

April 26/05

The new proof of the GCT

G-1

GCT = General Completeness Theorem:

stated on page 5 30 of

2015 April 07 Sketch Notes, PDF.

'New' proof here; old proof in the paper

"Generalized sketches..." JPAA 1997.

Re-statement:

Given $\mathcal{S} : \text{lfp category};$

$\mathcal{R} \subseteq \text{Arr}(\mathcal{S}_{\text{fp}});$

$[\mathcal{S}_{\text{fp}} = \text{full subcat of } \mathcal{S} \text{ on the fp objects}]$

$A \xrightarrow{\varphi} B$: relatively finite arrow;

we have a pushout

$$\begin{array}{ccc} U_0 & \xrightarrow{s} & V_0 \\ a \downarrow & \xrightarrow{\quad} & \downarrow b \\ A & \xrightarrow[\varphi]{} & B \end{array} \quad (*)$$

Assume: $\mathcal{R} \models \varphi$

Want: $\mathcal{R} \vdash^? \varphi$

We factor the 'terminal' arrow

$$A \xrightarrow{!X} 1$$

to the terminal object 1 of $\mathcal{S}$ ($\mathcal{S}$ is a complete category!) according to the Small Object Argument with $I = \mathbb{R}$:
(see p. A14 of "Good diagrams"). We obtain:

object X , arrow $A \xrightarrow{g} X$ such that:

$$\begin{array}{ccc} A & \xrightarrow{!X} & 1 \\ & \searrow \scriptstyle g & \nearrow \scriptstyle !X \\ & X & \end{array} \quad \ominus$$

(which part, of course, is empty!) and:

$$R \perp !X; \quad (*)$$

and: we have $A : P \longrightarrow \mathcal{S}$

$$g = \underbrace{\langle A \rangle}_{\text{composite of } A} = a_{\perp+} : A = A_0 \longrightarrow A_T = X$$

where A is κ_0 -good (for $x \in P$, $\{y : y \leq x\}$ is finite)

and κ_0 -directed — simply, directed.

⊗ means:

for all $r: U \rightarrow V$ in $\mathbb{R}$ and diagram

$$\begin{array}{ccc}
 U & \xrightarrow{m} & X \\
 r \downarrow & & \downarrow !X \\
 V & \xrightarrow{!V} & \perp
 \end{array}
 \quad (\ominus: \text{automatic})$$

there is $d: V \rightarrow X$ s.t.

$$\begin{array}{ccc}
 U & \xrightarrow{m} & X \\
 r \downarrow & \text{\scriptsize $\ominus$} \nearrow & \downarrow !X \\
 V & \xrightarrow{d(\ominus)} & \perp
 \end{array}
 \quad dr = m$$

For a given $r \in \mathbb{R}$, this means $X \models r$; since this holds for all $r \in \mathbb{R}$, we have $\boxed{X \models \mathbb{R}}$.

The assumption $\mathbb{R} \models \varphi$ means:

for all S : $S \models \mathbb{R} \Rightarrow S \models \varphi$.

Thus: we conclude that

$$X \models \varphi.$$

Applying this to the 'instantiation' $g: A \rightarrow X$, we have $h: B \Rightarrow X$ such that

$$\begin{array}{ccc}
 A & \xrightarrow{\varphi} & B \\
 & \text{\scriptsize $\ominus$} & \\
 g \swarrow & & \nwarrow h \\
 & X &
 \end{array}$$

$$h\varphi = g$$

Put this together with relative finiteness: $(*)$, p. 61

(1)

$$\begin{array}{ccc}
 U_0 & \xrightarrow{s} & V_0 \\
 a \downarrow & & \downarrow b \\
 A & \xrightarrow{\varphi} & B \\
 & \text{\scriptsize $\ominus$} & \\
 a_{\perp T} = g \swarrow & & \downarrow h \\
 & A_T = X &
 \end{array}
 \qquad hb: V_0 \longrightarrow A_T$$

Since V_0 is finitely presentable, and the diagram $A : P \rightarrow \mathcal{B}$ of which A_T is the colimit is directed, there exist $x_0 \in X$ and $k_0: V_0 \rightarrow A_{x_0}$ s.t.:

