5. The universal property of the slice

As before, $C$ is a coherent category, $X$ a fixed object of $C$; we consider $\Delta = \Delta_X : C \to C/X$ discussed above.

To abbreviate, let's write $1_D$ for $C/X$, and use $\overline{f}$ for the effect of $\Delta$: $\overline{Y} = \overline{\Delta(Y)}$; for $Y \to Z$, $\overline{f} = \overline{\Delta(f)}$.

The first and main observation is that the object $X$ has a global element $1_D \xrightarrow{c_X} X$ in $1_D$. As we know, $1_D$ can be chosen as $(X, 1_X)$; we have

\[ \begin{array}{ccc}
X & \xrightarrow{\langle 1_X, 1_X \rangle} & X \times X \\
\downarrow & \Theta & \downarrow \pi_1 \\
X & \xrightarrow{1_X} & X
\end{array} \]

Thus, $c_X = \Delta_X : (X, 1_X) \to (X \times X, \pi_1)$

usual notation
for $\langle 1_X, 1_X \rangle$
$c_X: 1_B \rightarrow X$ is a generic element of $X$ in the sense that I will now explain — although we do not use this fact explicitly in our further work.

Using the generic element $c_X$, and the objects and arrows of $C/X$ that come from $C$ via $F$, we can generate all of the structure of $C/X$. Let $(Y, y)$ be an arbitrary commutative object of $C/X$, and consider the following diagram in $C$:

```
Y \rightarrow Y \times X
|       | \downarrow y
|       | \downarrow y
Y \times X \leftarrow X \leftarrow X \times X
| \downarrow y \times X
X \rightarrow X \times X
```

The outside square in a pullback — the checking of this is omitted (not really; see pages 43.1 and following!).
This means that we have the following pullback diagram in \( C/X \):

\[
\begin{array}{ccc}
(Y, y) & \xrightarrow{\langle 1_Y, y \rangle} & Y \\
\downarrow y & & \downarrow y \\
1_D & \xrightarrow{c_X} & X
\end{array}
\]

Now, assume that we have an arbitrary coherent functor \( F : C \to \mathcal{E} \), and a global element \( 1_\mathcal{E} \xrightarrow{e} F(X) \) of \( F(X) \).

I claim that we can extend \( F \) to \( G : C/X \to \mathcal{E} \) such that \( G \) is coherent,

\[
G \circ \Phi = F
\]

\[
\begin{array}{ccc}
C & \xrightarrow{\Phi} & C/X \\
\downarrow \circ & & \downarrow G \\
F & \xrightarrow{\circ} & \mathcal{E}
\end{array}
\]
by a natural isomorphism
\[ \mu: G \Xi \rightarrow F \]
and also \( G \) maps \( e: 1_\mathcal{C} \rightarrow \mathcal{C}(X) \) modulo the isomorphism \( \mu \), that is:
\[
\begin{array}{ccc}
G(1) & \xrightarrow{G(e)} & G\Xi(X) \\
\downarrow & & \downarrow \\
1_\mathcal{C} & \xrightarrow{e} & \mathcal{C}(X)
\end{array}
\]
Commutes.
The pullback in \( \mathcal{C}/X \) on the previous page will have to be mapped to a pullback
\[
\begin{array}{ccc}
G(Y, y) & \xrightarrow{e} & G\Xi(Y) \\
\downarrow & & \downarrow \\
G(1) & \xrightarrow{G(e)} & G\Xi(X)
\end{array}
\]
which, combined with isomorphisms:

\[
G((Y,y)) \longrightarrow G\Xi(Y) \xrightarrow{\lambda_{Y}} F(Y)
\]

\[
! \downarrow \\
G\Xi(Y) \xrightarrow{=} \downarrow F(y)
\]

\[
G(1_{\mathcal{D}}) \longrightarrow G\Xi(X) \xrightarrow{\lambda_{X}} F(X)
\]

\[
! \downarrow \circlearrowright \\ \circlearrowleft \downarrow \lambda_{X}
\]

\[
1_{E} \longrightarrow F(X)
\]

gives rise to pullback in \(IE:\)

\[
G((Y,y)) \xrightarrow{!} F(Y)
\]

\[
! \downarrow \\
\sqrt{F(y)} \xrightarrow{(*)} F(X)
\]

\[
1_{E} \longrightarrow e \longrightarrow F(X)
\]

This means that \(G((Y,y))\) is determined, at least up an isomorphism, by the requirements on \(G\).

To actually define \(G\), we pick, for any object
\((Y, y) \) of \( C/X \), a pullback diagram \((*)\).

Thus, not only the object \( G((Y, y)) \), but also the morphism \( G((Y, y)) \to F(Y) \). It is seen that the action of \( G \) on morphisms \((Y, y) \to F((Z, z)) \) will be uniquely determined so that the diagrams

\[
\begin{align*}
G((Y, y)) & \to F(Y) \\
G(f) \downarrow & \cong \downarrow F(f) \\
G((Z, z)) & \to F(Z)
\end{align*}
\]

all commute.

Let me show why \( G \) preserves e.e. morphisms.

Let \( f : (Y, y) \to (Z, z) \) be e.e. in \( C/X \).

The reason why \( G(f) \) is e.e. is that \((*)\) is actually a pullback, and, by assumption on \( F \), \( F(f) \) is e.e., and \( \text{IE} \) is coherent. The fact that \((*)\) is a pullback comes from the fact that the two squares on the left in what follows compose to the quadruple on the right:
\[
\begin{align*}
G((y_1, y)) & \longrightarrow F(y) \\
G(y) & \downarrow \quad 1 \quad \downarrow F(y) \\
G((z, z)) & \longrightarrow F(z) \\
1_E & \longrightarrow F(X) \\
\end{align*}
\]

(since \( Y \xrightarrow{f} Z \), and there is just one \( Y \xrightarrow{g} Z \))

and we have the general (and widely used rule) that if in a situation like the above, with knowing that 1 commutes, and 2 and 3 are pullbacks, then so is 4.

