\to\Gamma$ constructed in the lemma is in $\langle \mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{J}], \mathbf{G}\rangle$, and φ is a pushout of it.

Theorem Assume:

κ is a regular cardinal, $\kappa \geq \aleph_1$;
$\mathbf{G}$ is a category such that $\#\mathbf{G}<\kappa$;
$\mathbf{A}$ is a locally κ-presentable category;
$\mathcal{I} \subseteq \mathrm{Arr}(\mathbf{A}_\kappa)$.

Then

$$\langle \mathcal{C}^{(\mathbf{A})}[\mathrm{Po}^{(\mathbf{A})}[\mathcal{I}]], \mathbf{G}\rangle = \mathcal{C}^{(\mathbf{A}^{\mathbf{G}})}[\mathrm{Po}^{(\mathbf{A})}[\langle \mathcal{C}^{(\mathbf{A})}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa], \mathbf{G}\rangle]] \ .$$

(The conclusion in words: every natural transformation between functors $\mathbf{G} \rightrightarrows \mathbf{A}$ whose components are transfinite composites of pushouts of $\mathcal{I}$-arrows is a transfinite composite of pushouts of natural transformations whose components are $<\kappa$-length transfinite composites of pushouts in $\mathbf{A}_\kappa$ of $\mathcal{I}$-arrows.)

Since the class is $\langle \mathcal{C}^{(\mathbf{A})}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa], \mathbf{G}\rangle$ is essentially small, Jacob Lurie's lemma is contained in the theorem.

Proof

$$\langle \mathcal{C}^{(\mathbf{A})}[\mathrm{Po}^{(\mathbf{A})}[\mathcal{I}]], \mathbf{G}\rangle \underset{\substack{\uparrow \\ \text{7. Prop.}}}{=} \mathcal{C}^{(\mathbf{A}^{\mathbf{G}})}[\langle \mathcal{C}^{(\mathbf{A})}[\mathrm{Po}^{(\mathbf{A})}[\mathcal{I}], <\kappa], \mathbf{G}\rangle]$$

Formula: LHS = $\mathcal{C}^{(\mathbf{A}^{\mathbf{G}})}[\mathrm{Po}^{(\mathbf{A}^{\mathbf{G}})}[\mathcal{J}]]$, where $\mathcal{J} = \langle K, \mathbf{G}\rangle$, where $K = \mathcal{C}^{(\mathbf{A})}[L, <\kappa]$, where $L = \mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}]$

$$= \mathcal{C}^{(\mathbf{A}^G)}[\mathrm{Po}^{(\mathbf{A})}[\mathcal{C}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa], G\rangle]]$$

↑

8. Prop.

$$= \mathcal{C}^{(\mathbf{A}^G)}[\mathrm{Po}^{(\mathbf{A}^G)}\langle\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{C}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa], G\rangle]]$$

↑

10. Prop. for $\mathcal{J} = \mathcal{C}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa]$

$$= \mathcal{C}^{(\mathbf{A}^G)}[\mathrm{Po}^{(\mathbf{A}^G)}[\langle\mathcal{C}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa], G\rangle]] \quad .$$

Since $\mathcal{I} \subseteq \mathrm{Arr}(\mathbf{A}_\kappa)$, and $\mathbf{A}_\kappa$ is closed under colimits of diagrams of size less than κ, we have $\mathcal{J} = \mathcal{C}[\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{I}], <\kappa] \subseteq \mathrm{Arr}(\mathbf{A}_\kappa)$ as well; thus, 10. Prop. is applicable. Obviously, $\mathrm{Po}^{(\mathbf{A}_\kappa)}[\mathcal{J}] = \mathcal{J}$, justifying the last step.