Good diagrams – continued

$\boxed{2.}$ 2.1 I will start with some <u>annotation</u> to the notes $\boxed{\text{TrivialExtensionVersion2 2012Nov03.pdf}}$

(the title heading is "Notes on the 'triple paper'", because of the use of the material in that paper). This material will be considered mathematically trivial by 'self-respecting' category theorists. On the other hand, it is not only useful, but, I think, even its details are of interest. My present 'annotation' is a summary in the style of the 'self-respecting' categorist. I will try to match the notation of the notes.

Let $\mathbb{X}$ be a (small) cocomplete category. P, Q and R will denote small categories; later: posets.

Suppose:

$$\begin{tikzcd} P \arrow[r, "\overline{\Phi}"] \arrow[rr, bend right, "A \underset{\text{def}}{=} B\overline{\Phi}"'] & Q \arrow[r, "B"] & \mathbb{X} \end{tikzcd}$$

We have the colimits colim A, colim B in $\mathbb{X}$, and a <u>canonical comparison morphism</u>

$$\operatorname{colim} A \xrightarrow{f} \operatorname{colim} B. \qquad (*)$$

We say that the diagram

$$B : Q \to \mathbb{X}$$

is a trivial extension of

$$A : P \longrightarrow \mathbb{X}$$

via $\Phi : P \to Q$ if f in $(*)$ is an isomorphism.

In the 'Notes', the words are different, but the effect is the same. In the present version, the 'canonical comparison' is something that is a bit loose: it must be defined using the colimit cocones for the two colimits: if $\alpha = \langle \alpha_x \rangle_{x \in P}$, $\beta = \langle \beta_y \rangle_{y \in Q}$ are those for colim A and colim B, then f is the unique arrow for which

$$f\alpha_x = \beta_{\Phi(x)}$$

for all $x \in P$. The pedantic reader may object to the present definition that none of the items colim A, colim B

α and β is uniquely determined — thus, logically speaking, one has to show that the definition is independent of the choice of those items — to which the self-respecting categorist will reply that those items are determined up to isomorphism, so the 'independence' is 'obvious'. In the 'trivial' notes being annotated, the definition is formulated in an elementary manner that does not provoke the said objection of the pedant.

After some 'pedantic' discussion (1.1, 1.2), on page 1.3 we find the following statement:

<u>Suppose</u>:

$$\begin{tikzcd} & Q \arrow[dr, "\Psi"] & \\ P \arrow[ur, "\Phi"] \arrow[rr, bend right, "\Theta"'] \arrow[rr, phantom, "="] & & R \end{tikzcd}$$

$$\Theta = \Psi\Phi$$

and let $A : P \to \mathbb{X}$, $B : Q \to \mathbb{X}$ and $C : R \to \mathbb{X}$.
<u>Suppose</u> B extends A via Φ ($A = B\Phi$), C extends B via Ψ, and, as a consequence C extends A via Θ.
Then if any two of the following three conditions hold: (i) B is a trivial extension of A (via Φ);

(ii) C is a trivial extension of B;

(iii) C is a trivial extension of A;

then the third one holds as well. The reason is that we have the corresponding comparisons:

$$\begin{tikzcd}
 & \operatorname{colim} B \arrow[dr, "f_\Psi"] & \\
\operatorname{colim} A \arrow[ur, "f_\Phi"] \arrow[rr, bend right, "f_\Theta"'] \arrow[rr, phantom, "="] & & \operatorname{colim} C
\end{tikzcd}$$

and the diagram <u>commutes</u>; and, of course, in a commutative triangle if two arrows are isomorphisms, then the third one is iso too. In the 'Notes', there are some ... pedantic words to the same effect ... .

(2.2) The main message is about the <u>existence</u> of trivial extensions with prescribed properties. The basic construction is the '<u>one-point</u> <u>trivial</u> <u>extension</u>', which has to be iterated to arrive at desired effects seen later. The one-point trivial extension is explained on pages (4.2) — (9). I will repeat the definition only partially.

Let P be a (small) category, and

Σ any sieve in P, i.e., Σ is a set of objects in P such that if $x \in \Sigma$ and $f: y \to x$, then $y \in \Sigma$.

We define a new category $Q = P[\Sigma]$ that

(i) contains P as a full subcategory;

(ii) Q contains exactly one object, denoted $[\Sigma]$, not in P;

and

(iii)
$$\hom_Q([\Sigma],[\Sigma]) = \{id_{[\Sigma]}\}$$
$$\hom_Q([\Sigma], x) = \emptyset \qquad (x \in P)$$
$$\hom_Q(x,[\Sigma]) = \begin{cases} \{[x]\}, \text{ a singleton, } & \text{if } x \in \Sigma \\ \emptyset & \text{if } x \notin \Sigma. \end{cases}$$

Given any diagram $A: P \to \mathbb{X}$, we define $B: Q \to \mathbb{X}$ such that (iv) B restricted to the subcategory P is A;

(v) $B([\Sigma]) \underset{\text{def}}{=} (\text{any chosen}) \operatorname{colim} A \upharpoonright \Sigma$

here, Σ meaning the full subcategory of P

and

(vi) $B([x])_{x\in\Sigma}$: colimit cocone:

$$\begin{cases} [x] : x \to [\Sigma] \text{ in } Q, \text{ for } x \in \Sigma; \\ B([x]) : A_x \longrightarrow \operatorname{colim} A \restriction \Sigma. \end{cases}$$

We $\boxed{\text{prove}}$ that B is a trivial extension of A via the inclusion $P \longrightarrow Q$.