I will not pursue the generality of \( e_X : 1 \xrightarrow{c/X} \mathfrak{E}(X) \) any more, since in the completeness proof this fact is not used.
6. Conservativeness

6.1 Let $C$, $D$ be categories, $M : C \rightarrow D$ a functor, $A$ a class of arrows in $C$.
Let us say that $M$ is conservative for $A$ if for all $f \in A$, $M(f)$ is an isomorphism in $D$
(iff and only if $f$ is an isomorphism in $C$).

Proposition: Assume $C$ has finite limits, and $M$ preserves finite limits. Assume $M$ is conservative for $A$ (the class of) monomorphisms in $C$. Then $M$ is conservative for $A$ (the class of) all arrows in $C$.

Proof: Consider an arrow $A \xrightarrow{f} B$ in $C$.

Consider:

\[ \xymatrix{ A \ar[r]^{m} & K \ar[r]^{b_{0}} & A \ar[r]^{f} & A \ar[r]^{f} & B \ar@{>}[u]_{b_{1}} \ar@/^2pc/[ll]^{1_{A}} \ar@/_2pc/[ll]_{f \cdot 1_{A}} } \]

Here, $(K, b_{0}, b_{1})$ is the so-called kernel-pair of $f$: 
the pullback of $f$ against itself. The morphism $m : A \to K$ is defined by using the universal property of $K$ such that $k_0m = k_1m = 1_A$. $m$ is a monomorphism:

If $m : B \xrightarrow{g} A$, and $mg = mh$, then $k_0mg = k_0mh$, hence $g = h$. Also, $m$ is an isomorphism, iff $f$ is a monomorphism. Namely, if $f$ is a monomorphism, then $k_0, k_1$ are isomorphisms (the pullback diagram is isomorphic to

$$
\begin{array}{ccc}
A & \xrightarrow{1_A} & A & \xrightarrow{f} & B \\
\downarrow & & \downarrow & \Rightarrow & \downarrow \\
A & \xrightarrow{f} & \overline{B} & \xrightarrow{} & B
\end{array}
$$

in this case), and so $m$ is an isomorphism. And if $m$ is an iso, so are $k_0$ and $k_1$, and $f$ is a mono.

Now, assume $M(f)$ is an isomorphism. Consider the image of $(*)$ under $M$ in $S$. It is the same construct from $M(f)$ in $S$ as $(*)$ was.
from \( f \) in \( C \). Therefore, \( M(m) \) is an isomorphism, since \( M(f) \) is, in particular, a monomorphism. But \( m \) is a mono, and \( M \) is conservative for monos; therefore, \( m \) is an isomorphism. But then, by what we said above, \( f \) is a monomorphism. Now, since \( M \) is conservative for monos, and \( M(f) \) is an isomorphism, it follows that \( f \) is an isomorphism.

I now generalize the concept a bit. Suppose \( M \) is a class of functors \( M : C \rightarrow S_M \) from the fixed category \( C \) to — possibly — variable categories \( S_M \). Then \( M \) is conservative for \( A \) if \( (A) \) or before

\[ \text{for all } f \in A, \quad M(f) \text{ is an isomorphism} \]
\[ \text{for all } M \in M \text{ iff } f \text{ is an isomorphism in } C. \]
The above proposition is generalized, with essentially unchanged proof, to the version obtained by replacing \( M \) by \( \hat{M} \) three times (first it should be: "every \( M \in M \) preserves finite limit"). This generalization is not really a generalization, since with given \( M \), I can consider the single

\[
\hat{M} : C \rightarrow \text{IT} S^M_{\hat{M}}
\]

and the "generalization" is reduced to the original version.

Looking at the "Gödel completeness theorem", p 10: by the present Proposition, it suffices to prove it for all monomorphisms \( f : A \rightarrow B \).

(6.2) **Lemma.** Let \( A \rightarrow^m B \) be a monomorphism in \( C \). Then \( \Phi_B = \Phi : C \rightarrow C / B \) is conservative for \( m \).
Proof. We use the 'generic element'

\[ C_B : 1 \rightarrow C/B \rightarrow \mathcal{O}(B) \]

- and, actually, this proof will be used, rather than the lemma itself when it comes to the crunch.

I start with the fact that in \( C \), for an arbitrary arrow \( m : A \rightarrow B \) in \( C \), the following in a pullback diagram - and for a change, I will verify it:

\[
\begin{array}{ccc}
A \times B & \xrightarrow{m \times B} & B \times B \\
\langle 1_A, m \rangle & \uparrow & \uparrow \Delta_B = \langle 1_B, 1_B \rangle \\
A & \xrightarrow{m} & B
\end{array}
\]

Suppose we have \( f : X \rightarrow A \times B \), \( g : X \rightarrow B \) such that

\[(m \times B)f = \Delta_B g : \]

is there \( h \) as required by \( f \) ?

\[
\begin{array}{ccc}
A \times B & \xrightarrow{m \times B} & B \times B \\
\langle 1_A, m \rangle & \uparrow & \uparrow \Delta_B \\
A & \xrightarrow{m} & B
\end{array}
\]

\[
\begin{array}{ccc}
x & \xrightarrow{?} & A \\
\theta \rightarrow & \uparrow & \Delta_B \\
& \uparrow \Theta & \\
g & \rightarrow & B
\end{array}
\]
The required commutativity $1 \otimes \eta$ when followed by the projection $\pi_0$ determines $h$:

$h \equiv h_0 = \pi_0 f$.

The assumed equality (box, previous page), when followed by the two projections $B \times B \xrightarrow{\pi_0, \pi'} B$ gives us:

\[
\begin{align*}
X \xrightarrow{f} B \\
\xrightarrow{\Delta_X} B \times B \\
\xrightarrow{\pi_0} B \\
\xrightarrow{f} B
\end{align*}
\]

\[
\begin{align*}
X \xrightarrow{g} B \\
\xrightarrow{\pi_0} B \\
\xrightarrow{f} B
\end{align*}
\]

\[
\begin{align*}
m \pi_0 f &= g \\
\pi_1 f &= g
\end{align*}
\]
We verify that \( h = \pi_0 f \) indeed satisfies

1 \( \Leftrightarrow \) and 2 \( \Leftrightarrow \):

\[
\langle l_A, m \rangle h = f \quad (1)
\]

\[
m h = g \quad (2)
\]

