This contains two propositions

(Proposition 1: p \[\square\]
Proposition 2: p. \[\square\])

towards showing that
the 'coherent logic'
definition of 'pre-topos'
is equivalent to the
Grothendieck-Verdier definition
Proposition 1. In a coherent category $C$, every extremal epi is regular.

Proof. Suppose in:

$$
\begin{array}{ccc}
R & \rightarrow & A \\
\downarrow & & \downarrow f \\
B & \rightarrow & B
\end{array}
$$

$R \xrightarrow{g_0} A \xrightarrow{g} B$ is a pullback and $f$ is an e.e.

We want: If $h$ is the coequalizer of $g_0$ and $g_1$.

To show this, assume

$$
\begin{array}{ccc}
R & \rightarrow & A \\
\downarrow & & \downarrow g \\
B & \rightarrow & C
\end{array}
$$

$g \circ f_0 = g \circ f_1$.

We need $h$ s.t.

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow h \\
C & \rightarrow & C
\end{array}
$$

$hf = g$.

The uniqueness of $h$ is true since $f$ is an epi.
Define \( S \) as the following "double pullback" (finite limit):

\[
\begin{array}{c}
\pi_1 \rightarrow B \\
\pi_0 \rightarrow X \\
\pi_2 \rightarrow C \\
\end{array}
\]

\[
\begin{array}{c}
\rightarrow B \\
\rightarrow A \\
\rightarrow C \\
\end{array}
\]

and \( G \) by:

\[
\begin{array}{c}
\langle \pi_0, \pi_1, \pi_2 \rangle \rightarrow A \times B \times C \\
\rightarrow \Theta \\
\rightarrow B \times C \\
\end{array}
\]

\[
\begin{array}{c}
e \\
\rightarrow G \\
m. \\
\end{array}
\]

\[\text{Claim: } n = \pi_0 \circ m \text{ is an isomorphism:}\]

\[
\begin{array}{c}
G \\
\rightarrow B \times C \\
\rightarrow \Theta \\
\rightarrow B \\
\end{array}
\]

\[n \overset{\text{def}}{=} \pi_0 \circ m \overset{\cong}{\rightarrow} \]

It suffices to show that \( M(n) \) is an isomorphism, for all \( M : C \rightarrow \text{Set} \) coherent.
Let $M: C \to \text{Set}$ be coherent.

Apply $M$ to all the above objects and arrows; let in the notation $\negl M\negl$.

Let us use $a, a', b, c, c'$ for elements of $A, B, C$ (i.e., $M(A), \ldots$), respectively.

The set $S$ is, or rather, can be chosen to be

$$S = \{ (a, b, c) : f(a) = b \land g(a) = c \}$$

and $S$, as a subset of $B \times C$ (with $M$ the inclusion) as

$$G = \{ (b, c) : \exists a \ [ f(a) = b \land g(a) = c]\}.$$ 

The map $\nu: G \to B$ maps

$$(b, c) \in G \to b \in B.$$ 

I claim that for all $b \in B$, there is a unique pair $(b, c) \in G$ with the first component the given $b$. Since $f: A \to B$ is surjective (!),

there is a $a \in A$ such that $f(a) = b$. Let $c = g(a)$. 

We have shown that \((d, c) \in G\). As to uniqueness,

Suppose \((d, c) \in G\) and \((d, c') \in G\).

We have \(a, a' \in A\) such that

\[ f(a) = b \quad \& \quad g(a) = c \quad \& \quad f(a') = b' \quad \& \quad g(a') = c'. \]

\[ \text{same } b \]

\(R\) is the set \(\{ (a, a') \in A \times A : f(a) = f(a') \}\);

\[ r_0((a, a')) = a, \quad r_1((a, a')) = a'. \] The assumption

\[ g_0 = g_1, \] means that

\[ f(a) = f(a') \implies g(a) = g(a'). \]

From \((\ast)\), we have \(f(a) = f(a')\); hence, also \(g(a) = g(a')\).

But then again, by \((\ast')\), \(c = c'\). This shows the desired uniqueness.

The claim is proved.

Back: take \(m^{-1} : B \to G\), and \(h\) the composite:

\[ B \xrightarrow{n^{-1}} G \xrightarrow{m} B \times C \xrightarrow{\pi_1} C; \]

let \(h = \pi_1 \circ m \circ n^{-1} : B \to C\).
Then \( f: A \to B \)
\[ g \downarrow \downarrow \downarrow \]
\[ h \]
\[ c \]

We again may use commutators. So assume we can in Set, and chase the diagram: let \( a \in A \):

\[ a \xrightarrow{f} f(a) = b \]

\[ n \leftrightarrow (b, c) \] such that \( g(a) = c \)

\[ g(a) \equiv c \]
Remark: Consider our claim above. In his claim, he makes a series of choices of objects and arrows, starting with the given one:

\[ f \rightarrow B, \quad A \rightarrow C, \]

and going onto \( S \rightarrow A \times B \times C, \quad B \times C \rightarrow G \), and corresponding arrows.

