The new proof of the GCT

GCT = General Completeness Theorem:

stated on page S $\boxed{30}$ of

2015 April 07 Sketch Notes. PDF.

('New') proof here; old proof in the paper

"Generalized sketches..." JPAA 1997.

Re-statement:

Given $\mathbb{S}$: lfp category;

$\mathbb{R} \subseteq Arr(\mathbb{S}_{fp})$;

[$\mathbb{S}_{fp}$ = full subcat of $\mathbb{S}$ on the fp objects]

$A \xrightarrow{\varphi} B$: relatively finite arrow;

we have a pushout

$$\begin{tikzcd} U_0 \arrow[r, "s"] \arrow[d, "a"'] & V_0 \arrow[d, "b"] \\ A \arrow[r, "\varphi"'] \arrow[ur, phantom, "\ulcorner"] & B \end{tikzcd} \qquad (*)$$

Assume: $\mathbb{R} \models \varphi$

Want: $\mathbb{R} \overset{?}{\vdash} \varphi$

We factor the ('terminal') arrow

$$A \xrightarrow{!_A} \mathbb{1}$$

to the terminal object $\mathbb{1}$ of $\mathbb{S}$ ($\mathbb{S}$ is a complete category!) according to the Small Object Argument with $I = \mathbb{R}$: (see p. A $\boxed{14}$ of "Good diagrams"). We obtain:

object X, arrow $A \xrightarrow{g} X$ such that:

$$\begin{tikzcd} A \arrow[rr, "!_A"] \arrow[dr, "g"'] & \arrow[d, phantom, "\circleddash"] & \mathbb{1} \\ & X \arrow[ur, "!_X"'] & \end{tikzcd}$$

(which part, of course, is empty!) and:

$$\mathbb{R} \perp !_X ; \qquad (*)$$

and: we have $A : P \longrightarrow \mathbb{S}$

$$g = \underbrace{\langle A \rangle}_{\text{composite of } A} = a_{\perp\top} : A = A_0 \longrightarrow A_\top = X$$

where A is $\aleph_0$-good (for $x \in P$, $\{y : y \leq x\}$ is finite)

and $\aleph_0$-directed – simply, directed.

$(*)$ means:

for all $r: U \to V$ in $\mathbb{R}$ and diagram

$$\begin{tikzcd} U \arrow[r, "m"] \arrow[d, "r"'] & X \arrow[d, "!_X"] \\ V \arrow[r, "!_V"'] & \mathbb{1} \end{tikzcd}$$

($\ominus$: automatic)

there is $d: V \to X$ s.t.

$$\begin{tikzcd} U \arrow[r, "m"] \arrow[d, "r"'] \arrow[dr, phantom, "\ominus"] & X \arrow[d, "!_X"] \\ V \arrow[r] \arrow[ur, "d"] \arrow[ur, phantom, "(\ominus)"'] & \mathbb{1} \end{tikzcd} \qquad dr = m$$

For a given $r \in \mathbb{R}$, this means $X \models r$; since this holds for all $r \in \mathbb{R}$, we have $\boxed{X \models \mathbb{R}}$.

The assumption $\mathbb{R} \models \varphi$ means:

$$\text{for all } S: \quad S \models \mathbb{R} \Rightarrow S \models \varphi.$$

Thus: we conclude that

$$X \models \varphi.$$

Applying this to the 'instantiation' $g: A \to X$, we have $h: B \to X$ such that

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "g"'] & \arrow[d, phantom, "="] & B \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

$$h\varphi = g$$

Put this together with relative finiteness: $(*)$, p G1

$$(1) \qquad \begin{tikzcd} U_0 \arrow[r, "s"] \arrow[d, "a"'] & V_0 \arrow[d, "b"] \arrow[dl, phantom, "\ulcorner"] \\ A \arrow[r, "\varphi"] \arrow[dr, "a_{\perp T} = g"'] \arrow[r, phantom, shift right=4ex, "="] & B \arrow[d, "h"] \\ & A_T = X \end{tikzcd} \qquad hb : V_0 \longrightarrow A_T$$

Since V_0 is finitely presentable, and the diagram $A : P \to \mathcal{S}$ of which A_T is the colimit is directed, there exist $x_0 \in X$ and $k_0 : V_0 \longrightarrow A_{x_0}$ s.t.:

$$(2) \qquad \begin{tikzcd} V_0 \arrow[rr, "hb"] \arrow[dr, "k_0"'] & \arrow[d, phantom, "="] & A_T \\ & A_{x_0} \arrow[ur, "a_{x_0 T}"'] & \end{tikzcd} \quad ; \quad hb = a_{x_0 T} k_0 .$$

Pre-compose this with $s : U_0 \longrightarrow V_0$

consider: (3) & (4):