(1) is the equality of two arrows in the product \( A \times B \rightarrow B \); it is equivalent to the conjunction of the following two:

\[
\pi_0 \langle l_A, m \rangle h = \pi_0 f \quad (1.1)
\]

\[
\pi_1 \langle l_A, m \rangle h = \pi_1 f \quad (1.2)
\]

(1.1) was used to get \( h = \pi_0 f \). But \( \pi_1 \langle l_A, m \rangle = m \)

Thus, (1.2) is reduced to

\[
m h = \pi_1 f \quad (1.2')
\]

With \( h = \pi_0 f \), (2) is \( m \pi_0 f = g \) — which is true (see previous page), and (1.2') is \( m \pi_0 f = \pi_1 f \) — and this holds, since both sides equal \( g \) (see previous page).
Next, we can complete the diagram to something taking place in \( C/B \):

\[
\begin{array}{ccc}
\Lambda \times \Delta & \xrightarrow{m \times \Delta} & \Delta \\
\downarrow & & \downarrow \\
\langle 1_A, \Delta \rangle & \xrightarrow{m} & \Delta \\
\uparrow & & \uparrow \\
\Lambda & \overset{m}{\rightarrow} & B
\end{array}
\]

since all four triangles commute.

Therefore, we have the following pullback in \( C/B \):

\[
\Phi(A) \quad \Phi(m) \quad \Phi(B)
\]

\[
\begin{array}{ccc}
\langle 1_A, m \rangle & \xrightarrow{p} & \Delta \\
\uparrow & & \uparrow \\
(A/m) & \xrightarrow{m} & 1_{C/B}
\end{array}
\]  

\((*)\)

This is the pullback in \( C/B \) that we are going to use again. However, the assertion of 6.2 lemma is clear:

if \( \Phi(m) \) is an isomorphism, so is its pullback \( m: (A/m) \rightarrow 1_{C/B} \) in \( C/B \) - and so is \( m: A \rightarrow B \) in \( C \).
I must put here the conservativeness argument that uses the last-mentioned pullback.

Proposition Suppose: $C$ is coherent,

$$m : A \rightarrow B$$

is a mono in $C$;

and we have

$$\Sigma : C/B \rightarrow S,$$

a coherent functor that is conservative for the following subobject of the terminal object of $C/B$:

$$\left\{ \begin{array}{c}
(A, m) \xrightarrow{m} 1_{C/B} = (B, 1_B) \\
A \xrightarrow{m} B \\
\Theta \\
m \downarrow \Theta \downarrow 1_B \\
B
\end{array} \right\}.$$

Then the composite $\Sigma \sigma_B : C \rightarrow S$ is conservative for $m : A \rightarrow B$ in $C$:

$$\left( C \xrightarrow{\sigma_B} C/B \xrightarrow{\Sigma} S \right).$$

Proof. Suppose $\Sigma \sigma_B (m)$ is an isomorphism in $S$.

To derive that $m : A \rightarrow B$ is an iso in $C$. 
Consider the pullback $(\Sigma)$ on $p_{\Sigma}(44)$, in $C/B$, and apply the coherent functor $\Sigma$ to it, obtaining the pullback:

$$\Sigma \Phi(A) \xrightarrow{\cong} \Sigma \Phi(B)$$

$$\Sigma(\langle l_A, m \rangle) \xrightarrow{\cong} \Sigma(A, m) \xrightarrow{\Sigma(m)} \Sigma(\delta_{C/B})$$

in the category $\mathcal{S}_\Sigma$.

Since the pullback of an isomorphism is an isomorphism,

$$\Sigma(m) : \Sigma(A, m) \rightarrow \Sigma(\delta_{C/B})$$

is an isomorphism.

We have assumed that $\Sigma$ is conservative for $m : (A, m) \rightarrow 1_{C/B}$.

Therefore, $m : (A, m) \rightarrow 1_{C/B} = (B, 1_B)$ is an isomorphism in $C/B$. Applying the forgetful functor

$$\Psi_B : C/B \rightarrow C$$

thus, we get that $m : A \rightarrow B$ is an isomorphism in $C$, as desired.
Lemma. Suppose $X$ has global support $\not\in X \to 1$ in e.e. (in $C$). Then $\pi_X : C \to C/X$ is conservative (for all arrows in $C$; enough for mono's in $C$).

Proof: $\pi_X$ is conservative. Let $m : A \to B$ be any mono in $C$. Having an inverse $g$ of $\pi_X(m)$ means

$$
\begin{array}{c}
A \times X & \xleftarrow{g} & B \times X \\
\downarrow m \times X & \mbox{and} & \downarrow \pi_1' \\
B & \rightarrow & \pi_1 \\
\end{array}
$$

$$(m \times X) \circ g = 1_{B \times X}, \quad \pi_1' \circ g = \pi_1. \quad \text{Let } f = \pi_0 \circ g.$$
for $\pi_0 : A \times X \rightarrow A$. We have

\begin{align*}
A \times X & \xrightarrow{m} B \\
B \times X & \xrightarrow{1_{B \times X}} B \times X \\
B & \xrightarrow{\pi_0} B
\end{align*}

and

\begin{align*}
B \times X & \xrightarrow{\Theta} A \\
B & \xrightarrow{\pi_0} B
\end{align*}

\[(\ast)\]

Now,

\begin{align*}
B \times X & \rightarrow X \\
\pi_0 & \downarrow B \\
\downarrow & \\
\downarrow & !B
\end{align*}

is a pullback. Therefore, since $!X$ is assumed to be e.i., so is $\pi_0$. From $(\ast)$, $m$ must be an isomorphism.
Lemma. Suppose $U$ and $V$ are subobjects of $1 = \mathcal{C}$ and $A \xrightarrow{m} B$ a monomorphism. Suppose $U \vee V = \mathcal{V}$. If $\Pi_U(m)$ and $\Pi_V(m)$ are both isomorphisms in $\mathcal{C}/U$, resp., in $\mathcal{C}/V$, then $m$ is an isomorphism.

Proof. Of course, the monomorphism $U \to 1$ is the unique $!_U : U \to 1$; the assumption is that $!_U$ is a monomorphism; the same for $!_V : V \to 1$.