These choices are not unique, but unique "up to isomorphism". The fact is that if the assertion of the claim, in being an isomorphism, is true by some one choice, then it is true in any other. This was then used when we have verified that \( M(N) \) was an isomorphism in Set, \( M(S) \) may be something else than the standard limit we supposed it to be, etc. — but the same "up to isomorphism".
Proposition 2

\( C : \text{pretopos (in the 'coherent' sense...)} \)

\( A \xrightarrow{f} B : \text{epi} \)

(i.e.,

\( \begin{array}{c}
A \xrightarrow{f} B \\
\xrightarrow{y} X : \text{vf = vf} \Rightarrow u = v \\
\end{array} \)

Assertion: \( A \xrightarrow{f} B \) is an extremal epi.

Suffices: for every \( M : C \rightarrow \text{Set coherent functor} \)

("model"),

\( M(f) \) is a surjective function

\( M(f) : M(A) \xrightarrow{?} M(B) \)

1. Let \( C \) be the disjoint sum \( B \cup B \), according to the "coherent" definition of "disjoint sum".

That is: we have

\[ \begin{array}{c}
\emptyset = B \times B \\
C \\
B \xrightarrow{i_0} \quad \xrightarrow{i_1} \quad B
\end{array} \]

\( i_0, i_1 : \text{mono}; \text{pullback: initial object.} \)

Writing \( B_0 \) for the subobject (defined by \((B, i_0)\) of \( C \),
\[ B_0 \in \text{Sub}(C), \quad \text{and similarly } \quad B_1 = (B, i_2), \quad B_1 \in \text{Sub}(C) \]

\[ B_0 \land B_1 = \bot \quad (= \bot_C \text{, bottom element of } \text{Sub}(C)) \]

Consider the arrows:

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
& \downarrow{i_0} & \downarrow{i_0} \\
A & \xrightarrow{\langle i_0, i_0 \rangle} & C \\
\end{array} \]

\[ \begin{array}{ccc}
A & \xrightarrow{\langle i_0, i_0 \rangle} & C \\
& \downarrow{i_0} & \downarrow{i_0} \\
A & \xrightarrow{m_0} & C \\
\end{array} \]

\[ \begin{array}{ccc}
A & \xrightarrow{e_0} & A_0 \\
\end{array} \]

\[ \text{and define } \quad e_0, A_0, m_0 \text{ similarly.} \]

Then:

\[ \begin{array}{ccc}
m_0 \times C & \xrightarrow{A_0 \times C} & C \times C \\
\end{array} \]

\[ \begin{array}{ccc}
m_1 \times C & \xrightarrow{A_1 \times C} & C \times C \\
\end{array} \]

\[ \begin{array}{ccc}
C \times m_0 & \xrightarrow{C \times A_0} & C \times C \\
\end{array} \]

\[ \begin{array}{ccc}
C \times m_1 & \xrightarrow{C \times A_1} & C \times C \\
\end{array} \]

Let the corresponding subobjects of \( C \times C \)

\[ \begin{array}{ccc}
A_{00}, A_{10}, A_{01}, A_{11} \end{array} \] respectively.
Let \( S_c \) be the subobject \((C, \langle l_c, l_c' \rangle) \in \text{Sub}(C)\). 

Finally, let 

\[ R = \Delta_C \cup (A_{00} \lor A_{10}) \lor (A_{01} \lor A_{11}) \]

When this construction is done in Set, we have:

\[ c R c' \iff c = c' \]

or

\[ (\exists a. c = i_0 f(a)) \lor (\exists a. c = i_1 f(a)) \quad (1) \]

and

\[ (\exists a. c' = i_0 f(a)) \lor (\exists a. c' = i_1 f(a)) \quad (2) \]

CLAIM: when in Set, \( R \) is an equivalence relation on \( C \).

