The new proof of the GCT

\[ \text{GCT = General Completeness Theorem} \]

\[ \text{stated on page 30 of} \]

\[ \text{2015 April 07 Sketch Notes, PDF.} \]

"New proof here; old proof in the paper

Re-statement:

Given \( X : \text{lfp category} \);

\( IR \subseteq \text{Arr}(X_{\text{fp}}) \);

\[ [X_{\text{fp}} = \text{full subcat of} \ X \ \text{on the fp objects}] \]

\( A \xrightarrow{q} B : \text{relatively finite arrow} \)

we have a pushout

\[ \begin{array}{ccc}
U_0 & \overset{s}{\rightarrow} & V_0 \\
\alpha \downarrow & & \downarrow \delta \\
A & \xrightarrow{q} & B
\end{array} \quad (*) \]

Assume: \( IR \vdash q \)

Want: \( IR \vdash q \)
We factor the terminal arrow

\[
A \xrightarrow{!_A} 1
\]

to the terminal object \(1\) of \(\mathcal{S}\) (\(\mathcal{S}\) is a complete category) according to the Small Object Argument with \(I = \mathbb{R}\). (See p. \#14 of "Good diagrams"). We obtain:

object \(X\) arrow \(A \xrightarrow{g} X\) such that:

\[
\begin{array}{ccc}
A & \xrightarrow{!_A} & 1 \\
\downarrow \circ & \downarrow \circ & \downarrow \circ \\
g & \downarrow & X
\end{array}
\]

(which part, of course, is empty!) and:

\[
R \xrightarrow{!_X} \quad (\ast)
\]

and: we have \(A : P \rightarrow \mathcal{S}\)

\[
g = \langle A \rangle = a_{\downarrow} : A = A_0 \xrightarrow{\text{composite of } A} A_T = X
\]

where \(A\) is \(\leq_0\)-good (for \(x \in P\), \(\exists y : y \leq x\) is finite)

and \(\leq_0\)-directed - simply, directed.

(\(\ast\) means:

\[
\ldots
\]
For all \( r : U \rightarrow V \) in \( IR \), and diagram

\[
\begin{array}{ccc}
U & \xrightarrow{m} & X \\
\downarrow r & & \downarrow !_X \\
V & \rightarrow & 1 \\
\end{array}
\]

\((?:\text{ automatic})\)

there is \( d : V \rightarrow X \) s.t.

\[
\begin{array}{ccc}
U & \xrightarrow{m} & X \\
\downarrow r & & \downarrow !_X \\
V & \rightarrow & 1 \\
\end{array}
\]

\(d \circ r = m\)

For a given \( r : U \rightarrow V \), this means \( X \models r \); since this holds for all \( r : U \rightarrow V \), we have \( X \models IR \).

The assumption \( IR \models \varphi \) means:

for all \( S : S \models IR \Rightarrow S \models \varphi \).

Thus: we conclude that

\( X \models \varphi \).

Applying this to the instantiation \( g : A \rightarrow X \), we have \( h : B \rightarrow X \) such that
Put this together with relative finiteness: (\(*\)), p G\ref{G1}

\[
\begin{array}{ccc}
U_0 & \xrightarrow{s} & V_0 \\
\downarrow a & \downarrow t & \downarrow b \\
A & \xrightarrow{\theta} & B \\
\downarrow \alpha_{T} & & \downarrow h \\
A_T = X & & A_T = X
\end{array}
\]

Since \(V_0\) is finitely presentable, and the diagram \(A : P \rightarrow S\) of which \(A_T\) is the colimit in directed, there exist \(x_0 \in X\) and \(k_0 : V_0 \rightarrow A_{x_0}\) s.t.:

\[
\begin{array}{ccc}
V_0 & \xrightarrow{h_0} & A_{x_0} \\
\downarrow k_0 & \downarrow \alpha_{x_0} & \downarrow \alpha_{x_0,T} \\
A_{x_0} & \xrightarrow{\theta} & A_T \\
\end{array}
\]

Pres-compose this with \(s : U_0 \rightarrow V_0\) the obvious.

Consider: (3) \& (4):
In (3), we have the commutativities indicated (not (yet) the one with a question mark) by (1) & (2) above — and by the 'diagram identity' $a_{\perp T} = a_{x_0 T} a_{\perp x_0}$. Therefore, in (4), $a_{x_0 T}$ equalizes $f_1$ & $f_2$: $a_{x_0 T} f_1 = a_{x_0 T} f_2$. $U_0$ is finitely presentable. Therefore, there is $x E P$, $x \geq x_0$ such that in

\[
\begin{align*}
U_0 & \xrightarrow{f_1} A_{x_0} \xrightarrow{a_{x_0 x}} A_x \\
& \xrightarrow{f_2} A_{x_0} 
\end{align*}
\]
\[ a_{x_0 x} \text{ coequalizes } f_1 \& f_2 : a_{x_0 x} f_1 = a_{x_0 x} f_2. \]

(This is the unique part in the definition of \( U \) being finitely presentable: the 1-1-ness of the canonical arrow being required to be an isomorphism.) What this means — and the self-respecting category would have been content with this — that in (3) "we may assume that \( \Theta \) is indeed true — more precisely, we do have the following commuting:

\[ U_0 \xrightarrow{s} V_0 \]

Now, look at:

\[ U_0 \xrightarrow{s} V_0 \]
to conclude that there is (unique) \( l: B \to A_x \)
so that the two indicated commutativities hold.

We only need the lower one:

\[
x = A_\perp \xrightarrow{a_{\perp x}} A_x \\
\phi \quad \equiv \quad \ell \\
B
\]

The arrow \( a_{\perp x}: A_\perp \to A_x \) is the
composite of the finite good diagram

\[ A \Gamma (x \perp) : (x \perp) \longrightarrow S \]

\((x \perp = \{ y \in P : y \leq x \}, a \text{ finite set, with the induced ordering from } P\) all whose links are in \( P_0(\mathbb{C} IR) \), pushout of
arrows in \( IR \).

Lemma. We can linearize the diagram \( A \Gamma (x \perp) : \\
there are: a finite linear order \( Q \). \)
and a diagram

\[ C : Q \rightarrow \$ \]

such that each link in B is in Po(1R),
and \( \langle C \rangle = c_{LT} = a_{LX} \)

in our given composite

(proof of Lemma: later).

Note that with this:

\[ A = C \_ \xrightarrow{c_{LT}} C \_ \]

we have exactly what we called a deduction

of \( \psi \) from 1R — i.e., 1R \( \vdash \psi \) as we wanted.

The Lemma is a special case of a general

linearization statement, stated and proved in


I reproduce the proof for the finite case

—which also shows all the essential features

of the general "infinite" case.

(to be continued...