Then $A \times U \xrightarrow{\Pi_U} A$ is a monomorphism, since

$$
\begin{array}{ccc}
A \times U & \xrightarrow{\Pi_U} & A \\
\downarrow & \cong & \downarrow \\
A \times !_U & \xrightarrow{\pi_0} & A \times 1
\end{array}
$$

Similarly for other projections that appear below.

Consider:

$$
\begin{array}{ccc}
A \times U & \xrightarrow{m \times U} & B \times U & \xrightarrow{\pi_U} & U \\
\downarrow & \pi_0 & \downarrow & \downarrow & \downarrow \\
A & \xrightarrow{m} & B & \xrightarrow{!_B} & 1 \\
\downarrow & \pi_0 & \downarrow & \downarrow & \downarrow \\
A \times V & \xrightarrow{m \times V} & B \times V & \xrightarrow{\pi_V} & V
\end{array}
$$
All of the four quadrangles are pull-backs
\[ \mathcal{F}_U(m) = m \times U; \quad \mathcal{F}_V(m) = m \times V; \] they are both isomorphisms. Hence the monomorphisms

\[ A \times U \xrightarrow{m \times U} B \times U \xrightarrow{\mathcal{F}_U} B, \quad \text{and} \quad B \times U \xrightarrow{\mathcal{F}_V} B \]

determine the same subobject of \( B \); call it \( \hat{U} \in \text{Sub}(B) \).

Similarly for \( \hat{V} = \mathcal{F}_V(m \times V) = \mathcal{F}_B(m \times V, \pi_0) \in \text{Sub}(B) \).

Now, \( \hat{U} = \mathcal{F}_B(m \times U, \pi_0) \in \text{Sub}(B) \) equals

\[ \hat{U} = (\pi_B^*)^* U \]

for the map \( !_B : B \rightarrow 1 \); similarly

\[ \hat{V} = (\pi_B^*)^* V. \]

Then \( \hat{U} \cup \hat{V} = (\pi_B^*)^* U \cup (\pi_B^*)^* V = \]

\[ \sup \text{ in } \text{Sub}(B) \]

\[ = (\pi_B^*)^* (U \cup V) = (\pi_B^*)^* (\mathcal{T}_A) \]

\[ = \mathcal{T}_B \]

But the subobject \( \hat{A} = [A, m] \) has the property

that \( \hat{A}, \hat{V} \in \hat{A} \), because \( \hat{U} = [A \times U, m \pi_0], \hat{V} = [A \times V, m \pi_0] \).
Therefore, \( \hat{U} \vee \bar{U} \leq \hat{A} \), i.e.,
\[
\overline{\bar{B}} \leq \hat{A}, \quad \overline{\bar{B}} = \bar{A}
\]
which means that \( m \) is an isomorphism.

\[\text{(6.5) Lemma.} \quad \Gamma = \mathcal{C}(1, -) : \mathcal{C} \to \text{Set} \]

is conservative for all monads \( U \xrightarrow{m} 1 \)

into \( 1 \).

\[\text{proof:}\] This is immediate: \( \Gamma(m) \) being an isomorphism means, at least, that there is an arrow \( \Gamma(1) \to \Gamma(U) \) in \( \text{Set} \). Since \( \Gamma(1) \cong 1_{\text{Set}} \cong \forall \times 1 \), this means that there is an element of \( \Gamma(U) \), i.e., an arrow \( 1_C \to U \) in \( \mathcal{C} \). We know (see middle of p.15, e.g.) that this means that \( m \) is an isomorphism.
The proof of Gödel completeness — our goal — requires the repeated application of the slice construction discussed above, and the assembling of infinitely many coherent categories and coherent functors into one final one, one which is complete and saturated in the sense given in section 3. The assembling is done, formally, in the way of a directed colimit of coherent categories and functors. The directed colimit is constructed entirely in the category of sets; e.g., the set of objects of the colimit is the colimit of the object sets Lab of other ingredient categories; the same for the set of arrows. The success of the operation (mainly, but not exclusively, the fact that the resulting category is in fact coherent) depends on a property of directed colimits in \( \text{Set} \), the category of sets, which can be stated by saying that directed
collimits commute with finite limits in \( \text{Set} \).
This is a very fundamental fact that has
far-reaching application in category theory,
and, in category theory after Saunders
Mac Lane's "bible", the "Categories for the
working mathematician" (1971). Said fundamental
fact is duly proved towards the end of the book
(Chapter IX: Special Limits, section 2; interchange of
Limits, Theorem 1, p. 211), so that one can
easily imagine a "second volume" of the
book, starting with the Gabriel-Ulmer theory
of locally presentable categories, in which said
fundamental fact is at the heart of the matter.

and going on to accessible categories if you like.
If you visit the spot in Mac Lane's book, you'll
find, I think, that the half-page proof is a
bit breezy - understandable, since the theorem
is not put to any serious use yet in the book.
I will studiously avoid using the theorem; instead,
I show the details of the construction of the
directed colimit, and some details of the proof of
Mac Lane's theorem details that I actually use. The Diligent Reader can then go to the book and acquire a better understanding of the Theorem (p.211), better than just by running through Mac Lane's words (unless the D.R. is a different intelligence from mine).

Let \((I, \leq)\) be a non-empty partially ordered set \((\leq\) is reflexive, transitive and anti-symmetric) which is also directed: for all \(i, j \in I\), there is \(k \in I\) such that \(i \leq k\) and \(j \leq k\):

\[
\begin{array}{ccc}
  & k & \\
\downarrow & \swarrow & \searrow \\
  i & & j
\end{array}
\]

I use the arrow notation \((i \to k\) for \(i \leq k\) since, of course, \((I, \leq)\) is in fact a (rather special) category.

Let \(\mathcal{A}\) be a category: think, first, if \(\mathcal{A} = \text{Set}\), but later others, e.g., \(\mathcal{A} = \text{Cat}\) or \(\mathcal{A} = \text{coh}\).
Let $\mathcal{A} : \mathcal{I} \rightarrow \mathcal{A}$ be a functor: or, as

$I = (I, \leq)$

we say a diagram; $\mathcal{A}$ consists of objects $\mathcal{A}(i) = A_i$ of $\mathcal{A}$, and arrows $\mathcal{A}(i \leq j) = a_{ij} : A_i \rightarrow A_j$ such that $a_{ii} : A_i \rightarrow A_i$ in $a_{ii} = 1_{A_i}$, and for $i \leq j \leq k$,

\[
\begin{array}{cccc}
A_i & \xrightarrow{a_{ij}} & A_j & \xrightarrow{a_{jk}} & A_k \\
\downarrow{a_{ik}} & & & & \downarrow{a_{ik}} \\
\end{array}
\]

commutes: $a_{ik} = a_{jk} \circ a_{ij}$.