(3)
$$\begin{tikzcd}
U_0 \arrow[r, "s"] \arrow[d, "a"'] \arrow[dr, phantom, "= ?"] & V_0 \arrow[d, "k_0"] \arrow[dr, bend left, "hb"] \arrow[dr, phantom, "="] & \\
A \arrow[r, "a_{\perp x_0}"'] \arrow[rr, bend right=40, "a_{\perp T} = g"'] \arrow[rr, phantom, bend right=20, "="] & A_{x_0} \arrow[r, "a_{x_0 T}"] & A_T
\end{tikzcd}$$

$$\begin{tikzcd}
U_0 \arrow[r, "s"] \arrow[d, "a"'] & V_0 \arrow[dd, bend left, "hb"] \\
A \arrow[dr, bend right] \arrow[r, phantom, "="] & \\
& A_T
\end{tikzcd}$$

(4)
$$\begin{tikzcd}
U_0 \arrow[r, shift left, "f_1 = ks"] \arrow[r, shift right, "f_2 = a_{\perp x_0} a"'] & A_{x_0} \arrow[r, "a_{x_0 T}"] & A_T
\end{tikzcd}$$

In (3), we have the commutativities indicated (not (yet) the one with a question mark) by (1) & (2) above – and by the 'diagram identity' $a_{\perp T} = a_{x_0 T}\, a_{\perp x_0}$. Therefore, in (4), $a_{x_0 T}$ coequalizes f_1 & f_2 : $a_{x_0 T} f_1 = a_{x_0 T} f_2$. U_0 is finitely presentable. Therefore, there is $x \in P$, $x \geq x_0$ such that: in

$$U_0 \overset{f_1}{\underset{f_2}{\rightrightarrows}} A_{x_0} \xrightarrow{a_{x_0 x}} A_x$$

$a_{x_0 x}$ coequalizes f_1 & f_2 : $a_{x_0 x} f_1 = a_{x_0 x} f_2$.

(This is the '<u>uniqueness part</u>' in the definition of $\mathcal{U}$ being finitely presentable: the 1-1-ness of the canonical arrow being required to be an isomorphism). What this means — and the self-respecting categorist would have been content with this — that in (3)'' we may assume that (=)? is indeed true — more precisely, we do have the following commuting:

$$\begin{tikzcd}
U_0 \arrow[r, "s"] \arrow[d, "a"'] \arrow[dr, phantom, "= \checkmark"] & V_0 \arrow[d, "k \overset{\mathrm{def}}{=} a_{x_0 x} k_0"] \\
A \arrow[r, bend right, "a_{1x}"'] & A_x
\end{tikzcd}$$

Now, look at:

$$\begin{tikzcd}
U_0 \arrow[r, "s"] \arrow[d, "a"'] & V_0 \arrow[d, "b"] \arrow[dr, bend left, "k"] & \\
A \arrow[r, "\varphi"] \arrow[drr, bend right, "= a_{1x}"'] & B \arrow[ul, phantom, "\ulcorner p.o."] \arrow[dr, dashed, "l"] & \\
& & A_x
\end{tikzcd}$$

to conclude that there is (unique) $\ell : \mathbb{B} \to A_x$

so that the two indicated commutativities hold.

We only need the lower one:

$$\begin{tikzcd}
\mathbb{A} = A_\perp \arrow[rr, "a_{\perp x}"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\textcircled{=}"] & A_x \\
& \mathbb{B} \arrow[ur, "\ell"'] &
\end{tikzcd}$$

The arrow $a_{\perp x} : \mathbb{A}_\perp \longrightarrow A_x$ is the composite of the finite good diagram

$$A \restriction (x\downarrow) : (x\downarrow) \longrightarrow \mathbb{S}$$

$(x\downarrow = \{y \in P : y \leq x\}$, a finite set, with the induced ordering from P $)$

all whose links are in $P_0(\mathbb{R})$, pushouts of arrows in $\mathbb{R}$.

Lemma. We can "linearize" the diagram $A \restriction (x\downarrow)$: there are: a finite linear order Q;

"and a diagram

$$C : Q \longrightarrow \mathcal{S}$$

such that each link in B is in $Po(\mathbb{R})$,

and $\langle C \rangle = c_{\perp T} = \underbrace{a_{\perp} x}_{\text{our given composite}}$

(proof of Lemma: later).

Note that with this:

$$\begin{tikzcd}
A = C_{\perp} \arrow[rr, "c_{\perp T}"] \arrow[dr, "\varphi"'] \arrow[rr, phantom, shift right=4ex, "\circledequal"] & & C_T \\
& B \arrow[ur, "\ell"'] &
\end{tikzcd}$$

we have exactly what we called a deduction of φ from $\mathbb{R}$ — i.e., $\mathbb{R} \vdash \varphi$ as we wanted. $\boxed{\text{end of proof of GCT}}$

The Lemma is a special case of a general linearization statement, stated and proved in "Rearranging ...": 2. Proposition, p 15.

I reproduce the proof for the finite case — which also shows all the essential features of the general "infinite" case.

(to be continued...)