2.3 We immediately note the following, when we apply the above to 'good' diagrams. Suppose

(i) $A : P \to \mathbb{X}$ is a good diagram — even κ-good. (every good diagram is κ-good for a large enough κ ...), and suppose that (ii) the subset $\Sigma \subset P$, assumed to be closed downward ($y \le x$ & $x \in \Sigma \Rightarrow y \in \Sigma$) (a sieve) has no maximum element, and (iii) $\#\Sigma < \kappa$.

Then: the one-point trivial extension $B : Q \to X$ constructed in 2.2 is again (iv) κ-good: Q is an 'end-extension' of P: $x \in P$, $y \in Q$ & $y \le x \Rightarrow y \in P$ which ensures both the well-foundedness of Q (with $\bot$ bottom in Q)

and the κ-smallness for Q as far as elements of P are concerned ($x_0 > x_1 > \cdots > x_n > \cdots$ for $x_0 \in P$, being entirely in P, will break off; and $\{y : y \leq x\}$ taken in P and in Q are the same sets if $x \in P$) — and, for the only additional element $[\Sigma]$ of Q, a descending chain

$$[\Sigma] > x_1 > x_2 > \cdots > x_n > \cdots$$

if infinite, will give an infinite one:

$$x_1 > x_2 > \cdots x_n > \cdots$$

in P, — and $\{y : y < [\Sigma]\} = \Sigma$ (!), hence by assumption (iii), the κ-goodness condition at $[\Sigma]$ also holds. Moreover: since we assumed that Σ has no maximum (top) element, $\boxed{[\Sigma] \text{ is a limit point in } Q.}$ Therefore, in the diagram $B : Q \to \mathbb{X}$ there are no new links compared to $A : Q \to \mathbb{X}$. Also, as I should have said earlier, by point (v) stating on p A 17, stating that $B([\Sigma])$ is a colimit of the diagram $A \upharpoonright \Sigma$, we see — importantly — that the "continuity condition

$$B([\Sigma]) = \operatorname{colim} B \upharpoonright \{x \in Q : x < [\Sigma]\}$$

with the appropriate colimit cocone – holds,

since $\{x \in Q : x < [\Sigma]\} = \Sigma$ and

$B \upharpoonright \Sigma = A \upharpoonright \Sigma$.

2.4

$\boxed{\text{Proposition}}$ Let $A : P \longrightarrow \mathbb{X}$ be a

κ-good diagram. Then there exists a

(i) trivial extension $B : Q \longrightarrow \mathbb{X}$ of A

via an

(ii) initial inclusion $\Phi : P \longrightarrow Q$

(Φ is an inclusion, and $x \le y$, $y \in P \Rightarrow x \in P$)

such that

(iii) $B : Q \longrightarrow \mathbb{X}$ is (still) κ-good;

(iv) $\boxed{Q \text{ is } \kappa\text{-directed}}$; and

(v) all points in $Q - P$ are limit points

hence, as a consequence, all links in B are links in A – no new links in B.

The proof is a – rather straightforward – transfinite iteration of the one-point extension applied to good diagrams. The proof is given in some detail in the notes cited on the starting page A $\boxed{15}$ of this instalment of the notes: $\boxed{\text{Trivial Extension Version 2 2012 Nov 03.pdf}}$. Section 3 (p. 9, computer page 11) is about directed colimits of trivial extensions in general: they are again trivial. This proof is entirely straightforward – but uses, again, directed colimits in Cat, used and emphasised in the notes $\boxed{\text{2015 March 23 Coherent Completeness Part2.PDF}}$, section 7., p. 49.1 / computer page 23.

In the notes $\boxed{\text{Trivial Extension ...}}$, the statement of (2.4) Proposition in section 4., p 12 / computer page 14 (prev. page) is $\boxed{\text{incomplete}}$ because, stupidly, condition (iv), $\boxed{Q \text{ is } \kappa\text{-directed}}$, is omitted – but the reader will see that that the proof is doing precisely that: without this condition, we can take $Q = P$!

There is a version 1 of the 'trivial' notes:

$\boxed{\text{2012 Oct 25 Trivi Extensions.pdf}}$ in which

said condition on Q is not omitted —

see item 2) page 12 / comp page 14.

Our Proposition, 2.4, is used in the

proof of the 'κ-good small object argument'

stated on pages A 13, A 14 of the previous

instalment of the present notes; the use of the

Proposition appears on page 8 of $\boxed{\text{Small Object} \dots}$.

When I pass from A there to A^+ there;

A^+ there is the B of our present Proposition.