Example: $I = \mathbb{N}$ = the set of natural numbers;
\(\leq\): usual ordering; \(\mathcal{A} = \text{Set}\);

$A_i$: set for each $i \in \mathbb{N}$

and let $a_{ij} : A_i \rightarrow A_j$ be on inclusion; $A_i \subseteq A_j$, and $a_{ij}$ maps $x \in A_i$ to itself in $A_j$.

Then, yes, we have a diagram $\mathcal{A} : \mathcal{I} \rightarrow \text{Set}$.

Now, it is very natural to consider

$\bigcup_{i \in \mathbb{N}} A_i$. 
the union of the sets, as a kind of "limit" (we will say, "colimit") of the diagram \(<A_i>_{i \in \mathbb{N}}\). This is an example for the directed colimit, we are advocating here. Note that in this example we have the further inclusions
\[ q_{i \to j} : A_i \to \bigcup_{j' \in \mathbb{N}} A_{j'} \]
and \( j \) even,
\[ A_i \quad \xrightarrow{q_{i \to j}} \quad A_j \]
\[ \Theta \]
\[ q_{i \to j} \downarrow \quad \downarrow q_{j' \to j} \]
\[ \bigcup_{j' \in \mathbb{N}} A_{j'} \]

commuting — as of course all diagrams consisting of inclusions will commute — if they are of the right shape. (always non-empty!)

Returning to a general directed poset \((I, \leq) = \mathbb{N}\) and a general \(I\)-based diagram \(A : I \to \text{Set}\) in the category of sets (\(\mathbb{A} = \text{Set}\)), now we construct a colimit.
\[ A_w = \text{colim} \ A_i \rightarrow \text{a set}, \text{ together with} \]

maps \( A_i \rightarrow A_w \) \text{ such that} \]

\[ a_{i,j}^{-1} \]

\[ a_{i,j} \]

\[ \Xi \]

\[ a_{i,m} \]

\[ a_{j,w} \]

\text{commutes.} \text{ We will do this construction for an arbitrary poset} (I, \leq), \text{ without assuming directedness, or even for an arbitrary category (small) } I \text{ and diagram (functor) } A : I \rightarrow \text{Set} \]

\text{but that would not be useful now, since the special features of this construction for the directed case are crucial for the applications. The colimit will have a universal property which determines it up to isomorphism — and which will also be very useful, although not enough; the universal property by itself will not suffice for our goals, the actual construction (or, if you wish, Mac Lane's Theorem 1 (p.210))} \]
is necessary to keep in mind.

Here is the construction.

For each $i \in I$ and $x \in A_i$, we will have an element $a_{i,x}$ in $A_{i,x}$; thus, we will have a morphism $(i,x) \rightarrow a_{i,x}(x)$; moreover, if

$$A_i \xrightarrow{a_{ij}} A_j$$

then because of the required commutativity $(\ast)$, p. 54, I will have to have, for $y = a_{ij}(x)$

$$(i,x) \xrightarrow{a_{ij}} \xrightarrow{a_{ij}} \xrightarrow{\text{some}} (j,y) \xrightarrow{a_{jy}}$$

i.e., $(i,x)$ and $(j,y)$ will have to be identified in $A_{i,x}$.

However, we cannot recognize the necessity of this kind of identification more generally: for any $i$ and $j$, there is at least one $k$ with $i \leq k$, $j \leq k$, so that $(i,x)$ and $(i,j)$ should be eventually identified if we see that $a_{ik}(x) = a_{jk}(x) = (\ast)$ in $A_{i,k}$.
Thus, if \((\ast)\), then \(a_{j \cdot i \cdot a}(x) = a_{j \cdot i \cdot a}(y)\).

I make this into a definition. Let \(B\) be the disjoint union of the sets \(A_i:\)

\[
B = \bigsqcup_{i \in I} A_i = \{(i, x) : i \in I, x \in A_i\}
\]

and define the binary relation \(R\) on \(B\) by:

\[
s_1 R s_2 \iff \exists \delta \exists i \exists j \exists k \exists x \exists y : \begin{align*}
    s_1 &= (i, x) \quad & s_2 &= (j, y) \\
    i &\leq k \quad & j &\leq k
\end{align*}
\]

and there is \((i, x)\) such that

\[
a_{j \cdot i \cdot a}(x) = a_{i \cdot j \cdot a}(y) \quad (\in A_k)
\]

CLAIM: \(R\) is an equivalence relation on \(B\).

Only transitivity is at issue.
Let:

\[(i, x) \ R (j, y) \ R (k, z) \quad (*)\]

We have: \(l, m, n \in I\) such that

\[\begin{array}{ccc}
i & \rightarrow & l \\
j & \rightarrow & m & \rightarrow & n \\
l & \rightarrow & m & \rightarrow & n \\
k & \rightarrow & l \\
\end{array}\]

in \(I\). (Remember: \(i \rightarrow l\) means \(i \leq l\));

and, the choice of \(l\) and \(m\) : by the assumption \((*)\).

So that

\[a_{il}(x) = a_{jl}(y) = u,\]

\[a_{jm}(y) = a_{km}(z) = v,\]

Let: \(w = a_{jn}(y)\). The "diagram" \((***)\) when the functor \(A\) is applied to it, becomes commutative (since, of course \((***)\) commutes in \(I\)!)
and that means that
\[ a_{ij} (x) = a_{ij} (z) = w \]

— and this exhaust the requirement for the relationship \((i, x) R (k, z)\) — as desired.