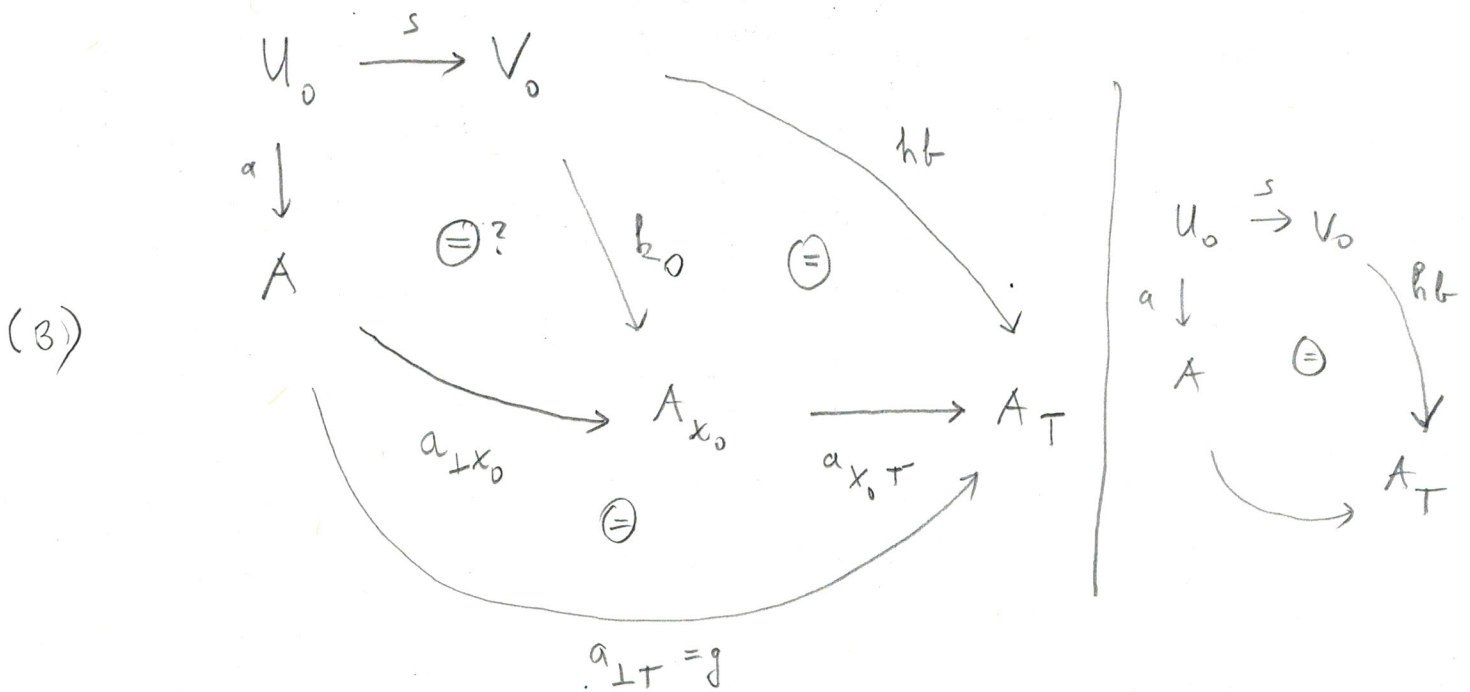
(2)

$$\begin{array}{ccc}
 V_0 & \xrightarrow{hb} & A_T \\
 & \text{\scriptsize $\ominus$} & \\
 k_0 \swarrow & & \nearrow a_{x_0 T} \\
 & A_{x_0} &
 \end{array}
 \qquad ; \quad hb = a_{x_0 T} k_0$$

Pre-compose this with $s: U_0 \rightarrow V_0$ the object (P)

$\perp$

consider: (3) & (4):



(4)

$$\begin{array}{ccccc}
 U_0 & \xrightarrow{f_1 = b_s} & A_{x_0} & \xrightarrow{a_{x_0 T}} & A_T \\
 & \xrightarrow{f_2 = a_{\perp x_0} a} & & & \\
 \end{array}$$