\( R \) is reflexive: \( c R c \)

\( R \) is symmetric: \( c R c' \implies c' R c \)

\( R \) is transitive:
Note:

\[ c \in R x' \iff c = c' \lor (B(c) \land \overline{B(c')}) \]

where \( B(c) \), \( B(c') \) are

abreviaching (1) & (2) &

the same predicate \( B(-) \)

figures in (1) & (2)

Now: \( c \in R x' \land c \in R x'' \implies c \in R x'' \)

\[ \iff c = c' \lor (B(c) \land \overline{B(c')}) \]

\&

\[ c = c'' \lor (B(c') \land \overline{B(c'')}) \]

\[ \implies c = c'' \lor (B(c) \land \overline{B(c'')}) \]

Case 1: \( c = c' \land \overline{c = c''} \): Then \( c = c'' \): OK

Case 2: \( c = c' \land B(c') \land B(c'') \): Then

\[ c = c' \land B(c') \implies B(c) \]

and so \( B(c) \land B(c'') \): OK

Case 3: \( B(c) \land B(c') \land B(c'') \): OK

Case 4: as Case 2.
Since for every model $M: C \rightarrow \mathcal{E}k$, $M(R) \subseteq M(\mathcal{E}k)$ is an equivalence relation, $R$ is an equivalence relation in the category $C$.

2. $C$ is a pretopos; $R \rightarrow C \times C$

\[
\xymatrix{ R \ar[r]^{r_0} \ar[d]^r \ar[r]^{r_1} & C } \]

has a quotient $C \xrightarrow{p} D$;

\[
\xymatrix{ R \ar[r]^{r_0} \ar[d]^r \ar[r]^{r_1} & C \ar[r]^p & D } \]

(kernel pair - diagram & $p$ is extremal epi).

I claim: $p_1 \circ f = p_1 \circ f$

\[
\xymatrix{ A \ar[r]^f & B \ar[r]^i_0 \ar[r]^{i_2} & C \ar[r]^p & D } \]
It suffices to show that this holds in any model.

When we are in Set, the fact that (*) holds in a kernel-pair diagram means that for the subset \( R \) in \( C \times C \),

\[
\text{that } (c, c') \in C \quad \text{implies} \quad p(c) = p(c') \iff c R c'.
\]

(Think of the pull-back

\[
\begin{array}{ccc}
R & \overset{r_0}{\to} & C & \overset{p}{\to} & D \\
\uparrow & & \uparrow & & \uparrow \\
\downarrow & & \downarrow & & \downarrow \\
C & \overset{r_1}{\to} & C & \overset{p}{\to} & D
\end{array}
\]

Let \( a \in A \), and \( c = i_0 f(a) \), \( c' = i_1 f(a) \).

Look at the definition of \( R \); p. 6. We see that:

\[ c R c'. \]

Therefore, \( p(c) = p(c') \);

\[ p_{i_0} f(a) = p_{i_1} f(a). \]
Since \( a \in A \) is arbitrary,

\[
p_i \circ f = p_i \downarrow f
\]

holds.

We have proved that \( p_i \circ f = p_i \downarrow f \) holds in models, hence, in the category \( C \).

Since \( f \) is an epi,

\[
p_i \circ \downarrow f = p_i \downarrow f
\]

in \( C \).

3. Now, let \( M : C \to \text{Set} \) be any model,

we show that \( M(f) : M(A) \to M(B) \)

is surjective. As usual, we write just \( f : A \to B \)

for \( M(f) \), etc.

Let \( b \in B \); we have \( p_i \circ \downarrow (b) = p_i \circ f \downarrow (b) \).

Thus, \( \bigcirc \downarrow \), \( p_i \downarrow \)(b) \( R_i \circ \downarrow (b) \).
However, $i_0(b) \neq i_1(b)$, since the subobjects 

\[(B, i_0), (B, i_1)\] 

of $C$ are disjoint.

Looking at the definition of $R$ (top, p. 47), we have

\[\mathcal{B}(\iota_0(b)) \text{ and } \mathcal{B}(\iota_1(b)) \text{ in } C,\]

i.e.: 

\[
\begin{cases}
\exists a. \ i_0(b) = i_0 f(a) \text{ or } \exists a. \ i_1(b) = i_1 f(a) \\
\text{and} \quad \exists a. \ i_1(b) = i_0 f(a) \text{ or } \exists a. \ i_2(b) = i_2 f(a)
\end{cases}
\]

\(2\) & \(3\) are impossible: $i_0$ & $i_1$ have disjoint images; thus $1$ & $4$ hold.

But $1$ is enough. We have $a \in A$ s.t. $i_0(b) = i_0 f(a)$.

Since $i_0$ is a monomorphism, $b = f(a)$.
This shows that
\[ f : A \to B \]

or rather,
\[ \text{M}(f) : \text{M}(A) \to \text{M}(B) \text{ in } \mathcal{C}, \]

is surjective.

Q.E.D.