I now define
\[ A_\infty = \colim_{i \in I} A_i = \colim_{i \in I} A_i \]

— abbreviated notation!

\[ \overset{\text{def}}{=} \mathcal{B} / R = \text{the set of equivalence classes of } \mathcal{B} R. \]

The equivalence class containing \((i, x)\) is written \([i, x]\). We put
\[
\begin{align*}
A_i & \xrightarrow{a_{ij}} A_j \\
x & \mapsto [i, x]
\end{align*}
\]

Then
\[
\begin{array}{ccc}
A_i & \xrightarrow{a_{ij}} & A_j \\
\downarrow{a_{i\infty}} & & \downarrow{a_{j\infty}} \\
A_\infty & \Theta & A_\infty
\end{array}
\]

\((i \leq j)\)
since, for \( x \in A_i \) and \( y = \alpha_{ij}(x) \), we have
\[
(i', x) R (j, y) \quad \text("by the choice } k = j \text{")}, \text{ thus}
\[
[i', x] = [j, y].
\]
Next, we formulate and verify the universal property of our construction — but for the formulation, we go to a general context, one that will have further special cases that are useful for us.

7.1 **Definition**

Let \( \mathcal{A} \) be a category, \( \mathcal{I} \) another category and \( \mathcal{A} : \mathcal{I} \to \mathcal{A} \) a functor (although we will call it a ‘diagram’).

\( \mathcal{A} \) is a cone on \( \mathcal{A} \) is given by an object \( B \) of \( \mathcal{A} \), and a natural transformation
\[
\begin{array}{ccc}
A & \xrightarrow{\gamma} & B^7 \\
\end{array}
\]

\( \gamma \).

i.e.
\[
\begin{array}{ccc}
I & \xrightarrow{\gamma} & \mathcal{A} \\
\downarrow \gamma & & \downarrow \gamma \\
B^7 & \to & B^7
\end{array}
\]

where \( B^7 : I \to \mathcal{A} \) is the constant functor.
with value $B$:

$$\text{B}^7(i) = B \quad \text{for all } i \in \text{Ob}(\mathbb{I})$$

$$\text{B}^7(i \downarrow x) = \text{id}_B \quad \text{for all } x \in \text{Arr}(\mathbb{I})$$

This means, in elementary terms, and using our previous notation, restricting ourselves to the special case of a poset $\mathbb{I} = (I, \leq)$, that:

$$\psi = \langle a_{ij} : A_i \to B \rangle_{i \in I}$$

such that (naturally!)

$$A_i \xrightarrow{a_{ij}} A_j$$

$$\psi_i \uparrow \Theta \uparrow \psi_j$$

$$B \xrightarrow{\text{id}_B} B$$

commutes.
(ii) The cocone $\Delta \rightarrow A^7$ is universal — and $A$ is a colimit of the diagram $\Delta$ if the following holds: for an arbitrary cocone $\Delta \rightarrow B^7$, there is a unique arrow $f : A \rightarrow B$ in $\Delta$ such that

$$
\begin{array}{ccc}
\Delta & \rightarrow & A^7 \\
\downarrow & \downarrow & \downarrow \\
\Delta & \rightarrow & B^7
\end{array}
$$

commutes (where $f^7 : A^7 \rightarrow B^7$ is the obvious natural transformation, all whose components are equal to $f : A \rightarrow B$).

In practice, the above means that there is a unique $f : A \rightarrow B$

$$
\begin{array}{ccc}
A_i & \rightarrow & A (-A_{\infty}) \\
\downarrow & \downarrow & \downarrow \\
\varphi_i & \rightarrow & B
\end{array}
$$

such that:

$$
\begin{array}{ccc}
A_i & \rightarrow & A_{\infty} \\
\downarrow & \downarrow & \downarrow \\
\varphi_i & \rightarrow & B
\end{array}
$$

commutes for all $i \in I$. 
Let us check that in our construction in Set above, we in fact have that $A_\infty = \varprojlim A_i$

$= \text{colim } A_i$, with the universal cocone

$y = \langle a_{i,i} : A_i \to A_\infty \rangle_{i \in I}$ (we also refer to $y_i : A_i \to A_\infty$ as $\text{(colimit) coprojection}$).

We need to define $f : A_\infty \to B$, i.e.,

$f([i,x])$ for all $i \in I$, $x \in A_i$. Obvious choice:

$$f([i,x]) = y_i(x) \quad (*)$$

dej

Is this well-defined? If $[i,x] = [j,y]$, we have $i \leq k$, $j \leq k$ and $a_{ik}(x) = a_{jk}(y) = z$.

But, by the naturality of $y : A \to B$, we have

$$\begin{array}{ccc}
A_k & \xrightarrow{a_{ik}} & A_i & \xrightarrow{y_i} & B \\
\downarrow{y_k} & & \downarrow{y_k} & & \\
A_j & \xleftarrow{a_{jk}} & A_j & \xrightarrow{y_j} & B
\end{array}$$
result, showing

\[ \varphi_i(x) = \varphi_{jk}(a_{ik}(x)) = \varphi_{jk}(z) \]

\[ \varphi_j(y) = \varphi_{jk}(a_{jk}(y)) = \varphi_{jk}(z) \]

Thus \( \varphi_i(x) = \varphi_j(y) \) — as desired.

Also, clearly, \( f \) must satisfy \( (\ast) \) — \( f \) is unique.

And, of course, \( (\ast) \), \( p61 \) holds true.

There is an "internal" characterization of the colimit cocone

\[ \gamma = \langle a_{ij} \rangle_{i \in I} : A \longrightarrow \vee_{i \in I} A_i \]

constructed on pages 55 and later:

1) "surjectivity": for any \( x \in \vee_{i \in I} A_i \), there are
   \( i \in I \) and \( u \in A_i \) such that \( x = a_{i\omega}(u) \);

2) "injectivity": whenever \( i, j \in I, u \in A_i, v \in A_j \), and \( a_{i\omega}(u) = a_{j\omega}(v) \), there \( i \leq k \) such that
   \( i \leq k \) and \( j \leq k \) and \( a_{ik}(u) = a_{jk}(v) \).
One can see that, in the proof of the universal property of \( 
\), all what we used was these facts about the construction.