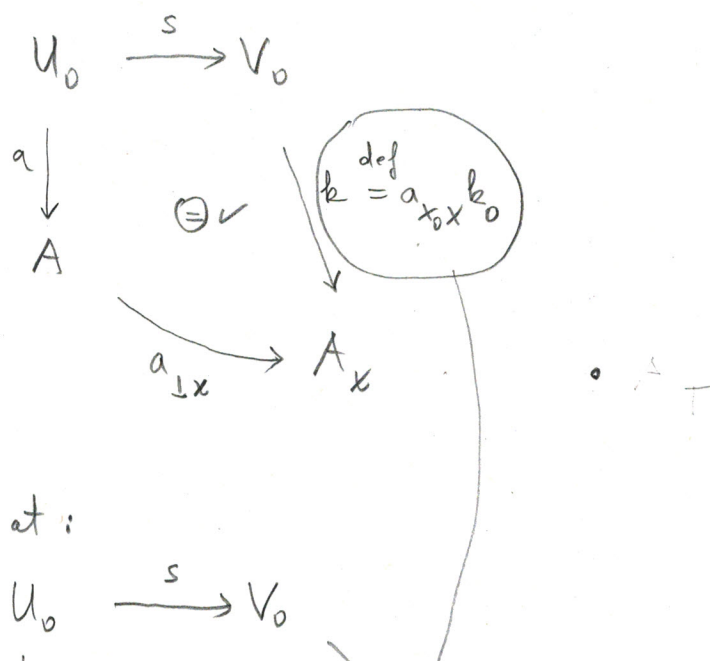
In (3), we have the commutativities indicated (not yet) the one with a question mark) by (1) & (2) above - and by the 'diagram identity' $a_{\perp T} = a_{x_0 T} a_{\perp x_0}$. Therefore, in (4), $a_{x_0 T}$ equalizes f_1 & f_2 : $a_{x_0 T} f_1 = a_{x_0 T} f_2$. U_0 is finitely presentable. Therefore, there is $x \in P$, $x \geq x_0$ such that: in

$$\begin{array}{ccccc}
 U_0 & \xrightarrow{f_1} & A_{x_0} & \xrightarrow{a_{x_0 x}} & A_x \\
 & \xrightarrow{f_2} & & & \\
 \end{array}$$

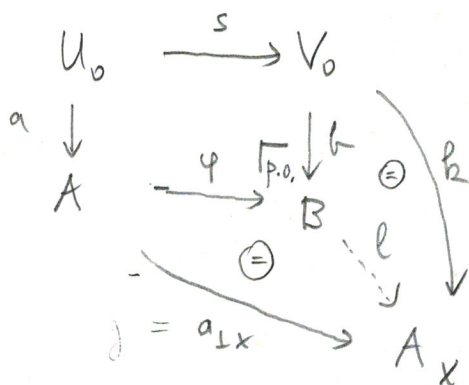
$a_{x_0 x}$ coequalizer f_1 & f_2 : $a_{x_0 x} f_1 = a_{x_0 x} f_2$.

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(this is the uniqueness part) in the definition of U being finitely presentable: the 1-1-ness of the canonical arrow being required to be an isomorphism). What this means — and the self-respecting categorist would have been content with this — that in (3) "we may assume that $\oplus?$ is indeed true — more precisely, we do have the following commuting:



Now, look at:



to conclude that there is (unique) $\ell: B \rightarrow A_x$
so that the two indicated commutativities hold.

We only need the lower one:

$$\begin{array}{ccc}
 X = A_{\perp} & \xrightarrow{a_{\perp x}} & A_x \\
 \searrow \varphi & \text{\textcircled{=}} & \nearrow \ell \\
 & B &
 \end{array}$$

The arrow $a_{\perp x} : A_{\perp} \rightarrow A_x$ is the
composite of the finite good diagram

$$A \upharpoonright (x \downarrow) : (x \downarrow) \longrightarrow \mathbb{S}$$

($x \downarrow = \{y \in P : y \leq x\}$, a finite set, with
the induced ordering from P)

all whose links are in $P_0(\mathbb{R})$, pushouts of
arrows in $\mathbb{R}$.

Lemma. We can 'linearize' the diagram $A \upharpoonright (x \downarrow)$:

there are: a finite linear order Q ,

(finite)

①

"and a diagram

$$C : Q \longrightarrow \mathbb{S}$$

such that each link in B is in $Po(\mathbb{R})$,

$$\text{and } \langle C \rangle = c_{\perp T} = \underbrace{a_{\perp}}_x$$

our given composite

(proof of Lemma: later).

Note that with this:

$$\begin{array}{ccc}
 A = C_{\perp} & \xrightarrow{c_{\perp T}} & C_T \\
 & \text{\textcircled{=}} & \\
 \downarrow \varphi & & \uparrow \ell \\
 & B &
 \end{array}$$

we have exactly what we called a deduction
of φ from $\mathbb{R}$ — i.e., $\mathbb{R} \vdash \varphi$ as we wanted.

end of proof of GCT

The Lemma is a special case of a general
linearization statement, stated and proved in
"Rearranging..." : 2. Proposition, §15.

I reproduce the proof for the finite case
— which also shows all the essential features
of the general "infinite" case.

(to be continued...)