We will prove — or, at least, sketch(?) a proof of — the fact that in the categories \( \mathcal{M} = \text{C}at \) (the category of small categories), and \( \mathcal{M} = \text{Coh} \) directed colimits exist — moreover that they are calculated "as in Set". The latter means that, for instance, the "object-set" of the colimit category is the "corresponding" colimit of the object-sets of the ingredient categories (as it already said above).

I state the theorem that we need: — no more and no less!
7.2 Theorem

Given a diagram

\[ C : \text{I} \rightarrow \text{Cat} \]

on a non-empty directed poset \( \text{I} = (I, \leq) \) (write \( C_i \) for \( C(i) \), \( i \in I \)), and \( F_{ij} : C_i \rightarrow C_j \) for \( C(i \leq j) \),

there is \( C_{\infty} \in \text{Cat} \) and a cocone

\[ \gamma = \langle F_{im} : i \in I \rangle : \downarrow I \rightarrow C_{\infty} \]

such that:

1) ("susceptibility"): for every \( X \in \text{Ob}(C_{i_0}) \)

there is \( i \) and \( X' \in \text{Ob}(C_i) \) such that \( C_{i_0}(X') = X_j \)

and similarly for arrows;

2) ("injectivity"): whenever \( i, j \in I \) and

\( X \in C_i \) and \( Y \in C_j \) and \( F_{i_0m}(X) = F_{j_0m}(Y) \), there is \( k \) such that \( i \leq k \) and \( j \leq k \) and

\( F_{ik}(X) = F_{jk}(Y) \) (\( \in C_k \)).

Observe that the theorem does not mention the notion "colimit"! However, in fact \( \gamma : \downarrow I \rightarrow C_{\infty} \)

is a colimit cocone — moreover, the properties described necessarily imply that this is so.
You will see that the proof of the theorem will make good use of the universal property of the colimits in \textit{Set}.

\begin{itemize}
\item \textbf{Corollary} Suppose, in addition to the data in the theorem, that each \( C_i \in \text{Coh} \), and each functor \( F_{ij} : C_i \to C_j \) is a coherent functor. Then \( C_0 \) is a coherent category, and each \( F_{iw} : C_i \to C_0 \) is a coherent functor.
\end{itemize}

The proof of the Corollary from the theorem is completely elementary, but somewhat tedious.

I postpone the proof of both the Theorem and Corollary to the Appendix. The following three little lemmas also belong to the same circle of ideas; they are also postponed.

\begin{itemize}
\item \textbf{Lemma (i)} Continuing the situation of the previous Corollary, let \( X \xrightarrow{!} 1 \) be an e.e. morphism in \( C_0 \) with codomain a terminal object of \( C_0 \). Then there are \( i \in I \) and an e.e. morphism \( !_{X'} : X' \to 1_{C_i} \) with terminal codomain in \( C_i \) such that \( F_{i,0} (!_{X'}) = !_X \).
\end{itemize}
(iii) Suppose $U \times V = T_{1 C_{i_0}}$. Then there is $i \in I$, $U', V' \in \text{Sub}(1_{C_{i_0}})$ such that $U' \xrightarrow{Fi_{i_0}} U$, $V' \xrightarrow{Fi_{i_0}} V$, and $U' \times V' = T_{1 C_{i}}$.

(iii) Let $i_0 \in I$, and $A \xrightarrow{f} B$ an arrow in $C_{i_0}$. Suppose that for all $i \geq i_0$, the functor $F_{i_0} : C_{i_0} \rightarrow C_i$ is conservative for $f$ ($F_{i_0}(f)$ iso $\Rightarrow f$ iso).

Then $C_{i_0\downarrow} : C_{i_0} \rightarrow C_{i_0}$ is also conservative for $f$. 
By $C$, I always mean a small coherent category.

8.1 Proposition. There exist: a small coherent category $C^*$, a coherent functor $\Sigma = \Sigma^*: C \to C^*$ with the following two properties:

1) $\Sigma$ is conservative (for all arrows in $C$);

2) for every $X \in \text{Ob}(C)$ with full support ($X \xrightarrow{id} C$, i.e.), the object $\Sigma(X)$ in $C^*$ has a global element.

8.2 Proposition. Given a monomorphism $A \xrightarrow{m} B$ in $C$,

there exist: a small coherent category $C^\# (= C^\#_{C,m})$ and a coherent functor $\Sigma = \Sigma^\#: C \to C^\#$

with the following two properties:

1) $\Sigma$ is conservative for the arrow $m$ in $C$;

2) for every pair $(U,V)$ of subobjects of $1_C$ in $C$ such that $U \cup V = \prod_{1_C}$, at least one

of the two subobjects $\Sigma(U)$, $\Sigma(V)$ of $\prod_{1_C^\#}$ is equal to the hop $\prod_{1_C^\#}$. 
Before saying anything about the proof of the propositions, I will use them to complete the proof of the completeness theorem (p. 10).

Let \( C \) be a small coherent category, \( m: A \rightarrow B \) a monomorphism in \( C \). We will prove:

there is \( M: C \rightarrow \text{Set} \)

\( M \) coherent, such that \( M \) is conservative for \( m \):

\( M(m) \) isomorphism \( \Rightarrow m \) is an isomorphism.

By (6.1) Proposition (p. 40), this will be sufficient.

Let \( C_0 = C/_{\text{is}} \), and \( \Phi = \Phi_{\text{is}}: C \rightarrow C_0 \).

Let \( \tilde{m} = \Phi(m) = m \times B : (A \times B, \pi_1) \rightarrow (B \times B, \pi_1) \).
Recursively, we define the coherent category $C_n$ for all $n \in \mathbb{N}$, together with the coherent functors $F_{k,n} : C_k \to C_n$ for $k \leq n$.

$(F_{n,0} \equiv \text{id}_{C_n})$ satisfying the following: (i) & (ii):

(i) $(F_{k,n})_{k \leq n}$ is compatible (a functor $\mathbb{N} \to \text{Coh}$):

$\begin{array}{ccc}
C_k & \xrightarrow{F_{k,n}} & C_n \\
\downarrow & \downarrow & \downarrow \\
F_{k,n} & \to & C_n
\end{array}$

$F_{k,n} = F_{k,n} \circ F_{k,n}$ ($k \leq n$)

(ii) $F_{0,n} : C_0 \to C_n$ is conservative for $n$.

$C_0$ has been defined.

Suppose $n \in \mathbb{N}$, and the data for $k \leq n$ have all been defined. We need to define $C_{n+1}$ and the functors $F_{k,n+1} : C_k \to C_{n+1}$ ($k \leq n+1$).

Starting with $C_n$, we apply the two "consecutive" "$C \to C^k$" and "$C \to C_{(m)}^k$" of the last two propositions, first the first, second the second, to arrive at $C_{n+1}$.
Write \( \hat{m}_n \overset{\text{def}}{=} F_{n_0} (m) \), a mono in \( C_n \).

Applying 8.1, we have

\[
\Sigma \hat{m}_n : C_n \rightarrow C_n^* ;
\]

applying 8.2, we have

\[
\Sigma \hat{m}_n : C_n^* \rightarrow (C_n^*)^# ;
\]

Take the composite of these functors, to obtain

\[
F_{n_0, n+1} : C_n \rightarrow C_{n+1}.
\]

(Thus, \( C_{n+1} = (C_n^*)^# \)).

As the composite of coherent functors, \( F_{n, n+1} \) is coherent.

Define, for \( k < n \), \( F_{k, n+1} = F_{n, n+1} \circ F_{k, n} \); the \( F_k, n+1 \)
are coherent and compatible (i)\( \circ \) with \( n+1 \) for \( n \).

Note that if \( F \) is conservative for \( m \) and \( G \) for \( F(m) \),
then \( G \circ F \) is conservative for \( m \). Since \( F_{n, n+1} \) is the
composite of functors with suitable conservativeness properties,
by also using the induction hypothesis (ii)\( \circ \) for \( n \), we get
(ii)\( \circ \) for \( n+1 \).
This completes the construction of the diagram

\[ C : \mathbb{N} \rightarrow \text{Coh} \]

where \( C(n) = C_n \), and \( C(1 \leq n) = F_{1n} \).

\( \mathbb{N} \) in the classical directed order.

We define

\[ C_\infty = \text{colim}_n C_n = \text{colim}_n C_n \]

with

\[ F_{n\infty} : C_n \rightarrow C_\infty \]

as the colimit coprojections.

7.2 Theorem with 7.3 Corollary (p.64, p.65) says that

\( C_\infty \) is a coherent category, \( F_{n\infty} \) is a coherent functor (\( n \in \mathbb{N} \)).

7.4 (iii) (p.66) implies that \( F_{\infty} : C_0 \rightarrow C_\infty \) is conservative for \( m \).

CLAIM. \( C_\infty \) is complete and saturated (3.1; pp.12, 13).

To see that \( C_\infty \) is saturated, let \( X \in C_\infty \) have full support: \( X \rightarrow \) \( 1 \in C_\infty \) is e.e. By 7.4, there are \( n \in \mathbb{N} \), \( X \in C_n \), and \( \chi' : X' \rightarrow 1_{C_n} \).
such that $F_{n,m}(\!\! x\!\!) = \!\! x\!\!$. By the construction
of $C^n_\alpha$ (see 8.1), the object $\Sigma^*_{C^n_\alpha}(X')$ has
a global element in $C^n_\alpha$. Therefore, the object

$$\Sigma^*_{C^n_\alpha, m_n}(\Sigma^*_{C^n_\alpha}(X'))$$

$$= F_{n,n+1}(X') \in C_{n+1}$$

also has a global element, and so does $X = F_{\alpha, \alpha}(X') = F_{n+1, \alpha}(F_{n,n+1}(X'))$ in $C_\alpha$, which

was to be proved.

The proof that $C_\alpha$ is complete is similar,
using 7.4 (ii) (p. 66).

The desired $M : C \rightarrow \text{Set}$ is the composite

$$\begin{array}{cc}
\xymatrix{ C \ar[r]^q & C/\mathbb{G} = C_0 \ar[r]^{F_{0,\alpha}} & C_\alpha \ar[r]^\Gamma & \text{Set}, }
\end{array}$$

where $\Gamma$ is the global sections functor $C_\alpha(1_{C_\alpha}, -)$. Since the three factors are all coherent — $\Gamma$ by
"Summing", p. 20) based on the CLAIM —, $M$ is a model.
Write $\Sigma$ for the composite

$$C/\mathcal{B} \xrightarrow{F_{0,m}} C_m \xrightarrow{\Gamma} \text{Set},$$

and remember $\tilde{m} := (A_j \xrightarrow{m} (B, 1_B))$ in $C/\mathcal{B}$.

$F_{0,m}$ is conservative for $\tilde{m}$, and $\Gamma$ is conservative for $F_{0,m}(\tilde{m})$, since $F_{0,m}(\tilde{m})$ is a monomorphism into a terminal object in $C_m$, and we have 6.5 lemma (p49).

Thus $\Sigma$ is conservative for $\tilde{m}$. We have the situation of 6.2 Proposition (p44.1), and we can conclude that $\Sigma \overline{\Phi}_B = M : C \to \text{Set}$ in conservative for $\tilde{m}$.

QED

The proofs of the Corollaries 8.1 and 8.2 are similar to the last proof — but they will use ordinals, indexings by ordinals, and transfinite recursion. For instance, for the proof of 8.1, we start with an enumeration

$$\langle X_\alpha \rangle_{\alpha < \alpha}$$

of all objects of $C$ with full support, and
Construct an ordinal-based diagram \( \langle C_\beta \rangle_{\beta<\alpha} \)

\[ F_{\gamma \beta} : C_\gamma \to C_\beta \ (\gamma < \beta < \alpha) \]

of coherent categories and functors, where the main step of the construction

\[ F_{\beta, \beta+1} : C_\beta \to C_{\beta+1} \ (\beta < \alpha) \]

will take \( C_{\beta+1} = C_\beta / \hat{X}_\beta \) for \( \hat{X}_\beta = F_{0, \beta} (X_\beta) \)

and \( F_{\beta, \beta+1} = \Phi(C_\infty) : C_\beta \to C_\beta / \hat{X}_\beta \)

thereby ensuring that \( \hat{X}_{\beta+1} = F_{\beta, \beta+1} (X_\beta) = F_{0, \beta+1} (X_\beta) \)

has a global element (see p. 39.1).